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ON S-CLOSED AND EXTREMALLY DISCONNECTED FUZZY TOPOLOGICAL SPACES

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Abstract. The concepts of a filter-base and s-convergence and θ-convergence of a filter-base in a fuzzy setting are defined and investigated. Fuzzy filter-base is used to characterize fuzzy S-closed and extremally disconnected spaces. Several other properties of these two types of spaces and comparison between different forms of compactness in fuzzy topology are established.

Introduction

The concept of filters in fuzzy set theory was introduced by Lowen and at the same time by Katsaras who studied in his work [11] fuzzy filters, ultra filters, clusters and the convergence of filters in fuzzy setting. In this paper we have developed the theory of filters a little further and introduced s-convergence and θ-convergence of a filter (filter-base).

We offer several characterizations of fuzzy S-closed and fuzzy extremally disconnected spaces in terms of fuzzy filter-bases and fuzzy nets. The results are parallel to ones which have been found in general topology. A systematic discussion of these properties in general topology is given in [7], [10], [16] and [17].

In the last section we study the implications between different forms of compactness and comparison with S-closedness and extremal disconnectedness in a fuzzy setting.

1. Preliminaries

Throughout the paper by (X, τ_X), or simply by X, we mean a fuzzy topological space (fts, shortly) of Chang [3]. A fuzzy singleton with support x and value α $(0 < \alpha \leqslant 1)$ will be denoted by x_α. Two fuzzy sets λ and μ are said to be q-coincident if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$ and by $\bar{q}$ we denote "is not q-coincident". A fuzzy set λ is said to be a q-neighbourhood (q-nbd) of x_α if there is a fuzzy open set μ such that $x_\alpha \, q \, \mu$ and $\mu \leqslant \lambda$, where $\lambda \leqslant \mu$ if $\lambda(x) \leqslant \mu(x)$,

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for all $x \in X$. We denote ny $N(x_\alpha)$ ($N_q(x_\alpha)$) the neighbourhood (q-nbd) system of x_α [12]. For a fuzzy set λ in an fts X by $\mathrm{Cl}\,\lambda$, $\mathrm{SCl}\,\lambda$, $\mathrm{Int}\,\lambda$, $\mathrm{supp}\,\lambda$ we denote the closure, semi-closure, interior and support of λ, respectively. By 0_X and 1_X we mean the constant fuzzy sets taking the values 0 and 1 on X, resp. For the defnition of fuzzy regularly open ($RO(X)$), regularly closed ($RC(X)$), semi-open ($SO(X)$) and semi-closed ($SC(X)$) sets we refer to Azad [2]. A fuzzy point $x_\alpha \in \mathrm{Cl}_\delta\,\lambda$ ($x_\alpha \in \mathrm{Cl}_\theta\,\lambda$) if $x_\alpha\; q\; U$ implies $U\; q\; \lambda$, for each $U \in RO(X)$ ($x_\alpha\; q\; U$ implies $\mathrm{Cl}\,U\; q\; \lambda$ for each $U \in \tau_X$ resp.). A fuzzy set λ is called δ-closed (θ-closed resp.) if $\lambda = \mathrm{Cl}_\delta\,\lambda$ ($\lambda = \mathrm{Cl}_\theta\,\lambda$). It is known [15] that for any set λ, $\lambda \leqslant \mathrm{SCl}\,\lambda \leqslant \mathrm{Cl}\,\lambda \leqslant \mathrm{Cl}_\delta\,\lambda \leqslant \mathrm{Cl}_\theta\,\lambda$.

An fts (X, τ_X) is said to be semi-regular if the fuzzy regularly open sets of X form a base for τ_X [2]. For (X, τ_X), X_S denotes the semiregularization of X, i.e. $X_S = (X, \tau_S)$ and $N_S(x_\alpha)$ the nbd-system of x_α in X_S. An fts X is called fuzzy almost regular (fuzzy regular, resp.) if every fuzzy regularly open (resp. fuzzy open) set λ of X can be expressed as a union of fuzzy regularly open (fuzzy open, resp.) sets U_α such that $\mathrm{Cl}\,U_\alpha \leqslant \lambda$, for all α. Fuzzy regularity implies fuzzy semi-regularity as well as fuzzy almost regularity (see [1] and [15]).

2. Nets and filters in fuzzy topology

Definition 2.1 [12] Let $(D, \geqslant)$ be a directed set. Let X be an ordinary set. Let $\mathcal{J}$ be the collection of all fuzzy points in X. A function $S\colon D \to \mathcal{J}$ is called a fuzzy net in X. For $n \in D$, $S(n)$ is often denoted by $x^n_{\alpha_n}$, where x^n is the support and α_n is the value of the n-th member of the fuzzy set. Hence the net S is often denoted by $\{x^n_{\alpha_n}, n \in D\}$ and it is called α-net if $\alpha_n \to \alpha$.

Definition 2.2 [12] A fuzzy net $\{x^n_{\alpha_n}\}$ is said to be q-coincident with $\lambda \in I^X$ if for each $n \in D$, $x^n_{\alpha_n}$ is q-coincident with λ; it is said to be eventually q-coincident (or q-final) with λ if there is an $m \in D$ such that if $n \in D$ and $n \geqslant m$, then $x^n_{\alpha_n}$ is q-coincident with λ; it is said to be frequently q-coincident (or q-cofinal) with λ if for each $m \in D$, there is an $n \in D$ such that $n \geqslant m$ and $x^n_{\alpha_n}$ is q-coincident with λ.

Definition 2.3. [12] A fuzzy net $\{x^n_{\alpha_n}\}$ in an fts X is said to converge to a fuzzy point x_α if $\{x^n_{\alpha_n}\}$ is eventuallly q-coincident with each q-nbd of x_α.

Definition 2.4. [11] Let $\mathcal{B}$ be a nonempty family of fuzzy subsets of I^X. Then $\mathcal{B}$ is called a base for a fuzzy filter on X (or a fuzzy filter-base) if the following two conditions are satisfied:

(1) $0_X \notin \mathcal{B}$;

(2) if $\lambda_1, \lambda_2 \in \mathcal{B}$, then $\lambda_1 \wedge \lambda_2 \in \mathcal{B}$.

If $\mathcal{B}$ has the property

(3) $\lambda \in \mathcal{B}$ and $\lambda \leqslant \mu$ implies $\mu \in \mathcal{B}$,

then $\mathcal{B}$ is called a fuzzy filter on X.

A maximal, with respect to set inclusion, fuzzy filter on X is called a fuzzy ultra-filter or a maximal filter (or filter-base). If $\mathcal{B}$ is a base for a fuzzy filter on

X, the collection $F_{\mathcal{B}} = \{\, \mu \in I^X : \exists \lambda \in \mathcal{B} \text{ with } \lambda \leqslant \mu \,\}$ is the fuzzy filter generated by $\mathcal{B}$. We say that a filter F_1 is finer than a filter F_2 if $F_2 < F_1$, i.e. if for each $\lambda \in F_2$, there exists $\mu \in F_1$ such that $\mu \leqslant \lambda$.

Bellow are listed some results on fuzzy filter bases (ffb, shortly) which one can prove in a straightforward manner.

THEOREM 2.1. (1) *Let F_1, F_2 be any ffb on fts X. Then the family $F_1 \vee F_2 = \{\, \lambda_1 \vee \lambda_2 : \lambda_1 \in F_1, \lambda_2 \in F_2 \,\}$ is an ffb on X.*

(2) *If $\lambda_1 \wedge \lambda_2 \neq 0$ for each $\lambda_1 \in F_1$ and each $\lambda_2 \in F_2$, then $F_1 \wedge F_2 = \{\, \lambda_1 \wedge \lambda_2 : \lambda_1 \in F_1, \lambda_2 \in F_2 \,\}$ is an ffb on X.*

(3) *A nonempty family $\mathcal{B} \subset I^X$ is an ffb on X iff for any finite collection $\{\lambda_i\}_1^n$ from $\mathcal{B}$, $\bigwedge_{i=1}^n \lambda_i \neq 0$.*

(4) *Let $\mathcal{B}$ is an ffb on X and let $f\colon X \to Y$ be a function. Then $f(\mathcal{B}) = \{f(\lambda) : \lambda \in \mathcal{B}\}$ is an ffb on Y. If f is onto and $\mathcal{B}$ is an ffb on Y, then $f^{-1}(\mathcal{B}) = \{f^{-1}(\mu) : \mu \in \mathcal{B}\}$ is an ffb on X.*

DEFINITION 2.5. A fuzzy point x_α is said to be a cluster point of a filter-base $\mathcal{B}$ (i.e. $F_{\mathcal{B}}$) if every q-nbd of x_α is q-coincident with each member of $\mathcal{B}$.

PROPOSITION 2.1. *A fuzzy point x_α $(0 < \alpha \leqslant 1)$ in an fts X is a cluster point of a filter-base $\mathcal{B}$ iff $x_\alpha \in \operatorname{Cl}\lambda$, for each $\lambda \in \mathcal{B}$.*

Proof. It is straightforward. This proposition is equivalent to the definition 3.2 [6]. ∎

DEFINITION 2.6. [6] A filter-base $\mathcal{B}$ is said to converge to x_α (denoted by $\mathcal{B} \to x_\alpha$) if every q-nbd of x_α contains a member of $\mathcal{B}$ and $x_\alpha \in \operatorname{Cl}\lambda$, for every $\lambda \in \mathcal{B}$.

THEOREM 2.2. *Let F be an ffb on fts X and x_α a fuzzy point. Then:*

(1) *$F \to x_\alpha$ iff $N_q(x_\alpha) < F$.*

(2) *If $\mathcal{B}$ is a base for the q-nbd system of x_α, then $\mathcal{B} \to x_\alpha$.*

(3) *If x_α is a cluster point of an ffb F on X and U is a q-nbd of x_α, then $G = \{\lambda \wedge U : \lambda \in F\}$ is finer than F and $G \to x_\alpha$.*

(4) *Let λ be a non-empty fuzzy set. If $F \to x_\alpha$ and there exists $\mu \in F$ such that $\mu \leqslant \lambda$, then $x_\alpha \in \operatorname{Cl}\lambda$.*

The proof is straightforward.

THEOREM 2.3. *Let x_α be a fuzzy point in an fts X. Then*

(1) *x_α is a cluster point of an ffb F iff there exists an ffb B finer than F and $B \to x_\alpha$.*

(2) *If $F \to x_\alpha$, then x_α is a cluster point of F.*

(3) *If x_α is a cluster point of a fuzzy ultra-filter F, then $F \to x_\alpha$.*

Proof. (1) $B = N_q(x_\alpha) \vee \{\lambda \wedge U : \lambda \in F, U \in N_q(x_a)\}$ is an ffb finer than F and $B \to x_\alpha$. Converesely, let $B \to x_\alpha$ and let $B < F$. Let $\lambda \in F$. Since $B < F$, there exists $\mu \in F$ such that $\mu \leqslant \lambda$; $x_\alpha \in \mathrm{Cl}\,\mu$ implies $x_\alpha \in \mathrm{Cl}\,\lambda$, for each $\lambda \in F$, i.e. x_α is a cluster point of F.

(2) It follows from (1).

(3) Since x_α is a cluster point of an ultra-filter F, that means that for each $U \in N_q(x_\alpha)$, $U\ q\ \lambda$, for each $\lambda \in F$, that implies $U \wedge \lambda \neq 0$ and hence $U \in F$. Therefore $F \to x_\alpha$. ∎

COROLLARY 2.1. *If x_α is a cluster point of a filter F_1 that is finer than F_2, then x_α is a cluster point of the filter F_2. If x_α is a limit of the filter F_2, then x_α is the limit of every filter F_1 finer than F_2.*

PROPOSITION 2.2. [11] *Let X, Y be fts's and x_α be a fuzzy point in X. If f is a mapping from X to Y, continuous at x_α, then for every filter-base $\mathcal{B}$, $\mathcal{B} \to x_\alpha$ implies $f(\mathcal{B}) \to f(x)_\alpha$.*

DEFINITION 2.7. We say that a fuzzy point $x_\alpha \in \mathrm{SCl}_\theta\,\lambda$ if for each semi open fuzzy set U, $x_\alpha\ q\ U$ implies $\mathrm{Cl}\,U\ q\ \lambda$.

DEFINITION 2.8. A fuzzy filter (or a filter-base) F is said to:

(1) δ-accumulate to x_α if $x_\alpha \in \mathrm{Cl}_\delta\,\lambda$, for each $\lambda \in F$ [6].

(2) s-accumulate to x_α if $x_\alpha \in \mathrm{SCl}_\theta\,\lambda$, for each $\lambda \in F$.

(1) θ-accumulate to x_α if $x_\alpha \in \mathrm{Cl}_\theta\,\lambda$, for each $\lambda \in F$.

DEFINITION 2.9. Let (X, τ_X) be an fts and let $N_S^q(x_\alpha)$ be the q-nbd filter in X_S, i.e. the filter generated by regular open fuzzy sets; $S(x_\alpha)$ be the filter generated by the family $SO(x_\alpha) = \{\mathrm{Cl}\,\lambda : x_\alpha\ q\ \lambda \in SO(\tau_X)\}$, and $X(x_\alpha)$ be the filter generated by $C(x_\alpha) = \{\mathrm{Cl}\,\lambda : x_\alpha\ q\ \lambda \in \tau_X\}$. We say that a filter (or a filter-base) F

(1) δ-converges to x_α if $N_S^q(x_\alpha) < F$;

(2) s-converges to x_α if $S(x_\alpha) < F$;

(3) θ-converges to x_α if $C(x_\alpha) < F$.

PROPOSITION 2.3. *A filter (or a filter-base) F in an fts X*

(1) *δ-converges to x_α iff every fuzzy regular open q-nbd of x_α contains some member of F and $x_\alpha \in \mathrm{Cl}_\delta\,\lambda$ for each $\lambda \in F$;*

(2) *s-converges to x_α iff for each fuzzy semi open q-nbd μ of x_α there is a $\lambda \in F$ such that $\lambda \leqslant \mathrm{Cl}\,\mu$ and $x_\alpha \in \mathrm{SCl}_\theta\,\lambda$, for each $\lambda \in F$;*

(3) *θ-converges to x_α iff for every open q-nbd μ of x_α there is $\lambda \in F$ such that $\lambda \leqslant \mathrm{Cl}\,\mu$ and $x_\alpha \in \mathrm{Cl}_\theta\,\lambda$ for each $\lambda \in F$.*

Proof. Easy. ∎

THEOREM 2.4. *A filter-base $\mathcal{B}$ (or a filter) on X s-accumulates (θ-accumulates, resp.) iff there exists a filter F finer than $\mathcal{B}$ which s-converges (θ-converges, resp.).*

Proof. Let x_α be an s-cluster point of $\mathcal{B}$. Then $x_\alpha \in \mathrm{SCl}_\theta \lambda$, for every $\lambda \in \mathcal{B}$. Hence for every $\mu \in SO(x_\alpha)$ and for each $\lambda \in \mathcal{B}$, $\mu \; q \; x_\alpha$ implies $\mathrm{Cl}\,\mu \; q \; \lambda$, i.e. $\mathrm{Cl}\,\mu \wedge \lambda \neq 0$. Therefore $\mathcal{B} \vee \{\mathrm{Cl}\,\mu : \mu \in SO(x_\alpha)\}$ is a system of generators for some filter F finer than $\mathcal{B}$ and it s-converges to x_α. The converse is obvious. Similarly for a θ-cluster point. ∎

3. S-closed and extremally disconnected fuzzy topological spaces

Definition 3.1. [3,5] A family $\{\lambda_\alpha : \alpha \in I\}$ of fuzzy open subsets of a fuzzy topological space (X, τ_X) is called a cover if $\bigvee\{\lambda_\alpha : \alpha \in I\} = X$. A fuzzy topological space is called compact (shortly FC) if every open cover has a finite subcover. An fts X is said to be fuzzy nearly (almost) compact (shortly FNC (FAC)) if every open cover contains a finite subfamily $\{\lambda_{\alpha_i} : i = 1, \dots, n\}$ such that $X = \bigvee_{i=1}^{n} \mathrm{Int}\,\mathrm{Cl}\,\lambda_{\alpha_i}$ ($X = \bigvee_{i=1}^{n} \mathrm{Cl}\,\lambda_{\alpha_i}$).

Definition 3.2. [15] An fts X is fuzzy δ-compact (denoted FδC) if every open cover contains a finite δ-open subcover.

Definition 3.3. [4] An fts X is S-closed if every fuzzy semi-open cover $\{\lambda_\alpha : \alpha \in I\}$ contains a finite subfamily $\{\lambda_{\alpha_i}\}_{i=1}^{n}$ such that $X = \bigvee_{i=1}^{n} \mathrm{Cl}\,\lambda_{\alpha_i}$.

Definition 3.4. [9] An fts X is extremally disconnected (denoted FED) if the closure of every fuzzy open set is open.

Using all these definitions it is easy to see that the following implications hold:

$$F\delta C \implies FC \implies FNC \implies FAC.$$

Theorem 3.1. *A semi-regular fts X is $F\delta C$ iff it is FC.*

Proof. The proof follows easily from the fact that $\tau_S = \tau_X$. ∎

Using results from [5] and [15] and theorem 3.1, we have:

Theorem 3.2. *a) If fts X is semi-regular, then $F\delta C \sim FC \sim FNC$;*

b) if X is fuzzy regular, then $F\delta C \sim FC \sim FNC \sim FAC$ ("$\sim$" means "is equivalent").

Theorem 3.3. *Let an fts (X, τ_X) be S-closed. Then for every family of fuzzy open sets $\{\lambda_\alpha : \alpha \in I\}$ of X with finite intersection property (FIP, shortly) it holds $\bigwedge_\alpha \mathrm{Cl}\,\lambda_\alpha \neq 0$.*

Proof. Let $\{\lambda_\alpha\}_{\alpha \in I}$ be a family of fuzzy open sets with FIP. If $\bigwedge_\alpha \mathrm{Cl}\,\lambda_\alpha = 0$, then $\bigvee_\alpha (1 - \mathrm{Cl}\,\lambda_\alpha) = 1_X$ and $\{1 - \mathrm{Cl}\,\lambda_\alpha\}_\alpha$ is a semi-open cover of X. Hence, there exists a finite subcollection $\{\lambda_{\alpha_i}\}_{i=1}^{n}$ such that $\bigvee_{i=1}^{n} \mathrm{Cl}(1 - \mathrm{Cl}\,\lambda_{\alpha_i}) = 1_X$. Therefore $\bigwedge_{i=1}^{n} \lambda_{\alpha_i} \leqslant \bigwedge_{i=1}^{n} (1 - \mathrm{Cl}(1 - \mathrm{Cl}\,\lambda_{\alpha_i})) = 0$, a contradiction. Hence $\bigwedge_\alpha \mathrm{Cl}\,\lambda_\alpha \neq 0$. ∎

Theorem 3.4. *If an fts (X, τ_X) is extremally disconnected and for any family of fuzzy open sets $\{\lambda_\alpha\}_\alpha$ of X with FIP it holds $\bigwedge_\alpha \mathrm{Cl}\,\lambda_\alpha \neq 0$, then X is S-closed.*

Proof. Let $\{\lambda_\alpha\}_{\alpha\in I}$ be a semi-open cover of X and suppose that X is not S-closed. Hence for each finite family $\{\lambda_{\alpha_i}\}_{i=1}^n$, $\bigvee_{i=1}^n \operatorname{Cl}\lambda_{\alpha_i} < 1$. Therefore $\bigwedge_{i=1}^n(1-\operatorname{Cl}\lambda_{\alpha_i}) \neq 0$, i.e. $\{1-\operatorname{Cl}\lambda_\alpha\}_\alpha$ is a family of fuzzy open sets with FIP. Hence $\bigwedge_\alpha \operatorname{Cl}(1-\operatorname{Cl}\lambda_\alpha) \neq 0$. Since X is FED and since $\operatorname{Cl}\lambda_\alpha = \operatorname{Cl}\operatorname{Int}\lambda_\alpha$, for each $\lambda_\alpha \in SO(X)$ it follows that $\operatorname{Cl}\lambda_\alpha$ is fuzzy open for each $\alpha \in I$, i.e. $1-\operatorname{Cl}\lambda_\alpha$ is fuzzy closed. Thus

$$\bigvee_\alpha \lambda_\alpha \leqslant \bigvee_\alpha \operatorname{Cl}\lambda_\alpha = \bigvee_\alpha (1-(1-\operatorname{Cl}\lambda_\alpha)) = \bigvee_\alpha (1-\operatorname{Cl}(1-\operatorname{Cl}\lambda_\alpha)) < 1_X,$$

a contradiction (since $\{\lambda_\alpha\}_\alpha$ is a cover of X). ∎

Proposition 3.1. *An fts X is S-closed iff any cover by regular closed sets has a finite subcover.*

Proof. It is trivial, since the closure of a semi-open set is regular closed, therefore semi-open. Even more, it is easy to see that $RC(X) = \{\operatorname{Cl}\lambda_\alpha : \lambda_\alpha \in SO(X)\}$. ∎

Corollary 3.1. *S-closedness is a fuzzy semiregular property, i.e. X is S-closed iff X_S is S-closed.*

Proof. The proof is straightforward since an fts and its semiregularization have the same fuzzy regular closed sets. ∎

Theorem 3.5. *For an fts X the following are equivalent:*

(1) *X is S-closed;*

(2) *each filter-base in X s-accumulates;*

(3) *every maximal filter-base on X s-converges.*

Proof. (2) $\iff$ (3). See theorem 2.1.

(1) $\iff$ (3). Let X be S-closed and let F be a maximal filter-base on X which doesn't s-converge. Therefore, it doesn't s-accumulate to any point. This implies that for each point x_α there exists a semi-open set $\mu_\alpha \in S(x_\alpha)$ and an element $\lambda_\alpha \in F$ such that $x_\alpha\ q\ \mu_\alpha$ and $\operatorname{Cl}\mu_\alpha\ \bar{q}\ \lambda_\alpha$. Without loss of generality, we may assume that $C = \{\mu_\alpha\}_{x_\alpha \in I^X}$ is a semi-open cover of X, because if $x_\alpha \notin \mu_\alpha$, then $x_{\alpha'} = x_{1-\alpha} \in \mu_\alpha$, for each $\alpha \in (0,1)$. Since X is S-closed, there exists a finite subcollection $\{\mu_{\alpha_i}\}_{i=1}^n$ such that $X = \bigvee_{i=1}^n \operatorname{Cl}\mu_{\alpha_i}$. Since F is a maximal filter-base there exists $\lambda \neq 0$, $\lambda \in F$ such that $\lambda \leqslant \bigwedge_{i=1}^n \lambda_{\alpha_i}$ and $\lambda_{\alpha_i}\ \bar{q}\ \operatorname{Cl}\mu_{\alpha_i}$, for each i. This implies that $\lambda\ \bar{q}\ \operatorname{Cl}\mu_{\alpha_i}$, for each $i = 1,\dots,n$, i.e. $\lambda\ \bar{q}\ \bigvee_{i=1}^n \operatorname{Cl}\mu_{\alpha_i} = X$, a contradiction, since $\lambda \neq 0$.

Conversely, suppose X is not S-closed and let $\{\lambda_\alpha\}_{\alpha\in I}$ be a semi-open cover of X such that $\bigvee_{i=1}^n \operatorname{Cl}\lambda_{\alpha_i} < X$ for every finite family $\{\lambda_{\alpha_i} : i = 1,\dots,n\}$. Then there exists an x_t such that $\operatorname{Cl}\lambda_{\alpha_i}(x_t) < t$, for each $1 \leqslant i \leqslant n$. Hence $\mu_{\alpha_i}(x_t) = (1-\operatorname{Cl}\lambda_{\alpha_i})(x_t) > 1-t$, i.e. $\mu_{\alpha_i}\ q\ x_t$ and $\mu_{\alpha_i} \in SO(X)$. Therefore $\bigwedge_{i=1}^n \mu_{\alpha_i} \neq 0$, for each finite intersection. Hence $\{\mu_\alpha\}_\alpha$ forms a filter-base $\mathcal{B}$ which s-accumulates to x_t. Then there exists a maximal filter-base F which s-converges to x_t, and this implies that $\bigvee_\alpha \operatorname{Cl}\lambda_\alpha \neq X$. Therefore $\bigvee_\alpha \lambda_\alpha \neq X$, a contradiction, since $\{\lambda_\alpha\}_\alpha$ is a cover of X. ■

It is known that the product of S-closed spaces is not necessarily S-closed, even in general topology. In fuzzy topology for fuzzy spaces X and Y, where X is product related to Y (cf. [2, Definition 3.7; Theorem 3.10; Theorem 4.6]) we have the following result:

Theorem 3.6. *Let (X, τ_X) and (Y, τ_Y) be S-closed fts's such that X is product related to Y. Then fuzzy topological product $X \times Y$ is S-closed.*

Proof. Let $\{\lambda_\alpha\}_\alpha$ and $\{\mu_\beta\}_\beta$ be semi-open covers for X, respectively Y. Then $\{\lambda_\alpha \times \mu_\beta\}_{\alpha,\beta}$ is fuzzy semi-open cover of $X \times Y$. For finite $n \in \mathbf{N}$ we have $\bigvee_{i=1}^n \operatorname{Cl}\lambda_{\alpha_i} \times \bigvee_{i=1}^n \operatorname{Cl}\mu_{\beta_i} = \bigvee_{i=1}^n \operatorname{Cl}(\lambda_{\alpha_i} \times \mu_{\beta_i})$, i.e. $X \times Y = \bigvee_{i=1}^n \operatorname{Cl}(\lambda_{\alpha_i} \times \mu_{\beta_i})$. Thus $X \times Y$ is S-closed. ■

Definition 3.5. [14] A function $f\colon X \to Y$ is said to be fuzzy semi-continuous (resp. irresolute) of $f^{-1}(\mu) \in SO(X)$, for any fuzzy open (resp. semi-open) set μ of Y.

Theorem 3.7. [4] *If $f\colon X \to Y$ is an irresolute surjection from an S-closed fts X to an fts Y, then Y is S-closed.*

Since fuzzy semi-continuous and almost open (i.e. $f^{-1}(\operatorname{Cl}\lambda) < \operatorname{Cl} f^{-1}(\lambda)$, for each $\lambda \in \tau_Y$) mapping implies that f is irresolute [6] we have the following corollary:

Corollary 3.2. *An image of an S-closed space under an almost open fuzzy continuous surjection is S-closed.*

Definition 3.6. A property of fts is called fuzzy semi-topological if it is preserved by fuzzy semi-homeomorphism (i.e. bijection and such that the images of semi-open sets are semi-open and the inverses of semi-open sets are semi-open).

Corollary 3.3. *To be an S-closed fts is a fuzzy semi-topological property.*

Theorem 3.8. *For an fts (X, τ_X) the following are equivalent:*

(1) *X is FED.*

(2) *If a filter base on X δ-converges, then it s-converges.*

(3) *A filter-base X s-converges iff it θ-converges.*

(4) *If a filter-base on X converges with respect to the topology τ_X, then it s-converges.*

Proof. We shall first prove the following lemma.

Lemma 3.1. *If an fts X is FED, then $\operatorname{SCl}\lambda = \operatorname{Cl}_\theta\lambda$, for each $\lambda \in SO(X)$.*

Proof of Lemma. $\operatorname{SCl}\lambda \leqslant \operatorname{Cl}\lambda \leqslant \operatorname{Cl}_\delta\lambda \leqslant \operatorname{Cl}_\theta\lambda$, for each $\lambda \in I^X$. We shall prove that $\operatorname{Cl}_\theta\lambda \leqslant \operatorname{SCl}\lambda$, for $\lambda \in SO(X)$. Let $x_\alpha \in \operatorname{SCl}\lambda$, then there exists $U \in SO(X)$ such that $x_\alpha\ q\ U$ and $U\ \bar{q}\ \lambda$, which implies that $U \leqslant \lambda^c$, where

$\lambda^c = 1 - \lambda$. Hence $\mathrm{Cl}\,U \leqslant \mathrm{Cl}(\lambda^c)$. Since X is FED, then $\mathrm{Cl}\,U$ is open. Therefore $\mathrm{Cl}\,U \leqslant \mathrm{Int}\,\mathrm{Cl}(\lambda^c) \leqslant \mathrm{SCl}(\lambda^c) = \lambda^c$. Hence $\mathrm{Cl}\,U \;\bar{q}\; \lambda$ and so $x_\alpha \notin \mathrm{Cl}_\theta\,\lambda$. ■

Corollary 3.4. *If X is FED, then* $\mathrm{SCl}\,\lambda = \mathrm{Cl}\,\lambda = \mathrm{Cl}_\delta\,\lambda = \mathrm{Cl}_\theta\,\lambda$, *for each* $\lambda \in SO(X)$.

Using the results above, the proof of theorem 3.8 is now straightforward. ■

Remark 3.1. From the above corollary and Lemma 7 and Theorem 12 in [9] we have the following: $\mathrm{SCl}\,\lambda = \mathrm{Cl}\,\lambda = \mathrm{Cl}_\delta\,\lambda = \mathrm{Cl}_\theta\,\lambda$, for every fuzzy set $\lambda \in SO(X) \cup PO(X)$, where $PO(X)$ is the set of all fuzzy pre-open sets in X. Also it is easy to prove that definition of rc-convergence [9] is equivalent to our definition of s-convergence.

Lemma 3.2. *Let X be fuzzy almost regular and S-closed. Then X is FED and fuzzy nearly compact (FNC).*

Proof. Suppose that X is not FED. Then there exists a fuzzy regular open set λ such that $\mathrm{Cl}\,\lambda(x) > \lambda(x)$ for some $x \in X$ and $\mathrm{Cl}\,\lambda \neq 1_X$. Let $x_\alpha \in \mathrm{Cl}\,\lambda$ and $x_\alpha \notin \lambda$. For every open q-nbd U of x_α, we have $U \wedge \lambda \neq 0$. Therefore $F = \{U \wedge \lambda : U \in N_q(x_\alpha)\}$ forms a filter-base in $\mathrm{Cl}\,\lambda$. Since $\mathrm{Cl}\,\lambda$ is S-closed relative to X, then $F \xrightarrow{s} y_\beta$, i.e. F s-converges to some point $y_\beta \in \mathrm{Cl}\,\lambda$. If $y_\beta \notin \lambda$, then $y_\beta \; q \; \lambda^c = 1 - \lambda \in RC(X)$, hence $1 - \lambda$ is semi-open and $1 - \lambda \in S(y_\beta)$. Therefore, there exists $\mu \in F$ such that $\mu \leqslant 1 - \lambda$, i.e. $\mu \;\bar{q}\; \lambda$. Since $F \to y_\beta$ in the usual sense and $y_\beta \in \mathrm{Cl}\,u \leqslant 1 - \lambda$, therefore every member of F is q-coincident with $1 - \lambda$, what is impossible. Hence $y_\beta \in \lambda$. Almost regularity and $\lambda \in RO(X)$ imply that $\lambda \in S(y_\beta)$, i.e. there exists $V \in RO(X)$ such that $y_\beta \; q \; V \leqslant \mathrm{Cl}\,V \leqslant \lambda$. Since $x_\alpha \notin \lambda$, then $x_\alpha \; q \; \mathrm{Cl}\,V^c = 1 - \mathrm{Cl}\,V$ is an open q-nbd of x_α. Since $N_q(x_\alpha) \to x_\alpha$, there exists a q-nbd U of x_α such that $(U \wedge \lambda) \leqslant 1 - \mathrm{Cl}\,V$, i.e. $(U \wedge \lambda) \;\bar{q}\; \mathrm{Cl}\,V$. But that contradicts the fact that $F \xrightarrow{s} y_\beta$. Therefore it must be $x_\alpha = y_\beta$, i.e. $\mathrm{Cl}\,\lambda = \lambda$. Hence X is FED.

To prove nearly compactness, let $\mathcal{U}$ be a maximal filter-base. S-closedness implies that $\mathcal{U}$ s-converges. Theorem 3.8.(3) implies that $\mathcal{U}$ θ-converges. Almost regularity implies that $\mathcal{U}$ δ-converges. Since X is fuzzy nearly compact (see Theorem 3.9 [13] and Theorem 2.3) iff every maximal filter-base δ-converges, then this completes the proof. ■

Corollary 3.5. *Let X be an almost regular fts. Then X is S-closed iff X is FNC and FED iff X_S is regular, compact and FED.*

Theorem 3.9. *Let fts (X, τ_X) be a fuzzy regular space. Then the following are equivalent:*

(1) *X is compact and FED;*

(2) *X is S-closed;*

(3) *X is FAC and FED.*

Proof. (1) $\Longrightarrow$ (2). See Theorem 3.4.

(2) $\Longrightarrow$ (3). That S-closed implies FAC, follows directly from the definitions. Using lemma 3.2 we have the proof of (2) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (1). The proof is straightforward. ∎

COROLLARY 3.6. (cf. [4, Theorem 3.7; Corollary 3.8]) *Let X be fuzzy regular and extremally disconnected. Then the following is valid:*

$$FC \sim F\delta C \sim S\text{-}closed \sim FAC \sim FNC.$$

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