МАТЕМАТИЧКИ ВЕСНИК
50 (1998), 1–4

UDK 515.126.4
оригинални научни рад
research paper

A FURTHER EXTENSION OF MAPS WITH NON-UNIQUE FIXED POINTS

Ljubomir Ćirić and Nikola Jotić

Abstract. A wider class of mappings in metric spaces, which have a non-unique fixed point, is introduced and investigated. Presented fixed point theorems include as special cases the corresponding theorems of Dhage, Pachpatte and the first author.

1. Introduction

Recently in [4] a class of some maps in metric spaces with a fixed point has been firstly introduced and studied. Further studies have been made by Achari [1,2], Basu [3], Dhage [5], Jain and Rajaria [6], Mishra [7], Pachpatte [8], Pathak [9,10] and Tasković [11].

The purpose of this paper is to investigate a wider class of maps which still have non-unique fixed points.

2. Main results

In this paper we shall use the notions of orbitally continuous mappings and orbitally complete spaces, which the first author introduced in 1971. We now suppose that these notions are well known, since they appear very often in the fixed point theory.

THEOREM 2.1. *Let (X,d) be a metric space and $T\colon X\to X$ a self-mapping which satisfies the following contractive condition:*

$$\min\Big\{ d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}, \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{\min\{d^2(Tx,Ty), d^2(x,Tx), d^2(y,Ty)\}}{d(x,y)} \Big\} - a\min\{d(x,Ty), d(y,Tx)\} \leqslant q\max\{d(x,y), d(x,Tx)\} \quad (1)$$

AMS Subject Classification: 54 H 25
Keywords and phrases: Cauchy sequence, complete spce, fixed point

for all $x \neq y$ in X, where $a \geqslant 0$ and $0 \leqslant q < 1$. Then $\{T^n x\}$ is a Cauchy sequence for each x in X. Furthermore, if X is complete and T is orbitally continuous, then T has at least one fixed point.

Proof. Let x in X be arbitrary. We shall show that a sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$; $x_0 = x$ is a Cauchy sequence. Suppose that $x_{n-1} \neq x_n$ for each $n = 1, 2, \dots$. From (1) for $y = Tx$ we have

$$\min\Big\{d(Tx,T^2x), d(x,Tx), \frac{d(x,Tx)[1+d(Tx,T^2x)]}{1+d(x,Tx)}, \frac{\min\{d^2(Tx,T^2x), d^2(x,Tx)\}}{d(x,Tx)}\Big\} \leqslant qd(x,Tx).$$

Hence

$$\min\Big\{d(Tx,T^2x), d(x,Tx), \frac{d(x,Tx)[1+d(Tx,T^2x)]}{1+d(x,Tx)}, \frac{d^2(Tx,T^2x)}{d(x,Tx)}\Big\} \leqslant qd(x,Tx). \tag{2}$$

Since $d(x,Tx) \leqslant qd(x,Tx)$ is impossible in the case $Tx \neq x$ (as $q < 1$) and as $\dfrac{d^2(Tx,T^2x)}{d(x,Tx)} \leqslant qd(x,Tx)$ gives $d(Tx,T^2x) \leqslant \sqrt{q}\, d(x,Tx)$ and as

$$\frac{d(x,Tx)[1+d(Tx,T^2x)]}{1+d(x,Tx)} \leqslant qd(x,Tx)$$

implies $d(Tx,T^2x) \leqslant qd(x,Tx)$, from (2) it follows

$$d(Tx,T^2x) \leqslant \sqrt{q}\, d(x,Tx). \tag{3}$$

By induction we have

$$d(T^n x, T^{n+1}x) \leqslant \sqrt{q}\, d(T^{n-1}x, T^n x) \leqslant qd(T^{n-2}x, T^{n-1}x) \leqslant \dots \leqslant \sqrt{q^n}\, d(x,Tx). \tag{4}$$

Hence, from the triangle inequality, for any $p \in \mathbf{N}$ we get

$$d(T^n, T^{n+p}x) \leqslant \sum_{k=n}^{n+p-1} d(T^k x, T^{k+1}x).$$

Then from (4) we obtain

$$d(T^n x, T^{n+p}x) \leqslant \sum_{k=n}^{n+p-1} \sqrt{q^k}\, d(x,Tx) \leqslant \frac{\sqrt{q^n}}{1-\sqrt{q}} d(x,Tx).$$

Since $0 \leqslant q < 1$ implies $\lim_{n\to\infty} \sqrt{q^n} = 0$, it follows that $\{T^n x\}$ is a Cauchy sequence.

Assume now that X is complete and T-orbitally continuous. Since $\{T^n x\}$ is a Cauchy sequence, there is some u in X such that $\lim_{n\to\infty} T^n x = u$. By orbital continuity of T it follows $Tu = \lim_{n\to\infty} TT^n x = u$, that is, u is a fixed point of T. ∎

COROLLARY 2.1. *Let $T\colon X \to X$ be an orbitally continuous self-mapping on T-orbitally complete metric space (X,d). If T satisfies the following condition:*

$$\min\{d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty)\} - a\min\{d(x,Ty), d(y,Tx)\} \leqslant q\max\{d(x,y), d(x,Tx)\}, \tag{5}$$

where $a \geqslant 0$ and $0 \leqslant q < 1$, then T has a fixed point.

REMARK 2.1. Corollary 2.1 is a generalization of the theorem of Dhage [3], where instead of contractive inequality (5) he considered the following inequality

$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\} - a\min\{d(x,Ty),d(y,Tx)\} \leqslant kd(x,y) + pd(x,Tx), \tag{5'}$$

where $0 \leqslant k + p < 1$.

Indeed, since $kd(x,y)+pd(x,Tx) \leqslant (k+p)\max\{d(x,y),d(x,Tx)\}$, the inequality (5') implies (5) with $q = k + p$.

COROLLARY 2.2. (Pathak [9]) *Let T be an orbitally continuous self-mapping of a T-orbitally complete metric space X such that*

$$\min\{d(x,Tx),d(Tx,Ty),d^2(x,y),d(x,Tx)d(y,Ty)\} \leqslant \frac{1}{2}h[d(x,Tx) + d(x,y)]\max\{d(x,Tx),d(x,y)\},$$

where $0 < h < 1$. Then T has a fixed point.

REMARK 2.2. Corollary 2.2 is a generalization of the main theorem of Pathak [9] where he has investigated mappings which satisfy the following contractive condition:

$$\min\{d(x,Tx)d(Tx,Ty),d^2(x,y),d(x,Tx)d(y,Ty)\} \leqslant \frac{1}{2}h[d(x,Tx) + d(x,y)]\max\{d(x,Tx),d(x,y)\} \tag{6}$$

with $h < 1$.

To see that (6) implies (5) we observe that

$$\min\{d^2(Tx,Ty),d^2(x,y),d^2(x,Tx),d^2(y,Ty)\} \leqslant \min\{d(x,Tx)d(Tx,Ty),d^2(x,y),d(x,Tx)d(y,Ty)\},$$

$$\frac{1}{2}h[d(x,Tx) + d(x,y)]\max\{d(x,Tx),d(x,y)\} \leqslant \frac{1}{2}h \cdot 2\max\{d(x,Tx),d(x,y)\}\times \times\max\{d(x,Tx),d(x,y)\} = h\max\{d^2(x,y),d^2(x,Tx)\}.$$

Therefore, (6) implies (5) with $q = \sqrt{h}$ and $a = 0$.

THEOREM 2.2. *Let T be an orbitally continuous self-mapping on a T-orbitally complete metric space X such that*

$$\min\{d(Tx,Ty),d(y,Ty)\} - a\min\{d(x,Ty),d(y,Tx)\} \leqslant q\max\{d(x,y),d(x,Tx),d(y,Ty)\}, \tag{7}$$

where $a > 0$, $0 < q < 1$. Then T has a fixed point.

Proof. Put $y = Tx$ in (7). Then we obtain

$$d(Tx,T^2x) \leqslant q\max\{d(x,Tx),d(Tx,T^2x)\}.$$

Hence, as $q < 1$, we have $d(Tx, T^2x) = 0$ (which means T has a fixed point), or $d(Tx, T^2x) \leqslant qd(x, Tx)$. Proceeding as in the proof of the Theorem 1 we conclude that T has a fixed point. ■

THEOREM 2.3. *Let $T\colon X \to X$ be an orbitally continuous mapping on X and let X be T-orbitally complete. If T satisfies the following condition:*

$$\min\{d^2(Tx,Ty), d(Tx,Ty)d(x,y), d^2(y,Ty)\} \\ - a\min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \leqslant qd(x,Tx)d(y,Ty) \quad (8)$$

for all x, y in X and $a > 0$, $0 < q < 1$, then for each x in X, the sequence $\{T^n x\}$ converges to a fixed point of T.

Proof. Put $y = Tx$ in (8). Then it becomes

$$\min\{d^2(Tx,T^2x), d(Tx,T^2x)d(x,Tx)\} \leqslant qd(x,Tx)d(Tx,T^2x). \quad (9)$$

If we assume that $T^n x \neq T^{n-1}x$ for all $n \in \mathbf{N}$, then (9) implies (as $q < 1$), $d(Tx, T^2x) \leqslant \sqrt{q}\, d(x, Tx)$, which is the relation (3). Therefore, we can follow the arguments presented in the proof of the Theorem 1. ■

COROLLARY 2.3. (Pachpatte [8]) *Let T be an orbitally continuous self-mapping on a T-orbitally complete metric space X. If T satisfies the following condition:*

$$\min\{d^2(Tx,Ty), d(x,y)d(Tx,Ty), d^2(y,Ty)\} \\ - a\min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \leqslant hd(x,Tx)d(y,Ty),$$

where $a > 0$, $0 < h < 1$, then T has a fixed point in X.

REFERENCES

[**1**] Achari, J., *On Ćirić's non-unique fixed points*, Mat. Vesnik **13** (1976), 255–257.

[**2**] Achari, J., *On the generalization of Pachpatte non-unique fixed point theorem*, Indian J. Pure Appl. Math. **13** (3) (1982), 299–302.

[**3**] Basu, T., *Extension of Ćirić's fixed point theorem in a uniform space*, Ranchi Univ. Math. J. **11** (1980), 109–115 (1982).

[**4**] Ćirić, Lj., *On some maps with a non-unique fixed point*, Publ. Inst. Math. (Beograd) **17** (31) (1974), 52–58.

[**5**] Dhage, B. C., *Some theorems for the maps with a non-unique fixed point*, Indian J. Pure Appl. Math. **16** (1985), 254–256.

[**6**] Jain, R. K. and Rajoriya, *Non-unique fixed points in orbitally complete metric space*, Pure Appl. Math. Sci. **25** (1987), 55–57.

[**7**] Mishra, S. N., *On fixed points of orbitally continuous maps*, Nantha Math. **12** (1979), 83–90.

[**8**] Pachpatte, B. G., *On Ćirić's type maps with a non-unique fixed point*, Indian J. Pure Appl. Math. **10** (8) (1979), 1039–1043.

[**9**] Pathak, H. K., *Some non-unique fixed points theorems for new class of mappings*, Ranchi Univ. Math. J. **17** (1986), 65–70.

[**10**] Pathak, H. K., *On some non-unique fixed point theorems for the maps of Dhage type*, Pure Appl. Math. Sci. **27** (1988), 41–47.

[**11**] Tasković, M., *Some new principles in fixed point theory*, Math. Japon, **35** (1990), 645–666.

(received 28.09.1995, in revised form 02.07.1997.)

Faculty of Mechanical Engineering, 27. marta 80, Belgrade, Yugoslavia