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THE COMPATIBILITY OF THE TANGENCY RELATIONS OF SETS IN GENERALIZED METRIC SPACES

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Abstract. In this paper the problem of the compatibility of the tangency relations $T_{l_i}(a_i, b_i, k, p)$ $(i = 1, 2)$ of sets of the classes $\tilde{M}_{p,k}$ and $A^*_{p,k}$ in a generalized metric space is considered. Some sufficient conditions for the compatibility of these relations of sets of the above classes are given here.

Introduction

Let E be an arbitrary non-empty set and let l be a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E. Let l_0 be the function defined by the formula

$$l_0(x, y) = l(\{x\}, \{y\}) \qquad \text{for } x, y \in E. \tag{1}$$

If we put some conditions on the function l, then the function l_0 defined by (1) will be a metric on the set E. By this reason the pair (E, l) can be treated as a certain generalization of a metric space and we shall call it (see [9]) a generalized metric space. Using (1) we may define in the space (E, l), similarly as in a metric space, the following notions: the sphere $S_l(p, r)$ and the ball $K_l(p, r)$ with the centre at the point p and the radius r.

Let $S_l(p, r)_u$ denote the so-called u-neighbourhood of the sphere $S_l(p, r)$ in the space (E, l) defined by the formula

$$S_l(p, r)_u = \begin{cases} \bigcup_{q \in S_l(p,r)} K_l(q, u), & \text{for } u > 0, \\ S_l(p, r), & \text{for } u = 0. \end{cases} \tag{2}$$

Let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r \to 0+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r \to 0+]{} 0. \tag{3}$$

We say that the pair (A, B) of sets A, B of the family E_0 is (a, b)-clustered at the point p of the space (E, l), if 0 is the cluster point of the set of all real numbers $r > 0$ such that the sets of the form $A \cap S_l(p, r)_{a(r)}$ and $B \cap S_l(p, r)_{b(r)}$ are non-empty.

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Let (see [9])

$$T_l(a,b,k,p) = \Big\{ (A,B) : A, B \in E_0, \text{ the pair } (A,B) \text{ is } (a,b)\text{-clustered at the point } p \text{ of the space } (E,l) \text{ and } \frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r\to 0+]{} 0 \Big\}. \tag{4}$$

If $(A,B) \in T_l(a,b,k,p)$ then we say that the set $A \in E_0$ is (a,b)-tangent of order k $(k > 0)$ to the set $B \in E_0$ at the point p of the space (E,l).

We shall call $T_l(a,b,k,p)$ defined by (4) the relation of (a,b)-tangency of order k at the point p, or shortly: the tangency relation of sets in the generalized metric space (E,l).

Two relations of the tangency $T_{l_1}(a_1,b_1,k,p)$ and $T_{l_2}(a_2,b_2,k,p)$ are called compatible if $(A,B) \in T_{l_1}(a_1,b_1,k,p)$ if and only if $(A,B) \in T_{l_2}(a_2,b_2,k,p)$ for $A, B \in E_0$.

We say that the set $A \in E_0$ has the Darboux property at the point p of the space (E,l), which we write: $A \in D_p(E,l)$ (see [3]), if there exists a number $\tau > 0$ such that the set $A \cap S_l(p,r)$ is non-empty for $r \in (0,\tau)$.

Let ρ be a metric on the set E and let A, B be arbitrary sets of the family E_0. Let us denote

$$\rho(A,B) = \inf\{\, \rho(x,y) : x \in A, y \in B \,\}, \quad d_\rho A = \sup\{\, \rho(x,y) : x,y \in A \,\}. \tag{5}$$

Let f be a subadditive increasing real function defined in a certain right-hand side neighbourhood of 0 such that $f(0) = 0$. By $\overline{F}_f$ we shall denote the class of all functions l fulfilling the conditions:

1° $l\colon E_0 \times E_0 \to \langle 0,\infty)$,

2° $f(\rho(A,B)) \leqslant l(A,B) \leqslant f(d_\rho(A \cup B))$ for $A, B \in E_0$.

Since

$$f(\rho(x,y)) = f(\rho(\{x\},\{y\})) \leqslant l(\{x\},\{y\}) \leqslant f(d_\rho(\{x\} \cup \{y\})) = f(\rho(x,y)),$$

then from here and from (1) it follows that

$$l_0(x,y) = l(\{x\},\{y\}) = f(\rho(x,y)) \quad \text{for } l \in \overline{F}_f \text{ and } x,y \in E. \tag{6}$$

It is easy to prove that the function l_0 defined by (6) is a metric on the set E.

In the present paper the problem of the compatibility of the tangency relations of sets of the classes $\tilde{M}_{p,k}$ and $A^*_{p,k}$ having the Darboux property at the point p of the space (E,l), for the functions l belonging to the class $\overline{F}_f$, is considered.

1. On the compatibility of the tangency relations of sets of the classes $\tilde{M}_{p,k}$

By A' we shall denote the set of all cluster points of the set $A \in E_0$. Let k be any fixed positive real number and let

$$\rho(x,A) = \inf\{\, \rho(x,y) : y \in A \,\}. \tag{1.1}$$

Let us put by definition (see [4])

$$\begin{aligned}\tilde{M}_{p,k} = \{\, A \in E_0 : p \in A' &\text{ and there exists } \mu > 0 \text{ scuh that for an arbitrary } \varepsilon > 0\\ &\text{there exists } \delta > 0 \text{ such that for every pair of points } (x,y) \in [A,p;\mu,k]\\ &\text{if } \rho(x,y) < \delta \text{ and } \frac{\rho(x,A)}{\rho^k(x,p)} < \delta \text{ then } \frac{\rho(x,y)}{\rho^k(x,p)} < \varepsilon \,\},\end{aligned} \tag{1.2}$$

where

$$[A,p;\mu,k] = \{\,(x,y) : x \in E, y \in A \text{ and } \mu\rho(x,A) < \rho^k(x,p) = \rho^k(y,p)\,\} \tag{1.3}$$

In the paper [4] the following lemma was proved.

LEMMA 1.1. *If*

$$\frac{a(r)}{r^{k+1}} \xrightarrow[r\to 0+]{} \alpha, \tag{1.4}$$

where $\alpha < \infty$, then for an arbitrary set $A \in \tilde{M}_{p,k} \cap D_p(E,\rho)$

$$\frac{1}{r^k} d_\rho(A \cap S_\rho(p,r)_{a(r)}) \xrightarrow[r\to 0+]{} 0. \tag{1.5}$$

From this lemma and from the fact that every function $l \in \overline{F}_f$ generates on the set E the metric defined by the formula (6) it follows that

$$\frac{1}{r^k} d_l(A \cap S_l(p,r)_{a(r)}) \xrightarrow[r\to 0+]{} 0, \tag{1.6}$$

for $l \in \overline{F}_f$ and $A \in \tilde{M}_{p,k} \cap D_p(E,l)$, when the function a fulfills the condition (1.4).

THEOREM 1.1. *If $l_i \in \overline{F}_f$ $(i = 1,2)$,*

$$\frac{a(r)}{r^{k+1}} \xrightarrow[r\to 0+]{} \alpha \quad \textit{and} \quad \frac{b(r)}{r^{k+1}} \xrightarrow[r\to 0+]{} \beta, \tag{1.7}$$

where $\alpha, \beta < \infty$, then for arbitrary sets of the classes $\tilde{M}_{p,k} \cap D_p(E,l)$ the tangency relations $T_{l_1}(a,b,k,p)$ and $T_{l_2}(a,b,k,p)$ are compatible.

Proof. Let us assume that $(A,B) \in T_{l_1}(a,b,k,p)$ for $A, B \in \tilde{M}_{p,k}$. Hence, from (2) and from the fact that (see (6))

$$l_1(\{x\},\{y\}) = l_2(\{x\},\{y\}) = l_0(x,y) \quad \text{for } x,y \in E, \tag{1.8}$$

it follows that the pair of sets (A,B) is (a,b)-clustered at the point p of the space (E,l_1) and

$$\frac{1}{r^k} l_1(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r\to 0+]{} 0. \tag{1.9}$$

From the inequality

$$d_\rho(A \cup B) \leqslant d_\rho A + d_\rho B + \rho(A,B) \quad \text{for } A,B \in E_0, \tag{1.10}$$

from the properties of the function f and from the fact that $l_1,\, l_2 \in \overline{F}_f$ we get

$$\left|\frac{1}{r^k} l_2(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) - \frac{1}{r^k} l_1(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)})\right| \leqslant$$
$$\frac{1}{r^k} f(d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)}))) - \frac{1}{r^k} f(\rho(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}))$$
$$\leqslant \frac{1}{r^k} f(d_\rho(A \cap S_l(p,r)_{a(r)}) + d_\rho(B \cap S_l(p,r)_{b(r)}) + \rho(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)})) -$$
$$- \frac{1}{r^k} f(\rho(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}))$$
$$\leqslant \frac{1}{r^k} f(d_\rho(A \cap S_l(p,r)_{a(r)})) + \frac{1}{r^k} f(d_\rho(B \cap S_l(p,r)_{b(r)})). \tag{1.11}$$

From the fact that f is an increasing function we obtain

$$\begin{aligned} f(d_\rho(A \cap S_l(p,r)_{a(r)})) &= f(\sup\{\, \rho(x,y) : x,y \in (A \cap S_l(p,r)_{a(r)})\, \}) \\ &= \sup\{\, f(\rho(x,y)) : x,y \in (A \cap S_l(p,r)_{a(r)})\, \} \\ &= \sup\{\, l_0(x,y) : x,y \in (A \cap S_l(p,r)_{a(r)})\, \} = d_l(A \cap S_l(p,r)_{a(r)}). \end{aligned} \tag{1.12}$$

Hence and from (1.6) it follows that

$$\frac{1}{r^k} f(d_\rho(A \cap S_l(p,r)_{a(r)})) \xrightarrow[r \to 0+]{} 0. \tag{1.13}$$

Analogously

$$\frac{1}{r^k} f(d_\rho(B \cap S_l(p,r)_{b(r)})) \xrightarrow[r \to 0+]{} 0. \tag{1.14}$$

From (1.9), (1.13), (1.14) and from the inequality (1.11) we get

$$\frac{1}{r^k} l_2(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r \to 0+]{} 0. \tag{1.15}$$

Since the functions $l_1, l_2 \in \overline{F}_f$ generate on the set E the same metric l_0 (see (6)), from the fact that the pair of sets (A,B) is (a,b)-clustered at the point p of the space (E, l_1), it follows that it is (a,b)-clustered at the point p of the space (E, l_2). Hence and from (1.15) it results that $(A,B) \in T_{l_2}(a,b,k,p)$.

If $(A,B) \in T_{l_2}(a,b,k,p)$, then similarly we prove that $(A,B) \in T_{l_1}(a,b,k,p)$. Hence it follows that the tangency relations $T_{l_1}(a,b,k,p)$ and $T_{l_2}(a,b,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,l)$. ■

Let a_i, b_i $(i = 1,2)$ be non-negative real functions defined in a certain right-hand side neighbourhood of 0 and fulfilling the condition

$$a_i(r) \xrightarrow[r \to 0+]{} 0 \quad \text{and} \quad b_i(r) \xrightarrow[r \to 0+]{} 0. \tag{1.16}$$

In the paper [7] the following theorem was proved.

Theorem 1.2. *If $l \in \overline{F}_f$ and*

$$\frac{a_i(r)}{r^{k+1}} \xrightarrow[r \to 0+]{} \alpha_i, \quad \frac{b_i(r)}{r^{k+1}} \xrightarrow[r \to 0+]{} \beta_i, \tag{1.17}$$

where $\alpha_i, \beta_i < \infty$ *for* $i = 1, 2$, *then for arbitrary sets of the classes* $\tilde{M}_{p,k} \cap D_p(E, l)$ *the tangency relations* $T_l(a_1, b_1, k, p)$ *and* $T_l(a_2, b_2, k, p)$ *are compatible.*

From the Theorems 1.1 and 1.2 it follows

Corollary 1.1. *If* $l_i \in \overline{F}_f$ *and the functions* a_i, b_i $(i = 1, 2)$ *fulfil the condition* (1.17), *then the tangency relations* $T_{l_1}(a_1, b_1, k, p)$ *and* $T_{l_2}(a_2, b_2, k, p)$ *are compatible in the classes of sets* $\tilde{M}_{p,k} \cap D_p(E, l)$.

2. On the compatibility of the tangency relations of sets of the classes $A^*_{p,k}$

Let (E, ρ) be a metric space. Let us put by definition (see [3])

$A^*_{p,k} = \{ A \in E_0 : p \in A'$ and there exists a number $\lambda > 0$ such that

$$\limsup_{[A,p;k]\ni(x,y)\to(p,p)} \frac{\rho(x,y) - \lambda\rho(x,A)}{\rho^k(x,p)} \leqslant 0 \Big\}, \quad (2.1)$$

where

$$[A, p; k] = \{ (x, y) : x \in E, y \in A \text{ and } \rho(x, A) < \rho^k(x, p) = \rho^k(y, p) \}. \quad (2.2)$$

In the paper [4] it was proved that $A^*_{p,k} \subset \tilde{M}_{p,k}$ for any $k > 0$ and $p \in E$. With this connection the Theorems 1.1, 1.2 mentioned in Section 1 of this paper are fulfilled in the classes of sets $A^*_{p,k} \cap D_p(E, l)$. It appears that these theorems will be true for sets of the classes $A^*_{p,k} \cap D_p(E, l)$ at slightly weaker conditions concerning the functions a, b, a_i, b_i $(i = 1, 2)$ appearing in the assumptions of the above theorems.

In the paper [3] the following lemma was proved:

Lemma 2.1. *If*

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0+]{} 0, \quad (2.3)$$

then for an arbitrary set $A \in A^*_{p,k} \cap D_p(E, \rho)$

$$\frac{1}{r^k} d_\rho(A \cap S_\rho(p, r)_{a(r)}) \xrightarrow[r\to 0+]{} 0. \quad (2.4)$$

From this lemma and from (6) it follows that

$$\frac{1}{r^k} d_l(A \cap S_l(p, r)_{a(r)}) \xrightarrow[r\to 0+]{} 0, \quad (2.5)$$

for $l \in \overline{F}_f$ and $A \in A^*_{p,k} \cap D_p(E, l)$, when the function a fulfils the condition (2.3).

Similarly as in the case of the classes of sets $\tilde{M}_{p,k}$, using (2.5) we can prove the following theorem.

Theorem 2.1. *If* $l_i \in \overline{F}_f$ $(i = 1, 2)$,

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0+]{} 0, \quad \textit{and} \quad \frac{b(r)}{r^k} \xrightarrow[r\to 0+]{} 0, \quad (2.6)$$

then for arbitrary sets of the classes $A^*_{p,k} \cap D_p(E,l)$ *the tangency relations* $T_{l_1}(a,b,k,p)$ *and* $T_{l_2}(a,b,k,p)$ *are compatible.*

Let a_i, b_i $(i = 1,2)$ be non-negative real functions defined in a certain right-hand side neighbourhood of 0 and fulfilling the condition (1.16). In the paper [8] the following theorem was proved.

THEOREM 2.2 *If* $l \in \overline{F}_f$ *and*

$$\frac{a_i(r)}{r^k} \xrightarrow[r\to 0+]{} 0 \quad \text{and} \quad \frac{b_i(r)}{r^k} \xrightarrow[r\to 0+]{} 0 \quad \text{for } i = 1,2, \tag{2.7}$$

then for arbitrary sets of the classes $A^*_{p,k} \cap D_p(E,l)$ *the tangency relations* $T_l(a_1,b_1,k,p)$ *and* $T_l(a_2,b_2,k,p)$ *are compatible.*

From the Theorems 2.1 and 2.2 the following corollary results.

COROLLARY 2.1. *If the functions* a_i, b_i $(i = 1,2)$ *fulfil the condition* (2.7) *and* $l_i \in \overline{F}_f$, *then the tangency relations* $T_{l_1}(a_1,b_1,k,p)$ *and* $T_{l_2}(a_2,b_2,k,p)$ *are compatible in the classes of sets* $A^*_{p,k} \cap D_p(E,l)$.

Let id denotes the identity function defined in a right-hand side neighbourhood of 0. If we put $f = id$, then the class $\overline{F}_{id}$ of the function l is equal to the class F^*_ρ (see [3], [4]). From here it results that all theorems about the problem of the compatibility of the tangency relations of sets for the functions of the class F^*_ρ given in the papers [3] and [4] follow from the theorems of this paper.

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