МАТЕМАТИЧКИ ВЕСНИК
49 (1997), 279–281

UDK 517.938
оригинални научни рад
research paper

# NOTE ON L-A PAIR FOR THE KOWALEVSKAYA GYROSTAT IN A MAGNETIC FIELD

**Vladimir Dragović**

## 1. Introduction

The methods of algebro-geometric integration have been developed in the first place for solving nonlinear partial differential equations such as Korteveg-de Vries, Sine-Gordon, Kadomtsev-Petriashvili, ... They were applied also to the classical, mechanical integrable systems. The Kowalevskaya top is one of the most celebrated [1].

The first L-A pair for Kowalevskaya top (KT) was found by Perelomov in 1981 [2]. In 1984 Bogoyavlensky modified this L-A pair for the system with magnetic field included [3]. Three years later, Reyman and Semenov-Tian-Shansky obtained L-A pair with spectral parametar for generalized KT called Kowalevskaya gyrostat (KG) [4]. Bobenko and Kuznetsov have noticed that removing the first column and the first row of the last Lax matrix one can get the Lax matrix for Goryachev-Chapligin gyrostat [5] (GCG).

In this note we start from Reyman and Semenov-Tian-Shansky L-A pair, in order to get L-A pairs for KG and GCG with magnetic field. The resulting matrices have all the symmetries necessary for procedure of algebro-geometric integration described in [6].

## 2. The Kowalevskaya gyrostat

The Kowalevskaya gyrostat in a magnetic field is a system given by the Hamiltonian

$$H = \frac{1}{2}\left(M_1^2 + M_2^2 + 2M_3^2 + 2\gamma M_3\right) - p_i - \delta_2.$$

Corresponding algebra is generated by $M_i, p_i, \delta_i$ and relations

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \{M_i, p_j\} = \epsilon_{ijk} p_k, \{M_i, \delta_j\} = \epsilon_{ijk}\delta_k$$

(Other brackets are 0).

*AMS Subject Classification*: 35 Q 60, 78 A 99

Communicated at the 4th Symposium on Mathematical Analysis and Its Applications, Aranđelovac 1997.

The equations of motion are:

$$\dot{M}_1 = M_2M_3 + \gamma M_2 + \delta_3 \qquad \dot{p}_1 = 2p_2M_3 - p_3M_2 + \gamma p_2$$
$$\dot{M}_2 = -M_1M_3 - \gamma M_1 - p_3 \qquad \dot{p}_2 = p_3M_1 - 2p_1M_3 - \gamma p_1$$
$$\dot{M}_3 = p_2 - \delta_1 \qquad \dot{p}_3 = p_1M_2 - p_2M_1$$

$$\dot{\delta}_1 = 2\delta_2M_3 - \delta_3M_2 + \gamma\delta_2$$
$$\dot{\delta}_2 = \delta_3M_1 - 2\delta_1M_3 - \gamma\delta_1$$
$$\dot{\delta}_3 = \delta_1M_2 - \delta_2M_1$$

Using standard notation $p_\pm = p_1 \pm ip_2, M_\pm = M_1 \pm iM_2$ we have:

Proposition. *The system is equivalent to*

$$\dot{L}(\lambda) = -\left[L(\lambda), A(\lambda)\right],$$

*where*

$$L(\lambda) = i\begin{bmatrix} -\gamma & \frac{p_- - i\delta_-}{\lambda} & M_- & \frac{-p_3+i\delta_3}{\lambda} \\ \frac{p_+ - i\delta_+}{\lambda} & \gamma & \frac{p_3+i\delta_3}{\lambda} & -M_+ \\ M_+ & \frac{-p_3+i\delta_3}{\lambda} & -T & \frac{-p_+ + i\delta_+}{\lambda} + 2\lambda \\ \frac{p_3+i\delta_3}{\lambda} & -M_- & \frac{p_- + i\delta_-}{\lambda} & T \end{bmatrix}$$

*and*

$$A(\lambda) = \frac{i}{2}\begin{bmatrix} T & 0 & M_- & 0 \\ 0 & -T & 0 & -M_+ \\ M_+ & 0 & -T & -2\lambda \\ 0 & -M_- & 2\lambda & T \end{bmatrix}$$

*where* $T = 2M_3 + \gamma$.

Lemma. *The matrices* $L(\lambda)$ *satisfy the relations:*

$$L(-\lambda) = \begin{bmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} L(\lambda) \begin{bmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}$$
$$L(\lambda)^T = -\begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix} L(\lambda) \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

*where the Pauli matrices* $\sigma_i$ *are*

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The equations of motion are linearizable on the Jacobian of the spectral curve $\Gamma$ defined by $\Gamma : \det(L(\lambda) - \mu E) = 0$. According to Lemma, there are two commuting involutions $\tau_1, \tau_2$ on $\Gamma$

$$\tau_1(\lambda, \mu) = (-\lambda, \mu), \tau_2(\lambda, \mu) = (\lambda, -\mu).$$

So, procedure of algebro-geometric integration is the same as in the case of Kowalevskaya gyrostat (see [6]).

### 3. The Goryachev-Chapligin case

The Goryachev-Chapligin gyrostat in a magnetic field is described by the Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2 + 4\gamma M_3) - 2p_1 - 2\delta_2.$$

It is integrable under the conditions:

$$M_1 p_1 + M_2 p_2 + M_3 p_3 = 0$$
$$M_1 \delta_1 + M_2 \delta_2 + M_3 \delta_3 = 0$$

Corresponding L-A pair is given by the formulas:

$$L = i\begin{bmatrix} \frac{2}{3}\gamma & \frac{p_3+i\delta_3}{\lambda} & -M_+ \\ \frac{-p_3+i\delta_3}{\lambda} & -T-\frac{2}{3}\gamma & \frac{-p_++i\delta_+}{\lambda} - 2\lambda \\ -M_- & \frac{p_-+i\delta_-}{\lambda} + 2\lambda & T \end{bmatrix}$$

$$A = i\begin{bmatrix} -M_3 - T & 0 & -M_+ \\ 0 & -T & -2\lambda \\ -M_- & 2\lambda & T+\frac{2}{3}\gamma \end{bmatrix}$$

where $T = 2M_3 + \frac{2}{3}\gamma$. The matrices $L(\lambda)$ have the property

$$L(-\lambda) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} L(\lambda) \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Further integration repeats the steps of integration without magnetic field.

REFERENCES

[**1**] B. A. Dubrovin, I. M. Krichever, S. P. Novikov, *Integrable systems I*, in *Dynamical systems IV*, pp. 173–281, Springer-Verlag, 1990.

[**2**] A. M. Perelomov, *Lax representation for the system of S. Kowalevskaya type*, Comm. Math. Phys. **81**, 2 (1981), 239–243.

[**3**] O. I. Bogoyavlensky, *Integrable Euler equations on Lie algebras ...* , Izv. AN SSSR **48**, 5 (1984), 883–938.

[**4**] A. G. Reyman, Semenov-Tian-Shansky, Funkts. Anal. Pril. **22**, 2 (1988), 87–88.

[**5**] A. I. Bobenko, V. B. Kuznecov, *Lax representation and new formulae for the Goryachev-Chapligin top*, J. Phys. A **21** (1988), 1999–2006.

[**6**] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, V. B. Matveev, *Algebro-geometric approach to nonlinear integrable equations*, p. 337, Springer Series in Nonlinear Dynamics, 1994.

(received 29.09.1997.)

Mathematical Institute SANU, Kneza Mihaila 35/I, 11000 Belgrade, Yugoslavia