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SOME PROPERTIES OF A CLASS OF POLYNOMIALS

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Abstract. In the paper [2], R. André-Jeannin studied a class of polynomials $U_n(p,q;x)$. In this paper we consider a new class of polynomials $U_{n,m}(p,q;x)$ and determine the coefficients $c_{n,k}(p,q)$ of these introduced polynomials. Also, we define the polynomials $f_{n,m}(p,q;x)$, which are the rising diagonal polynomials of $U_{n,m}(p,q;x)$.

1. Introduction

In the paper [2], R. André-Jeannin studied a class of polynomials $U_n(p,q;x)$. These polynomials are given by

$$U_n(p,q;x) = (x+p)U_{n-1}(p,q;x) - qU_{n-2}(p,q;x), \quad n \geq 2,$$

with starting polynomials $U_0(p,q;x)=0$ and $U_1(p,q;x)=1$. The particular cases of these polynomials are: Fibonacci polynomials, Pell polynomials ([6]), Fermat polynomials of the first kind ([5], [3]), Morgan-Voyce polynomials of the second kind ([1]), Chebyschev polynomials of the second kind ([5]). In this paper, we consider a more general class of polynomials $U_{n,m}(p,q;x)$, where n, m are nonnegative integers. These polynomials are given by the following recurrence relation

$$U_{n,m}(p,q;x) = (x+p)U_{n-1,m}(p,q;x) - qU_{n-m,m}(p,q;x), \quad n \geq m, \tag{1.1}$$

with starting polynomials:

$$U_{0,m}(p,q;x) = 0, \quad U_{n,m}(p,q;x) = (x+p)^{n-1}, \quad n = 1,2,\dots,m-1. \tag{1.2}$$

The parameters p and q are arbitrary real numbers. Note that the polynomials $U_{n,3}(p,q;x)$ are studied in [4].

Let us denote by $\alpha_1,\ \alpha_2,\ \dots,\ \alpha_m$ the real or complex numbers, such that

$$\sum_{i=1}^{m} \alpha_i = p, \quad \sum_{i<j} \alpha_i\alpha_j = 0, \quad \dots, \quad \alpha_1\alpha_2\cdots\alpha_m = (-1)^m q. \tag{1.3}$$

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Also, in this paper, we define the polynomials $f_{n,m}(p,q;x)$, which are the rising diagonal polynomials of $U_{n,m}(p,q;x)$.

2. Polynomials $U_{n,m}(p,q;x)$

Let us write $U_{n,m}(x)$ instead of $U_{n,m}(p,q;x)$. From (1.1) and (1.2), we find the first $m+2$ terms of the sequence $\{U_{n,m}(x)\}$:

$$\begin{gathered} U_{0,m}(x)=0,\ U_{1,m}(x)=1,\ U_{2,m}(x)=x+p,\dots,\ U_{m,m}(x)=(x+p)^{m-1}, \\ U_{m+1,m}(x)=(x+p)^m-q. \end{gathered} \tag{2.1}$$

From (2.1) and by induction on n, we can say that there is a sequence $\{c_{n,k}(p,q)\}$, $n\geq 0$, $k\geq 0$, of numbers such that

$$U_{n+1,m}(x)=\sum_{k\geq 0} c_{n,m,k}(p,q)\,x^k, \tag{2.2}$$

where $c_{n,k}(p,q)=0$ for $n<k$, and $c_{n,n}(p,q)=1$.

The main purpose of this section is to determinate the coefficients $c_{n,k}(p,q)$.

Comparing the coefficients of x^k in two members of (2.2), by (1.1), we get

$$c_{n,k}(p,q)=c_{n-1,k-1}(p,q)+pc_{n-1,k}(p,q)-qc_{n-m,k}(p,q), \tag{2.3}$$

for $n\geq m$, and $k\geq 1$. Now, we are going to prove the following result.

LEMMA 2.1. *For all $k\geq 0$, we have*

$$(1-pt+qt^m)^{-(k+1)}=\sum_{n\geq 0} d_{n,k}(p,q)\,t^n, \tag{2.4}$$

where

$$d_{n,k}(p,q)=\sum_{r=0}^{[n/m]}(-1)^r\,q^r\binom{k+n-(m-1)r}{k}\binom{n-(m-1)r}{r}p^{n-mr}. \tag{2.5}$$

Proof. Firstly, let us define the generating function of the sequence $U_{n,m}(x)$ by

$$f(x,t)=\sum_{n\geq 0} U_{n+1,m}(x)\,t^n. \tag{2.6}$$

From (1.1) and (2.6), we find

$$f(x,t)=\left(1-(x+p)t+qt^m\right)^{-1}. \tag{2.7}$$

Hence, from (2.6) and (2.7), we get

$$\frac{\partial^k f(x,t)}{\partial x^k}=k!t^k\left(1-(x+p)t+qt^m\right)^{-(k+1)}=\sum_{n\geq 0} U^{(k)}_{n+1+k,m}(x)\,t^{n+k}. \tag{2.7'}$$

For $x = 0$ in (2.7'), we get

$$d_{n,k}(p,q) = \frac{1}{k!}U^{(k)}_{n+1+k,m}(p,q;0) = \frac{1}{k!}U^{(k)}_{n+1+k,m}(0,q;p).$$

From (2.4), we obtain

$$\begin{aligned}\sum_{n\geq 0} d_{n,k}\, t^n &= (1-pt+qt^m)^{-(k+1)}\\ &= \sum_{n\geq 0}(-1)^n\frac{(k+n)!}{k!n!}t^n(p-qt^{m-1})^n\\ &= \sum_{n\geq 0} t^n \sum_{r\geq 0}\frac{q^n(k+n-(m-1)r)!p^{n-mr}}{k!r!(n-mr)!}\\ &= \sum_{n\geq 0} t^n \sum_{r=0}^{[n/m]}(-1)^r q^r\binom{n+k-(m-1)r}{k}\binom{n-(m-1)r}{r}p^{n-mr}.\end{aligned}$$

Comparing coefficients of t^n, from the last equalities, we get (2.5). This completes the proof. ■

THEOREM 2.1. *The coefficients $c_{n,k}(p,q)$ are given by the following formula*

$$c_{n,k}(p,q) = \sum_{r=0}^{[(n-k)/m]}(-1)^r q^r\binom{n-(m-1)r}{k}\binom{n-k+(m-1)r}{r}p^{n-k-mr}. \quad (2.8)$$

Proof. Firstly, from (1.1), we deduce

$$U_{n+1,m}(p,q;x) = U_{n+1,m}(0,q;x+p). \quad (2.9)$$

Using (2.2), from (2.9) we have

$$c_{n,k}(p,q) = \frac{1}{k!}U^{(k)}_{n+1,m}(p,q;0) = \frac{1}{k!}U^{(k)}_{n+1,m}(0,q;p).$$

From the last equalities and (2.4), we get

$$c_{n+k,k}(p,q) = \frac{1}{k!}U^{(k)}_{n+1+k,m}(p,q;0) = d_{n,k(p,q)}. \quad (2.9')$$

Then, from (2.9'), we get

$$c_{n,k}(p,q) = \sum_{r=0}^{[(n-k)/m]}(-1)^r q^r\binom{n-(m-1)r}{k}\binom{n-k-(m-1)r}{r}P^{n-k-mr},$$

which completes the proof. ■

THEOREM 2.2. *The coefficients $c_{n,k}(p,q)$ satisfy the following relation*

$$c_{n,k+1}(p,q) = \frac{1}{k+1}\frac{\partial c_{n,k}(p,q)}{\partial p}. \quad (2.10)$$

Proof. Supposing that $n \geq 1$, and using (2.9), we see that

$$U_{n,m}^{(k)}(p,q;x) = U_{n,m}^{(k)}(0,q;x+p),$$

where the superscript in parentheses denotes the k-th derivative with respect to x. Using Taylor's formula and (2.2), we get

$$c_{n,k}(p,q) = \frac{1}{k!} U_{n+1,m}^{(k)}(0,q;p). \tag{2.10'}$$

Differentiating $(2.10')$ with respect to p (q is fixed), we get

$$\frac{\partial c_{n,k}(p,q)}{\partial p} = \frac{1}{k!} U_{n+1,m}^{(k+1)}(0,q;p) = (k+1)c_{n,k+1}(p,q).$$

Hence, we deduce that

$$c_{n,k+1}(p,q) = \frac{1}{k+1} \frac{\partial c_{n,k}(p,q)}{\partial p},$$

which completes the proof. ∎

Now we mention some particular cases:

(i) If $m = 2$, then (2.8) becomes (see [2])

$$c_{n,k}(p,q) = \sum_{r=0}^{[(n-k)/2]} (-1)^r q^r \binom{n-r}{k} \binom{n-k-r}{r} p^{n-k-2r}.$$

(ii) For $m = 3$, (see [4]), (2.8) yields

$$c_{n,k}(p,q) = \sum_{r=0}^{[(n-k)/3]} (-1)^r q^r \binom{n-2r}{k} \binom{n-k-2r}{r} p^{n-k-3r}.$$

Also, the last formula can be written in the following form:

$$c_{n,k}(p,q) = \sum_{r=0}^{[(n-k)/3]} (-1)^r q^r \binom{n-3r}{k} \binom{n-2r}{r} p^{n-k-3r}.$$

(iii) If $k = 0$, from (2.8), we get

$$c_{n,0}(p,q) = \sum_{r=0}^{[n/m]} (-1)^r q^r \binom{n-(m-1)r}{r} = U_{n+1,m}(p,q,0).$$

3. Determination of $c_{n,k}(p,q)$ as a polynomial in $(\alpha_1, \alpha_2, \ldots, \alpha_m)$

We are going to prove the following theorem.

THEOREM 3.1. *The coefficients $c_{n,k}(p,q)$ are given by*

$$c_{n,k}(p,q) = \sum_{i_1+\cdots+i_m=n} \binom{k+i_1}{k}\binom{k+i_2}{k}\cdots\binom{k+i_m}{k}\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_m^{i_m}. \tag{3.1}$$

Proof. Using (1.3) and (2.4) we get

$$\begin{aligned}\sum_{n\geq 0} d_{n,k}(p,q)\,t^n &= (1-pt+qt^m)^{-(k+1)}\\ &= (1-\alpha_1 t)^{-(k+1)}\cdot(1-\alpha_2 t)^{-(k+1)}\cdots(1-\alpha_m t)^{-(k+1)},\end{aligned}$$

so that

$$\begin{aligned}&\sum_{n\geq 0} d_{n,k}(p,q)\,t^n =\\ &= \sum_{n\geq 0} t^n \sum_{i_1+\cdots+i_m=n} \binom{k+i_1}{k}\binom{k+i_2}{k}\cdots\binom{k+i_m}{k}\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_m^{i_m},\end{aligned}$$

where

$$d_{n,k}(p,q) = \sum_{i_1+\cdots+i_m=n} \binom{k+i_1}{k}\binom{k+i_2}{k}\cdots\binom{k+i_m}{k}\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_m^{i_m}.$$

From (2.9′) and by the last equality, we get

$$\begin{aligned}c_{n,k}(p,q) &= d_{n-k,k}(p,q) =\\ &= \sum_{i_1+\cdots+i_m=n-k} \binom{k+i_1}{k}\binom{k+i_2}{k}\cdots\binom{k+i_m}{k}\alpha_1^{i_1}\cdots\alpha_m^{i_m}.\end{aligned}$$

This completes the proof. ∎

We mention some particular cases of (3.1):

(i) For $m=2$, from (3.1) we get the well-known equality (see [2])

$$c_{n,k}(p,q) = \sum_{i+j=n-k} \binom{k+i}{k}\binom{k+j}{k}\alpha_1^i\alpha_2^j.$$

(ii) For $m=3$, equality (3.1) becomes

$$c_{n,k}(p,q) = \sum_{i+j+s=n-k} \binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k}\alpha_1^i\alpha_2^j\alpha_3^s.$$

(iii) If $k=0$, by (3.1), we get

$$c_{n,0}(p,q)=\sum_{i_1+\cdots+i_m=n}\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_m^{i_m}=U_{n+1,m}(p,q,0).$$

4. Rising diagonal polynomials

In this section we define and study the polynomials $f_{n,m}(p,q;x)$. These polynomials are the rising diagonal polynomials of the polynomials $U_{n,m}(p,q;x)$. Hence, we have

$$f_{n+1,m}(p,q;x)=\sum_{k=0}^{[n/m]}c_{n-k,k}(p,q)x^k, \tag{4.1}$$

where $f_{0,m}(p,q;x)=0$.

Now, we are going to write the coefficients $c_{n,k}(p,q)$ in the following form

Table 4.1.

n/k	0	1	2	$\ldots$	$m-1$	m	$m+1$	$\ldots$
1	1	0	0	$\ldots$	0	0	0	$\ldots$
2	p	1	0	$\ldots$	0	0	0	$\ldots$
3	p^2	$2p$	1	$\ldots$	0	0	0	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
$m-1$	p^{m-2}	$(m-2)p^{m-3}$	$\binom{m-2}{2}p^{m-4}$	$\ldots$	0	0	0	$\ldots$
m	p^{m-1}	$(m-1)p^{m-2}$	$\binom{m-1}{2}p^{m-3}$	$\ldots$	1	0	0	$\ldots$
$m+1$	p^m	mp^{m-1}	$\binom{m}{2}p^{m-2}$	$\ldots$	mp	1	0	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

If we put $f_{n,m}(x)$ instead of $f_{n,m}(p,q;x)$, then from table 4.1, we get the first five terms of the sequence $\{f_{n,m}(p,q;x)\}$:

$$\begin{gathered}f_{0,m}(x)=0,\ f_{1,m}(x)=1,\ f_{2,m}(x)=p,\ f_{3,m}(x)=p^2+x,\\ f_{4,m}(x)=p^3+2px.\end{gathered} \tag{4.2}$$

In general, the following theorem holds:

THEOREM 4.1. *The polynomials $f_{n,m}(x)$ satisfy the following recurrence relation*

$$f_{n+1,m}(x)=pf_{n,m}(x)+xf_{n-1,m}(x)-qf_{n+1-m,m}(x),\quad n\geq m-1. \tag{4.3}$$

Proof. From (4.2), we see that (4.3) holds for $n = 4$. By induction on n, supposing that (4.3) is true for $n \geq 4$, by (4.1) and (2.3) we get

$$\begin{aligned}
f_{n+1,m}(x) &= c_{n,0}(p,q) - q c_{n-m,0}(p,q) + \sum_{k=1}^{[n/m]} c_{n-k,k}(p,q)x^k \\
&= p \sum_{k=0}^{[(n-1)/m]} c_{n-1-k,k}(p,q)x^k + x \sum_{k=0}^{[(n-2)/m]} c_{n-2-k,k}(p,q)x^k \\
&\quad - q \sum_{k=0}^{[(n-m)/m]} c_{n-m-k,k}(p,q)x^k \\
&= p f_{n,m}(x) + x f_{n-1,m}(x) - q f_{n+1-m,m}(x).
\end{aligned}$$

Now, the statement (4.3) follows immediately from the last equalities. ■

REMARK 4.1. For $m = 2$ in (4.3) we have the polynomials $f_n(p,q;x)$ (see [2]). Namely, we get the following recurrence relation

$$f_n(x) = p f_{n-1}(x) + (x - q) f_{n-2}(x).$$

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