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APPLICATIONS OF GENERALIZED CHARACTERISTIC EQUATION OF LINEAR DELAY DIFFERENCE EQUATIONS

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Abstract. Linear delay difference equations as well as generalized characteristic equations and their importance at oscillation of all solutions of considered difference equations are studied in this paper. Some results of comparison of solutions of difference equations and inequalities as well as conditions for the existence of positive solutions are presented as the application of the generalized characteristic equation.

1. Introduction

In the work by G. Ladas and I. Győri in [1] as an open problem appears to obtain some discrete analogues of the results related to generalized characteristic equation of the linear delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0, \quad t_0 \le t \le T$$

where $t_0 < T \le \infty$ and for $i = 1, 2, \dots, n$, $p_i \in C[[t_0, T), R]$, $\tau_i \in C[[t_0, T), R^+]$.

For this reason consider the linear delay difference equation

$$a_{n+1} - a_n + \sum_{i=1}^{m} P_i(n)a_{n-k_i(n)} = 0, \quad n \in \mathbf{N}^* \tag{1}$$

where $\mathbf{N}^* = \{\, n \in \mathbf{N} : n_0 \le n < M,\ n_0 < M \le \infty \,\}$ and $\mathbf{N}$ is the set of positive integers. Let

$$\{P_i(n)\} \text{ be a sequence of real numbers for } i = 1, 2, \dots, m,\ n \in \mathbf{N}^* \tag{2}$$

$$\{k_i(n)\} \text{ be a sequence of positive integers for } i = 1, 2, \dots, m,\ n \in \mathbf{N}^*. \tag{3}$$

Define

$$n_{-1} = \min_{1 \le i \le m} \left\{ \inf_{n_0 \le n < M} \{n - k_i(n)\} \right\}. \tag{4}$$

With equation (1) one associates an initial condition

$$a_n = \Phi_n, \ \text{ for } n = n_{-1}, n_{-1} + 1, \dots, n_0,\ \Phi_n \in \mathbf{R}. \tag{5}$$

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The unique solution of the IVP (1) and (5) is denoted by $a(\Phi)_n$ and it exists for $n \in \mathbf{N}^*$.

Define for $i = 1, 2, \dots, m$ and $n \in \mathbf{N}^*$ the sequences

$$h_i(n) = \min\{n_0, n - k_i(n)\}, \quad H_i(n) = \max\{n_0, n - k_i(n)\}.$$

The generalized characteristic equation associated with the difference equation (1) is the equation

$$\lambda_n - 1 + \sum_{i=1}^{m} P_i(n) \frac{\Phi_{h_i(n)}}{\Phi_{n_0}} \prod_{j=H_i(n)}^{n-1} \frac{1}{\lambda_j} = 0 \quad \text{for} \quad n \in \mathbf{N}^*. \tag{6}$$

The next result will be used in the proofs of the following theorems. For proof see [4].

THEOREM 1.1. *Assume that (2) and (3) hold and let $\Phi_{n_0} > 0$. Then the following statements are equivalent:*

(1) the solution of the initial value problem (1) and (5) is positive for $n \in \mathbf{N}^*$,

(2) the generalized characteristic equation (6) has a positive solution for $n \in \mathbf{N}^*$,

(3) there exist real sequences $\{\beta_n^*\}$ and $\{\gamma_n^*\}$ such that $0 < \beta_n^* \le \gamma_n^*$ for $n \in \mathbf{N}^*$ and such that for any real sequence $\{\delta_n\}$ with $\beta_n^* \le \delta_n \le \gamma_n^*$ for $n \in N^*$ the following inequalities hold:

$$\beta_n^* \le S\delta_n \le \gamma_n^* \quad \textit{for} \quad n \in \mathbf{N}^*$$

where

$$S\delta_n \equiv 1 - \sum_{i=1}^{m} P_i(n) \frac{\Phi_{h_i(n)}}{\Phi_{n_0}} \prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j}.$$

2. Comparisson results

Consider, now, the delay difference equation

$$y_{n+1} - y_n + \sum_{i=1}^{m} Q_i(n) y_{n-k_i(n)} = 0 \quad \text{for} \quad n \in \mathbf{N}^* \tag{7}$$

and the delay difference inequalities

$$x_{n+1} - x_n + \sum_{i=1}^{m} P_i(n) x_{n-k_i(n)} \le 0 \quad \text{for} \quad n \in \mathbf{N}^*, \tag{8}$$

$$z_{n+1} - z_n + \sum_{i=1}^{m} R_i(n) z_{n-k_i(n)} \ge 0 \quad \text{for} \quad n \in \mathbf{N}^*. \tag{9}$$

The next result is a discrete analogue to the Theorem 3.2.1. in [1] formulated for differential equations and inequalities.

THEOREM 2.1. *Suppose that* $\{P_i(n)\}$, $\{Q_i(n)\}$ *and* $\{R_i(n)\}$ *are nonnegative real sequences, and* $\{k_i(n)\}$ *is a sequence of positive integers for* $i = 1, 2, \ldots, m$, $n \in \mathbf{N}^*$ *and that* $P_i(n) \geq Q_i(n) \geq R_i(n)$ *for* $n \in \mathbf{N}^*$. *Assume that* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *are solutions of (8), (7) and (9), respectively, such that*

$$z_{n_0} \geq y_{n_0} \geq x_{n_0}, \quad x_n > 0 \quad \text{for} \quad n \in \mathbf{N}^*, \tag{10}$$
$$\frac{x_n}{x_{n_0}} \geq \frac{y_n}{y_{n_0}} \geq \frac{z_n}{z_{n_0}} > 0 \quad \text{for} \quad n = n_{-1}, n_{-1}+1, \ldots, n_0 - 1.$$

Then

$$z_n \geq y_n \geq x_n \quad \text{for} \quad n \in \mathbf{N}^*. \tag{11}$$

Proof. Set $\alpha_n = x_{n+1}/x_n$, $\beta_n = y_{n+1}/y_n$, $\gamma_n = z_{n+1}/z_n$ for $n \in \mathbf{N}^*$. Then, by using the previous notation, it follows that

$$\alpha_n - 1 + \sum_{i=1}^{m} P_i(n) \frac{x_{h_i(n)}}{x_{n_0}} \prod_{j=H_i(n)}^{n-1} \frac{1}{\alpha_j} \leq 0 \quad \text{for} \quad n \in \mathbf{N}^*,$$
$$\beta_n - 1 + \sum_{i=1}^{m} Q_i(n) \frac{y_{h_i(n)}}{y_{n_0}} \prod_{j=H_i(n)}^{n-1} \frac{1}{\beta_j} = 0 \quad \text{for} \quad n \in \mathbf{N}^*.$$
$$\gamma_n - 1 + \sum_{i=1}^{m} R_i(n) \frac{z_{h_i(n)}}{z_{n_0}} \prod_{j=H_i(n)}^{n-1} \frac{1}{\gamma_j} \geq 0 \quad \text{for} \quad n \in \mathbf{N}^*.$$

It can be shown by induction and by using the Theorem 1.1. that

$$\alpha_n \leq \beta_n \leq \gamma_n \quad \text{for} \quad n \in \mathbf{N}^*. \tag{12}$$

For the first part of the inequality (12) let $\{\delta_n\}$ be an arbitrary sequence of real numbers such that $\alpha_n \leq \delta_n \leq 1$ for $n \in \mathbf{N}^*$. Then

$$\begin{aligned}\alpha_n &\leq 1 - \sum_{i=1}^{m} P_i(n) \frac{x_{h_i(n)}}{x_{n_0}} \prod_{j=H_i(n)}^{n-1} \frac{1}{\alpha_j} \\ &\leq 1 - \sum_{i=1}^{m} Q_i(n) \frac{y_{h_i(n)}}{y_{n_0}} \prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j} \equiv S\delta_n \leq 1.\end{aligned}$$

Then, the statement (3) of the Theorem 1.1. is true (with $\beta_n^* = \alpha_n$ and $\gamma_n^* \equiv 1$) so by the same theorem the equation

$$\delta_n - 1 + \sum_{i=1}^{m} Q_i(n) \frac{y_{h_i(n)}}{y_{n_0}} \prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j} = 0$$

has exactly one solution for $n \in \mathbf{N}^*$ and the solution of this equation is between α_n and 1 for $n \in \mathbf{N}^*$. But $\{\beta_n\}$ is the solution of the same equation for $n \in \mathbf{N}^*$. Hence, $\alpha_n \leq \beta_n \leq 1$ for $n \in \mathbf{N}^*$, the first part of (12) is established.

To prove the second part of inequality (12) it can be shown by induction that $\alpha_n \leq \gamma_n$ for $n \in \mathbf{N}^*$. Then, in similar way as before (with $\beta_n^* = \alpha_n$ and $\gamma_n^* = \gamma_n$) the second part of the inequality (12) can be proved, too.

Because of the equalities

$$x_n = x_{n_0} \prod_{j=n_0}^{n-1} \alpha_j, \quad y_n = y_{n_0} \prod_{j=n_0}^{n-1} \beta_j, \quad z_n = z_{n_0} \prod_{j=n_0}^{n-1} \gamma_j,$$

(10) and (12) imply that (11) hold and the proof is complete. ∎

3. Existence of positive solutions

Set, now, that $P_i^*(n) = \max\{0, P_i(n)\}$ for $i = 1, 2, \ldots, m$, $n \in \mathbf{N}^*$,

$$k = \max_{1 \le i \le m} \left\{ \sup_{n_0 \le n < M} \{k_i(n)\} \right\}, \quad k(n) = \max_{1 \le i \le m} \{k_i(n)\} \quad \text{for} \quad n \in \mathbf{N}^*,$$
$$F = \{\, \{\Phi_j\} : \Phi_{n_0} > 0,\ 0 < \Phi_j \le \Phi_{n_0} \text{ for } j = n_{-1}, n_{-1}+1, \ldots, n_0 \,\}.$$

The next results are discrete analogues of Theorems 3.3.1., 3.3.2. and 3.3.3. in [1] and they extend results from [1] and [3].

Theorem 3.1. *Assume that (2), (3) and (4) hold, and let*

$$\sum_{i=1}^{m} P_i^*(n) \frac{(k+1)^{k+1}}{k^k} \le 1 \quad \textit{for} \quad n \in \mathbf{N}^*. \tag{13}$$

Then, for every $\{\Phi_n\} \in F$, solution $a(\Phi)_n$ of difference equation (1) remains positive for $n \in \mathbf{N}^*$.

Proof. Consider the difference equation

$$b_{n+1} - b_n + \sum_{i=1}^{m} P_i^*(n) b_{n-k_i(n)} = 0 \quad \text{for} \quad n \in \mathbf{N}^*. \tag{14}$$

Now, one can show that the statement (3) of Theorem 1.1. is true for any sequence $\{\delta_n\}$ such that

$$0 < \beta_n^* \equiv \frac{k}{k+1} \le \delta_n \le 1 \equiv \gamma_n^* \quad \text{for} \quad n \in \mathbf{N}^*. \tag{15}$$

Because of $n - H_i(n) \le k_i(n) \le k$, and in view of (15) it follows that

$$\prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j} \le \prod_{j=H_i(n)}^{n-1} \frac{k+1}{k} \le \frac{(k+1)^k}{k^k}. \tag{16}$$

By means of (13), (15) and (16), the next inequalities follow:

$$1 \ge 1 - \sum_{i=1}^{m} P_i^*(n) \frac{\Phi_{h_i(n)}}{\Phi_{n_0}} \prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j} \equiv S\delta_n \ge 1 - \sum_{i=1}^{m} P_i^*(n) \frac{(k+1)^k}{k^k} \ge \frac{k}{k+1}.$$

Therefore, the solution $b(\Phi)_n$ of (14) is positive for $n \in \mathbf{N}^*$. Since the solution $a(\Phi)_n$ of (1) is also a solution of inequality

$$a_{n+1} - a_n + \sum_{i=1}^{m} P_i^*(n) a_{n-k_i(n)} \ge 0 \quad \text{for} \quad n \in \mathbf{N}^*,$$

and by using the Theorem 2.1. it follows that $a(\Phi)_n \geq b(\Phi)_n > 0$ for $n \in N^*$, and the proof is complete. ■

THEOREM 3.2. *Assume that (2), (3) and (4) hold and there exists a real number* $\mu \in (0,1)$ *such that*

$$\sum_{i=1}^{m} |P_i(n)|(1-\mu)^{-k_i(n)} \leq \mu \quad \text{for} \quad n \in \mathbf{N}^*. \tag{17}$$

Then, for every $\{\Phi_n\} \in F$, *the solution* $a(\Phi)_n$ *of (1) remains positive for* $n \in \mathbf{N}^*$.

Proof. One must show that the statement (3) of Theorem 1.1. is true for any sequence $\{\delta_n\}$ such that

$$0 < \beta_n^* \equiv 1-\mu \leq \delta_n \leq 1+\mu \equiv \gamma_n^* \quad \text{for} \quad n \in \mathbf{N}^*, \tag{18}$$

because of $n - H_i(n) \leq k_i(n)$ and from (18) it follows that

$$\prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j} \leq \prod_{j=H_i(n)}^{n-1} \frac{1}{1-\mu} \leq (1-\mu)^{-k_i(n)}. \tag{19}$$

Now, in view of (17), (18) and (19) it follows that

$$\begin{aligned} 1-\mu &\leq 1 - \sum_{i=1}^{m} |P_i(n)|(1-\mu)^{-k_i(n)} \leq 1 - \sum_{i=1}^{m} |P_i(n)| \prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j} \\ &\leq 1 - \sum_{i=1}^{m} P_i(n) \frac{\Phi_{h_i(n)}}{\Phi_{n_0}} \prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j} \equiv S\delta_n \\ &\leq 1 + \sum_{i=1}^{m} |P_i(n)| \prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j} \leq 1 + \sum_{i=1}^{m} |P_i(n)|(1-\mu)^{-k_i(n)} \leq 1+\mu \end{aligned}$$

for $n \in \mathbf{N}^*$, and the proof is complete. ■

THEOREM 3.3. *Assume that (2), (3) and (4) hold, and that* $0 \leq k_1(n) \leq k_2(n) \leq \cdots \leq k_m(n)$ *for* $n \in \mathbf{N}^*$,

$$\sum_{i=1}^{s} P_i(n) \leq 0 \quad \text{for} \quad s = 1,2,\ldots,m,\ n \in \mathbf{N}^*. \tag{20}$$

Then, the equation (1) has a positive increasing solution for $n \in N^*$.

Proof. Let $\Phi_j = 1$ for $j = n_{-1}, n_{-1}+1, \ldots, n_0$. One can claim that the statement (3) of Theorem 1.1. will be true if $\{\delta_n\}$ is any sequence such that

$$\beta_n^* \equiv 1 \leq \delta_n \leq 1 + \sum_{i=1}^{m} |P_i(n)| = \gamma_n^* \quad \text{for} \quad n \in \mathbf{N}^*. \tag{21}$$

Because of $H_1(n) \geq H_2(n) \geq \cdots \geq H_m(n)$, (20) and (21) it holds

$$\begin{aligned} 1 &\leq 1 + \left\{ -\sum_{i=1}^{m} P_i(n) \right\} \prod_{j=H_m(n)}^{n-1} \frac{1}{\delta_j} \leq 1 - \sum_{i=1}^{m} P_i(n) \prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j} \equiv S\delta_n \\ &\leq 1 + \sum_{i=1}^{m} |P_i(n)| \prod_{j=H_i(n)}^{n-1} \frac{1}{\delta_j} \leq 1 + \sum_{i=1}^{m} |P_i(n)| \quad \text{for} \quad n \in \mathbf{N}^*. \end{aligned}$$

Therefore, by Theorem 1.1., the solution $a_n = a(\Phi)_n$ of equation (1) is positive for $n \in \mathbf{N}^*$. Moreover, $a_n = \prod_{j=n_0}^{n-1} \lambda_j$ for $n \in N^*$, where $\{\lambda_n\}$ is a positive solution of characteristic equation associated with equation (1) between β_n and γ_n for $n \in \mathbf{N}^*$. Hence, $\{a_n\}$ is an increasing solution of (1) and the proof is complete. ∎

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