МАТЕМАТИЧКИ ВЕСНИК
50 (1998), 23–29

UDK 515.122.2
оригинални научни рад
research paper

ON NEARLY PARACOMPACT SPACES VIA REGULAR EVEN COVERS

M. N. Mukherjee and Atasi Debray

Abstract. The concept of near paracompactness is a well known weaker form of paracompactness, first introduced by Singal and Arya [6], followed by its further pursuit by many others. In the present paper, the same concept has been investigated in terms of a certain type of cover, called regular even cover. A number of characterizations of almost regular nearly paracompact spaces, including those analogous to the famous Michael's results on a regular paracompact space, have been achieved.

Paracompactness is one of the most celebrated concepts of general topology, and many of its variant forms have so far been investigated. Of the variegated generalizations of paracompactness found in the literature, mention may be made of near paracompactness, almost paracompactness, mildly paracompact spaces, m- and mn-paracompactness etc. A somewhat detailed survey of some weaker and stronger forms of paracompatness may be found in [5]. In [6], Singal and Arya introduced and studied nearly paracompact spaces. The behaviour of such a space in relation to near compactness has been seen to be similar to the role of paracompactness vis-à-vis compactness. Near paracompactness has further been taken up for study in [1] and [4]. All these investigations have furnished the generalized versions of many of the well known results on paracompactness.

This paper is a continuation of the study of nearly paracompact spaces in terms of regular even covers, a concept introduced here. We ultimately achieve a number of characterizations of such a space in an almost regular space, some of which present the analogues of the famous Michael's theorem on paracompact spaces in the presence of the axiom of regularity.

In what follows, by a space X we shall mean a topological space (X,τ). For a subset A of X, $\operatorname{int} A$ and $\operatorname{cl} A$ will stand respectively for interior and closure of A in (X,τ). The diagonal Δ_X of X is, as usual, given by $\Delta_X = \{(x,x) : x \in X\}$. For two covers $\mathcal{U}$ and $\mathcal{V}$ of X, we shall write $\mathcal{U} < \mathcal{V}$ to mean that $\mathcal{U}$ is a refinement

AMS Subject Classification: 54 D 20

Keywords and phrases: Nearly paracompact spaces, almost regular spaces, regular even cover, strongly locally finite cover, strongly discrete cover.

of $\mathcal{V}$, where we assume that a refinement of a cover is necessarily a cover of X. If V is any subset of $X \times X$ and $A \subseteq X$, then by $V[A]$ we shall denote the set $\bigcup\{V[x] : x \in A\}$, where $V[x] = \{y \in X : (x,y) \in V\}$. We recall that a set A in a space X is called regular open if $A = \operatorname{int}\operatorname{cl} A$, and complements of regular open sets are called regular closed. The class of all regular open sets in a space (X,τ) is a base for a topology on X coarser than τ, called the semiregularization topology on X and is denoted by τ_s [2]. The members of τ_s are called δ-open sets of X [8] and the complements of such sets are known as δ-closed sets [8]. The following result, to be used in the sequel, can be proved in a straightforward manner (see [3] for details).

Theorem 1. *For a topological space* (X,τ), $(\tau \times \tau)_s = \tau_s \times \tau_s$.

A family $\mathcal{U}$ of subsets in a space X will be called strongly locally finite (strongly discrete) if for each $x \in X$ there exists a regular open set V containing x such that V intersects at most finitely members (at most one member) of $\mathcal{U}$. A cover $\mathcal{U}$ of X is said to have a σ-strongly locally finite (σ-strongly discrete) refinement $\mathcal{V}$ if $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$, where each $\mathcal{V}_n$ is a strongly locally finite (resp. strongly discrete) family and $\mathcal{V} < \mathcal{U}$. A set A in a space X is called a δ-neighbourhood (henceforth abreviated as nbd.) of a point x of X if there is a δ-open set U such that $x \in U \subseteq A$. If, in addition, A itself is δ-open, we shall term A as a δ-open nbd. of x. According to Singal and Arya [6], a topological space X is called nearly paracompact if every regular open cover of X has a locally finite open refinement. Let us now set the following definition:

Definiton 2. A cover $\mathcal{U}$ of a topological space X is said to be regular even, if there exists some δ-nbd. V of the diagonal Δ_X in the product space $X \times X$ such that $\{V[x] : x \in X\} < \mathcal{U}$.

Theorem 3. *Let $\mathcal{U}$ be a regular open cover of a topological space X, which has a regular closed locally finite (strongly locally finite) refinement $\mathcal{R}$. Then $\mathcal{U}$ is a regular even cover.*

Proof. For each $A \in \mathcal{R}$, we can choose $U_A \in \mathcal{U}$ such that $A \subseteq U_A$. Define $V_A = (U_A \times U_A) \cup ((X \setminus A) \times (X \setminus A))$, for each $A \in \mathcal{R}$. Then each V_A is clearly a δ-open nbd. of the diagonal Δ_X in $X \times X$. Now, for each $x \in A$, we first show that $V_A[x] = U_A$. In fact, $x \in A \implies x \notin (X \setminus A)$. So, for all $y \in X$,

$$(x,y) \notin (X \setminus A) \times (X \setminus A). \tag{1}$$

Now, $y \in V_A[x] \implies (x,y) \in V_A \implies (x,y) \in U_A \times U_A$ (by (1)) $\implies y \in U_A$. Again, $y \in U_A \implies (x,y) \in U_A \times U_A$ (since $A \subseteq U_A$) $\implies (x,y) \in V_A \implies y \in V_A[x]$. Hence $V_A[x] = U_A$, for each $x \in A$. Put $V = \bigcap\{V_A : A \in \mathcal{R}\}$. Then for each $x \in X$, there exists $A \in \mathcal{R}$ such that $x \in A$ and then $V[x] \subseteq V_A[x] = U_A$. Consequently, $\{V[x] : x \in X\} < \mathcal{U}$.

Now, we are left to show that V is a δ-nbd. of Δ_X in $X \times X$. Let us choose any $(x,x) \in \Delta_X$ ($x \in X$). By local finitness (strong local finitness) of $\mathcal{R}$, there exists an open (resp. a regular open) set W_x containing x in X such that $W_x \cap A \neq \emptyset$ for at most finitely many $A \in \mathcal{R}$, say for $A_1, A_2, \ldots, A_k$. Our claim is:

$$(\operatorname{int}\operatorname{cl} W_x \times \operatorname{int}\operatorname{cl} W_x) \cap V_{A_1} \cap V_{A_2} \cap \cdots \cap V_{A_k} \subseteq V. \tag{2}$$

Indeed, for any $A \in \mathcal{R} \setminus \{A_1, A_2, \ldots, A_k\}$, we have: $W_x \cap A = \emptyset \implies W_x \subseteq X \setminus A \implies \operatorname{int}\operatorname{cl} W_x \subseteq \operatorname{int}\operatorname{cl}(X \setminus A) = X \setminus A$. Thus $(\operatorname{int}\operatorname{cl} W_x \times \operatorname{int}\operatorname{cl} W_x) \subseteq (X \setminus A) \times (X \setminus A) \subseteq V_A$, for all $A \neq A_1, A_2, \ldots, A_k$, i.e. $(\operatorname{int}\operatorname{cl} W_x \times \operatorname{int}\operatorname{cl} W_x) \subseteq \{V_A : A \in \mathcal{R}, A \neq A_1, A_2, \ldots, A_k\}$, from which (2) follows. Now, the left side of (2) is a δ-open set containing (x,x) so that the arbitrariness of (x,x) ensures that V is a δ-nbd. of Δ_X. ∎

Definition 4. [7] A topological space X is said to be almost regular if for any regular closed set F and any point $x \in X \setminus F$, there exists disjoint open sets containing F and x respectively.

The following theorem was obtained by Singal and Arya in [6]:

Theorem 5. *In an almost regular space X the following hold:*

(a) If a regular open cover of X has a locally finite refinement then it has a closed locally finite refinement.

(b) If a regular open cover of X has a locally finite closed refinement then it has a locally finite regular closed refinement.

Remark 6. The above theorem remains equally valid (without much changes in the proof) if each of the locally finite refinements involved in the statements is a strongly locally finite refinement.

Theorem 7. *Let X be a topological space such that each regular open cover of X is regular even. If U is a δ-nbd. of the diagonal Δ_X in $X \times X$, then there is a symmetric δ-open nbd. V of the diagonal Δ_X such that $V \circ V \subseteq U$, where $V \circ V = \{(x,y) \in X \times X : (x,z), (z,y) \in V,$ for some $z \in X\}$.*

Proof. As U is a δ-nbd of Δ_X, we have (using Theorem 1) that for each $x \in X$, there is a regular open nbd. W_x of x with $W_x \times W_x \subseteq U$. Since $\mathcal{W} = \{W_x : x \in X\}$ is a regular open cover of X, it is a regular even cover of X and consequently, there is some δ-nbd. V_1 of the diagonal Δ_X in $X \times X$ such that $\{V_1[x] : x \in X\} < \mathcal{W}$ and hence $V_1[x] \times V_1[x] \subseteq U$. Obviously, we can regard V_1 to be δ-open. Let $V = V_1 \cap V_1^{-1}$. Clearly V is symmetric and contains Δ_X. Now we see that if f_x $(x \in X)$ and g are the maps given by $f_x\colon X \to X \times X$, $g\colon X \times X \to X \times X$ such that $f_x(y) = (y,x)$ and $g(x,y) = (y,x)$, where X and $X \times X$ are equipped with the topologies τ_s and $\tau_s \times \tau_s$ respectively, then $f_x \circ p_2 = g$, $p_2\colon X \times X \to X$ being the second projection map. Now, p_2 and f_x being continuous, so is g. Thus for a δ-open set U of X (i.e. $U \in \tau_s$), $g^{-1}(U)$ $(= U^{-1})$ is also δ-open. Hence V_1 being a δ-open set, so is V_1^{-1} and hence so is $V_1 \cap V_1^{-1}$, i.e., V. Thus V is a symmetric δ-open nbd. of Δ_X such that $V[x] \times V[x] \subseteq U$, for all $x \in X$. Since $V \circ V = \bigcup_{x \in X}(V[x] \times V[x])$ (V being symmetric) it follows that $V \circ V \subseteq U$. ∎

Theorem 8 *Let X be a topological space such that each regular open cover is regular even and let $\mathcal{R}$ be a strongly locally finite (strongly discrete) family of subsets of X. Then there is a δ-open nbd. W of the diagonal in $X \times X$ such that $\{W[A] : A \in \mathcal{R}\}$ is strongly locally finite (resp. strongly discrete).*

Proof. As $\mathcal{R}$ is strongly locally finite (strongly dicrete), for each $x \in X$, there is some regular open set V_x containing x such that $V_x \cap R \neq \emptyset$ for at most finitely many (resp. one) members R of $\mathcal{R}$. Then $\mathcal{V} = \{V_x : x \in X\}$ is a regular open cover of X and hence is regular even. Then by definition, there is some δ-nbd. U of Δ_X such that $\{U[x] : x \in X\} < \mathcal{V}$. Again, U being a δ-nbd. of Δ_X, by Theorem 7, there is some symmetric δ-open nbd. W of Δ_X such that $W \circ W \subseteq U$. Now, we observe that for any $A \subseteq X$,

$$(W \circ W)[x] \cap A = \emptyset \implies W[x] \cap W[A] = \emptyset. \tag{1}$$

In fact, $y \in W[x] \cap W[A] \implies (x, y) \in W$ and $(z, y) \in W$, for some $z \in A \implies (x, y), (y, z) \in W$ (W being symmetric) $\implies (x, z) \in W \circ W \implies z \in (W \circ W)[x] \cap A$, establishing (1).

Now, for each $x \in X$, $(W \circ W)[x] \subseteq U[x] \subseteq V$, for some $V \in \mathcal{V}$. By construction of $\mathcal{V}$, V can meet at most finitely many members of $\mathcal{R}$ (resp. one member of $\mathcal{R}$). Consequently, $(W \circ W)[x]$ can meet at most finitely many (resp. at most one) members $A \in \mathcal{R}$. In view of (1), it follows that $W[x]$ meets at most finitely many (resp. one) members like $W[A]$, $A \in \mathcal{R}$. Now W being a δ-open nbd. of Δ_X, $W[x]$ is a δ-open nbd. of x, since $W[x] = f_x^{-1}(W)$, where $f_x \colon (X, \tau_s) \to (X \times X, \tau_s \times \tau_s)$, given by $f_x(y) = (x, y)$ (for $y \in X$) is continuous. Thus for some regular open set T containing x, $T \subseteq W[x]$ and clearly T meets at most finitely many (resp. one) members like $W[A]$, $A \in \mathcal{R}$. Hence, $\{W[A] : A \in \mathcal{R}\}$ is a strongly locally finite (resp. strongly discrete) collection. ∎

Remark 9. If V is a regular open or a δ-open subset in $X \times X$, then $V[x]$ is δ-open in X, for each $x \in X$. So for any $A \subseteq X$, $V[A] = \bigcup_{x \in A} V[x]$ is also δ-open in X, and hence open in X. By the last theorem, each member of a strongly locally finite (or strongly discrete) family can be enlarged to an open set such that the latter collection is also strongly locally finite (resp. strongly discrete).

Corollary 10. *Let X be a topological space such that each regular open cover of X is regular even. If each regular open cover $\mathcal{U}$ of X has a strongly locally finite (strongly discrete) refinement $\mathcal{R}$, then each regular open cover $\mathcal{U}$ has a strongly locally finite (resp. strongly discrete) open refinement.*

Proof. By Theorem 8, there is a δ-open nbd. V of the diagonal Δ_X such that $\{V[A] : A \in \mathcal{R}\}$ is strongly locally finite. This family may not be a refinement of $\mathcal{U}$. For that, we choose $U_A \in \mathcal{U}$, for each $A \in \mathcal{R}$, such that $A \subseteq U_A$. Construct $W_A = U_A \cap V[A]$. Then $\{W_A : A \in \mathcal{R}\}$ is the required family. ∎

Theorem 11. *If X is a topological space such that each regular open cover of X is regular even, then every regular open cover of X has an open σ-strongly* *discrete refinement (i.e. an open refinement which can be expressed as a countable union of strongly discrete families).*

Proof. In view of Corollary 10, it is enough to find a σ-strongly discrete refinement of a given regular open cover $\mathcal{U}$.

Since $\mathcal{U}$ is a regular even cover, there exists some δ-open nbd. V od Δ_X such that $\{V[x] : x \in X\} < \mathcal{U}$. Define $V_0 = V$. Then by induction, applying Theorem 7, we construct for each natural number n, a symmetric δ-open nbd. V_n of the diagonal Δ_X such that $V_n \circ V_n \subseteq V_{n-1}$. Let $U_1 = V_1$ and define inductively $U_{n+1} = U_n \circ V_{n+1}$, for $n = 2, 3, \ldots$. It is easy to verify that $U_n \subseteq V_0 = V$, for all natural numbers n. As $\{V_0[x] : x \in X\} = \{V[x] : x \in X\} < \mathcal{U}$, it follows that $\{U_n[x] : x \in X\} < \mathcal{U}$, for each natural number n. By Zermelo's theorem choose some well-ordering $<$ in X. For each natural number n and each $x \in X$, let $U_n^*(x) = U_n[x] \setminus \bigcup\{U_{n+1}[y] : y < x\}$. Now for each $y \in X$, $V_{n+1}[U_n^*(y)]$ is a δ-open nbd. of y. To verify that for each natural number n, the family $\mathcal{U}_n = \{U_n^*(x) : x \in X\}$ is strongly discrete, it suffices to verify that $U_n^*(x) \cap V_{n+1}[U_n^*(y)] = \emptyset$ whenever $x \neq y$. In fact, suppose, if possible, $p \in U_n^*(x) \cap V_{n+1}[U_n^*(y)]$ $(y \neq x)$. Then $p \in U_n^*(x) = U_n[x] \setminus \bigcup\{U_{n+1}[y] : y < x\}$ and $p \in V_{n+1}[z]$, for some $z \in U_n^*(y)$. Thus

$$p \in U_n[x] \text{ and } p \notin U_{n+1}[y], \text{ for all } y < x \quad (y \in X), \tag{1}$$

and

$$(z, p) \in V_{n+1} \text{ for some } z \in U_n[y] \text{ but } z \notin \bigcup\{U_{n+1}[q] : q < y\}. \tag{2}$$

Then

$$(y, p) \in U_n \circ V_{n+1} = U_{n+1}, \text{ i.e. } p \in U_{n+1}[y]. \tag{3}$$

From (1) and (3), we get $y > x$ and consequently (by (2))

$$z \notin U_{n+1}[x]. \tag{4}$$

Again, $(x, p) \in U_n$ and $(p, z) \in V_{n+1}$ (since each V_n is symmetric) so that $(x, z) \in U_n \circ V_{n+1} = U_{n+1}$, i.e. $z \in U_{n+1}[x]$, a contradiction to (4). Thus our contention is established.

Now, $\{U_n[z] : z \in X\}$ is a cover of X, for each n. We finally show that $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ is a cover of X. Let $x \in X$. For each natural number n, choose the first point $y(n)$ (say) of X such that $x \in U_n[y(n)]$. Let $y(k) = \min\{y(n) : n = 1, 2, \ldots\}$. Then $x \notin U_n[z]$, for $z < y(k)$ and for each n. Put $y = y(k)$. We see that $x \in U_k[y]$ and $x \notin U_{k+1}[z]$ for all $z < y$. Thus $x \in U_k^*(y)$, proving that $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ is a cover of X. This completes the proof. ∎

THEOREM 12. *If each regular open cover of a space X has an open σ- (strongly) locally finite refinement, then each regular open cover of X has a (strongly) locally finite refinement.*

Proof. Let $\mathcal{U}$ be a regular open cover of X. Let $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbf{N}\}$ (where $\mathbf{N}$ stands for the set of all natural numbers) be an open σ- (strongly) locally finite refinement of $\mathcal{U}$, where each $\mathcal{V}_n$ is (strongly) locally finite. For each $n \in \mathbf{N}$, and each $V \in \mathcal{V}_n$, let $V_n^* = V \setminus \bigcup\{U : U \in \mathcal{V}_k \text{ for } k < n\}$ (resp. $V_n^* = \operatorname{int} \operatorname{cl} V \setminus \bigcup\{\operatorname{int} \operatorname{cl} U : U \in \mathcal{V}_k \text{ for } k < n\}$). Let $\mathcal{W} = \{V_n^* : n \in \mathbf{N}, V \in \mathcal{V}_n\}$. Let $x \in X$. If n is the first positive integer such that $x \in W$ (resp. $x \in \operatorname{int} \operatorname{cl} W$) for some $W \in \mathcal{V}_n$, then clearly $x \in W_n^*$. Consequently, $\mathcal{W}$ is a cover of X. Obviously, $\mathcal{W} < \mathcal{U}$. Now to show that $\mathcal{W}$ is (strongly) locally finite, let $x \in X$, and n be the first positive integer such that $V \in \mathcal{V}_n$, $x \in V$ (resp. $x \in \operatorname{int} \operatorname{cl} V$). Then V (resp. $\operatorname{int} \operatorname{cl} V$) is an open (regular open) nbd. of x. We observe that for all $k > n$ and for each $P \in \mathcal{V}_k$, $P_k^* \cap V = \emptyset$ (resp. $P_k^* \cap \operatorname{int} \operatorname{cl} V = \emptyset$). So V can intersect at most the elements of $\mathcal{V}_k$, for $k \leqslant n$. Now each $\mathcal{V}_k$ ($k \leqslant n$) is (strongly) locally finite. So there is a nbd. (regular open nbd.) U_k of x such that U_k can meet at mose finitely many members of $\mathcal{V}_k$ ($k \leqslant n$). Then $S_x = V \cap U_1 \cap U_2 \cap \cdots \cap U_n$ (resp. $S_x = \operatorname{int} \operatorname{cl} V \cap U_1 \cap U_2 \cap \cdots \cap U_n$) is an open (regular open) nbd. of x such that S_x can meet at most finitely many members of $\bigcup_{n=1}^{\infty} \mathcal{V}_n$, i.e., of $\mathcal{V}$. Hence S_x can meet at most finitely many members of $\mathcal{W}$, i.e., $\mathcal{W}$ is (strongly) locally finite. ∎

Now, we are in a position to state the main theorem of this paper as follows.

THEOREM 13. *If X is an almost regular space, then the following are equivalent:*

(i) *X is nearly paracompact.*

(ii) *Each regular open cover of X has a strongly locally finite refinement.*

(iii) *Each regular open cover of X has a locally finite refinement.*

(iv) *Each regular open cover of X has a locally finite closed refinement.*

(v) *Each regular open cover of X has a locally finite regular closed refinement.*

(vi) *Each regular open cover of X is regular even.*

(vii) *Each regular open cover of X has an open σ-strongly discrete refinement.*

($viii$) *Each regular open cover of X has an open σ-strongly locally finite refinement.*

(ix) *Each regular open cover of X has an open σ-locally finite refinement.*

Proof. (i) $\implies$ (ii). Let $\mathcal{U}$ be any regular open cover of X. Then there is some open locally finite refinement $\mathcal{V}$ of $\mathcal{U}$. For each $x \in X$, there exists some open nbd. U_x of x such that $U_x \cap V \neq \emptyset$, for at most finitely many $V \in \mathcal{V}$, say V_1, V_2, $\ldots$, V_k. Then $\operatorname{cl} U_x \cap V = \emptyset$ for $V \in \mathcal{V} \setminus \{V_1, V_2, \ldots, V_k\}$ and hence $\operatorname{int} \operatorname{cl} U_x \cap V = \emptyset$ for $V \in \mathcal{V} \setminus \{V_1, V_2, \ldots, V_k\}$. Thus $\mathcal{V}$ is a strongly locally finite refinement of $\mathcal{U}$.

The implications "(ii) $\implies$ (iii)", "(vii) $\implies$ ($viii$)" and "($viii$) $\implies$ (ix)" are obvious, while the implications "(iii) $\implies$ (iv)", "(iv) $\implies$ (v)", "(v) $\implies$ (vi)", "(vi) $\implies$ (vii)" and "($viii$) $\implies$ (ii)" follow from Theorems 5(i), 5(ii), 3, 11 and 12 respectively.

(ii) $\implies$ (i). Let $\mathcal{U}$ be any regular open cover of X and $\mathcal{R}$ be its strongly locally finite refinement. As $\mathcal{R} < \mathcal{U}$, choose for each $A \in \mathcal{R}$, some $U_A \in \mathcal{U}$ such that $A \subseteq U_A$. Now, X being almost regular, by Theorems 3 and 5, $\mathcal{U}$ is regular even. So, by Theorem 8, there is a δ-open nbd. V of the diagonal Δ_X in $X \times X$ such that $\{V[A] : A \in \mathcal{R}\}$ is strongly locally finite. Put $W_A = V[A] \cap U_A$. Then $\{W_A : A \in \mathcal{R}\}$ is a locally finite open refinement of $\mathcal{U}$.

Finally, we see that $(ix) \implies (iii)$ is an immediate consequence of Theorem 12. ∎

REFERENCES

[**1**] S. P. Arya, *A note on nearly paracompact spaces*, Mat. vesnik **8(23)** (1971), 113–115.

[**2**] D. E. Cameron, *Maximal QHC-spaces*, Rocky Mountain Jour. Math. **7**, 2 (1977), 313–322.

[**3**] L. L. Herrington, *Properties of nearly compact spaces*, Proc. Amer. Math. Soc. **45** (1974), 431–436.

[**4**] T. Noiri, *A note on nearly paracompact spaces*, Mat. vesnik **5(18)(33)** (1981), 103–108.

[**5**] M. K. Singal, *Some generalizations of paracompactness*, Proc. Kanpur Topological Conference (1968), 245–263.

[**6**] M. K. Singal and S. P. Arya, *On nearly paracompact spaces*, Mat. vesnik **6(21)** (1969), 3–16.

[**7**] M. K. Singal and S. P. Arya, *On almost regular spaces*, Glasnik Mat. **4(24)** (1969), 89–99.

[**8**] N. V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl. (2) **78** (1968), 103–118.

(received 11.11.1997.)

Department of Pure mathematics, University of Calcutta, 35, Ballygunge Circular Road, Calcutta–700 019, India