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CONVERGENCE OF A FINITE DIFFERENCE METHOD FOR THE HEAT EQUATION — INTERPOLATION TECHNIQUE

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Abstract. In this paper we show how the theory of interpolation of function spaces can be used to establish convergence rate estimates for finite difference schemes. As a model problem we consider the first initial-boundary value problem for the heat equation with variable coefficients in a domain $(0,1)^2 \times (0,T]$. We assume that the solution of the problem and the coefficients of equation belong to corresponding Sobolev spaces. Using interpolation theory we construct a fractional–order convergence rate estimate which is consistent with the smoothness of the data.

1. Introduction

For a class of finite difference schemes for parabolic initial-boundary value problem, estimates of the convergence rate consistent with the smoothness of data, are of major interest, i.e.

$$\|u-v\|_{W_2^{r,r/2}(Q_{h\tau})} \le C(h+\sqrt{\tau})^{s-r}\|u\|_{W_2^{s,s/2}(Q)}, \quad s \ge r. \tag{1}$$

Here $u = u(x,t)$ denotes the solution of the original initial-boundary value problem, v denotes the solution of corresponding finite difference scheme, h and τ are discretisation parameters, $W_2^{s,s/2}(Q)$ denotes a Sobolev space, $W_2^{s,s/2}(Q_{h\tau})$ denotes a discrete Sobolev space, and C is a positive generic constant, independent of h, τ and u. If parameters h and τ satisfy the condition $k_1 h^2 \le \tau \le k_2 h^2$, $k_1, k_2 = const > 0$, then we obtain the estimate

$$\|u-v\|_{W_2^{r,r/2}(Q_{h\tau})} \le Ch^{s-r}\|u\|_{W_2^{s,s/2}(Q)}, \quad s \ge r. \tag{2}$$

For problems with variable coefficients the constant C depends on the norms of coefficients.

A standard technique for the derivation of such estimates is based on the Bramble-Hilbert lemma [2]. In this paper we expose an alternative technique,

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based on the theory of interpolation of Banach spaces. Estimate (2) was derived in the paper [3], by the same technique, for $r = 2$ and $2 \le s \le 4$ in the domain $Q = (0,1) \times (0,T]$. In this paper we derive estimate (2) for $r = 2$ and $2 \le s \le 4$ in the domain $Q = (0,1)^2 \times (0,T]$.

2. Interpolation of Banach spaces

In this paper we use the K-method of real interpolation [10,12]. Let $\{A_1, A_2\}$ be an interpolation pair. Define the functional

$$K(t,a) = K(t,a,A_1,A_2) = \inf\{\, \|a_1\|_{A_1} + t\|a_2\|_{A_2} \mid a \in A_1 + A_2, a = a_1 + a_2, a_i \in A_i \,\}.$$

It is obvious that, for a fixed $t \in (0,\infty)$, $K(t,a)$ is a norm in $A_1 + A_2$, equivalent to the standard norm $\|a\|_{A_1+A_2}$. For $0 < \theta < 1$, $1 \le q < \infty$, let us define the space $(A_1, A_2)_{\theta,q}$ as follows:

$$(A_1,A_2)_{\theta,q} = \left\{ a \in A_1 + A_2 : \|a\|_{(A_1,A_2)_{\theta,q}} = \left(\int_0^\infty \left(t^{-\theta} K(t,a) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\},$$

and for $q = \infty$

$$(A_1,A_2)_{\theta,\infty} = \left\{ a \in A_1 + A_2 : \|a\|_{(A_1,A_2)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K(t,a) < \infty \right\}.$$

The space $(A_1, A_2)_{\theta,q}$ is an interpolation space. The corresponding interpolation functor $\mathcal{F}(\{A_1, A_2\}) = (A_1, A_2)_{\theta,q}$ is of the type θ, i.e.

$$\|L\|_{(A_1,A_2)_{\theta,q} \to (B_1,B_2)_{\theta,q}} \le \|L\|^{1-\theta}_{A_1 \to B_1} \|L\|^{\theta}_{A_2 \to B_2},$$

An analogous assertion holds true for bilinear operators:

Lemma 1. *Let $A_1 \subset A_2$, $B_1 \subset B_2$ and $C_1 \subset C_2$ and let $L\colon A_2 \times B_2 \to C_2$ be a continuous bilinear form whose restriction on $A_1 \times B_1$ is a continuous maping with values in C_1. Then L is continuous maping from $(A_1, A_2)_{\theta,p} \times (B_1, B_2)_{\theta,q}$ into $(C_1, C_2)_{\theta,r}$, $0 < \theta < 1$, $1/r = 1/p + 1/q - 1 \ge 0$, and*

$$\|L\|_{(A_1,A_2)_{\theta,p} \times (B_1,B_2)_{\theta,q} \to (C_1,C_2)_{\theta,r}} \le \|L\|^{1-\theta}_{A_1 \times B_1 \to C_1} \|L\|^{\theta}_{A_2 \times B_2 \to C_2}.$$

As an example of interpolation spaces, let us consider the Sobolev spaces W_p^s [1]. For noninteger positive s one sets $W_p^s(\mathbb{R}^n) = B_{p,p}^s(\mathbb{R}^n)$, where B_{pp}^s is a Besov space [12].

For $0 \le s_1,\ s_2 < \infty$, $s_1 \ne s_2$, $0 < \theta < 1$, $1 \le q < \infty$ we have [12]:

$$\left(W_p^{s_1}(\mathbb{R}^n), W_p^{s_2}(\mathbb{R}^n) \right)_{\theta,q} = B_{p,q}^s(\mathbb{R}^n)\,,\ s = (1-\theta)s_1 + \theta s_2\,.$$

In such a way, for $q = p$ and noninteger $s = (1-\theta)s_1 + \theta s_2$, we obtain

$$\left(W_p^{s_1}(\mathbb{R}^n), W_p^{s_2}(\mathbb{R}^n) \right)_{\theta,p} = W_p^s(\mathbb{R}^n)\,,\ s = (1-\theta)s_1 + \theta s_2\,.$$

For $p = 2$ this relation holds without restrictions, i.e.:

$$(W_2^{s_1}(\mathbb{R}^n), W_2^{s_2}(\mathbb{R}^n))_{\theta,2} = W_2^s(\mathbb{R}^n)\,.$$

Hence, $W_2^s(\mathbb{R}^n)$ are interpolation spaces. The same result holds for Sobolev spaces in a domain Ω with sufficiently smooth boundary.

Let us define the anisotropic Sobolev space $W_2^{s,s/2}(Q)$, $Q = \Omega \times I$, $I = (0,T)$, as follows [5]: $W_2^{s,s/2}(Q) = L_2(I, W_2^s(\Omega)) \cap W_2^{s/2}(I, L_2(\Omega))$, with the norm

$$\|f\|_{W_2^{s,s/2}(Q)} = \left(\int_0^T \|f(t)\|^2_{W_2^s(\Omega)} dt + \|f\|^2_{W_2^{s/2}(I,L_2(\Omega))} \right)^{1/2} .$$

These spaces are interpolation spaces, too. For $s_1, s_2, r_1, r_2 \geq 0$, $0 < \theta < 1$, we have [8,12]

$$(W_2^{s_1,r_1}(Q), W_2^{s_2,r_2}(Q))_{\theta,2} = W_2^{s,r}(Q)\,,\ s = (1-\theta)s_1 + \theta s_2,\ r = (1-\theta)r_1 + \theta r_2 .$$

3. Initial-boundary value problem and its approximation

Let us consider the first initial-boundary value problem for parabolic equation with variable coefficients in the cylinder $Q = \Omega \times (0,T] = (0,1)^2 \times (0,T]$:

$$\begin{aligned} \frac{\partial u}{\partial t} - \sum_{i,j=1}^{2} D_i(a_{ij} D_j u) &= f\,, \quad (x,t) \in Q,\\ u &= 0\,, \quad (x,t) \in \partial\Omega \times [0,T],\\ u(x,0) &= u_0(x)\,, \quad x \in \Omega, \end{aligned} \tag{3}$$

We assume that the generalized solution of the problem (3) belongs to the Sobolev space $W_2^{s,s/2}(Q)$, $2 \leq s \leq 4$, with the right-hand side $f(x,t)$ which belongs to $W_2^{s-2,s/2-1}(Q)$. Consequently, the coefficients $a_{ij} = a_{ij}(x)$ belong to the space of multipliers $M\left(W_2^{s-1,(s-1)/2}(Q)\right)$, i.e. it is sufficient that [9]:

$$\begin{aligned} &a_{ij} \in W_2^{s-1}(\Omega)\,, && \text{for } \ 2 < s \leq 4\,,\\ &a_{ij} \in W_2^{1+\delta}(\Omega)\,,\ \delta > 0\,, && \text{for } \ s = 2\,. \end{aligned}$$

Let $\bar\omega$ be the uniform mesh in $\overline{\Omega} = [0,1]^2$ with the step size h, $\omega = \bar\omega \cap \Omega$, $\gamma = \bar\omega \cap \partial\Omega$. Let θ_τ be the uniform mesh in $(0,T)$ with the step size τ, $\theta_\tau^+ = \theta_\tau \cup \{T\}$, $\bar\theta_\tau = \theta_\tau \cup \{0,T\}$. We define the uniform mesh in Q: $Q_{h\tau} = \omega \times \theta_\tau$, $Q_{h\tau}^+ = \omega \times \theta_\tau^+$ and $\overline{Q}_{h\tau} = \bar\omega \times \bar\theta_\tau$. We assume that the condition:

$$k_1 h^2 \leq \tau \leq k_2 h^2\,, \quad k_1, k_2 = \text{const} > 0$$

is satisfied. We define finite differences in the usual manner:

$$v_{x_i} = \frac{v^{+i} - v}{h} = v_{\bar x_i}^{+i}\,, \quad v_t(x,t) = \frac{v(x,t+\tau) - v(x,t)}{\tau} = v_{\bar t}(x,t+\tau)\,,$$

where $v^{\pm i}(x,t) = v(x \pm h r_i, t)$, and r_i is the unit vector along the x_i axis. We also define the Steklov smoothing operators:

$$T_i^+ f(x,t) = \int_0^1 f(x + hx' r_i, t)\, dx' = T_i^- f(x + h r_i, t)\,,$$

$$T_i^2 f(x,t) = T_i^+ T_i^- f(x,t) = \int_{-1}^{1} (1 - |x'|) f(x + hx' r_i, t)\, dx'\,,$$

$$T_t^+ f(x,t) = \int_0^1 f(x, t + \tau t')\, dt' = T_t^- f(x, t+\tau)\,.$$

We approximate problem (3) with the following finite–difference scheme:

$$\begin{aligned} v_{\bar t} + L_h v &= T_1^2 T_2^2 T_t^- f\,, && \text{in } \ Q_{h\tau}^+\,,\\ v &= 0\,, && \text{on } \ \gamma\times\bar\theta_\tau\,,\\ v &= u_0\,, && \text{on } \ \omega\times\{0\}\,, \end{aligned} \tag{4}$$

where

$$L_h v = -0.5 \sum_{i,j=1}^{2} \left((a_{ij} v_{\bar x_j})_{x_i} + (a_{ij} v_{x_j})_{\bar x_i}\right).$$

The finite-difference scheme (4) is the standard symmetric scheme with the averaged right-hand side. Note that for $s \le 4$ the right-hand side may be a discontinuous function, so without averaging the scheme is not well defined.

4. Convergence of the finite-difference scheme

Let u be the solution of the initial-boundary value problem (3) and v the solution of the finite difference scheme (4). The error $z = u - v$ satisfies the conditions

$$\begin{aligned} z_{\bar t} + L_h z &= \sum_{i,j=1}^{2} \eta_{ij} + \varphi\,, && \text{in } \ Q_{h\tau}^+\,,\\ z &= 0\,, && \text{on } \ \omega\times\{0\}\,,\\ z &= 0\,, && \text{on } \ \gamma\times\bar\theta_\tau\,, \end{aligned} \tag{5}$$

where

$$\eta_{ij} = T_1^2 T_2^2 T_t^- \left(D_i(a_{ij} D_j u)\right) - 0.5((a_{ij} u_{\bar x_j})_{x_i} + (a_{ij} u_{x_j})_{\bar x_i})\,, \quad \varphi = u_{\bar t} - T_1^2 T_2^2 u_{\bar t}\,.$$

We define the discrete inner products:

$$(v,w)_\omega = (v,w)_{L_2(\omega)} = h^2 \sum_{x\in\omega} v(\cdot,t) w(\cdot,t)\,,$$

$$(v,w)_{Q_{h\tau}} = (v,w)_{L_2(Q_{h\tau})} = h^2 \tau \sum_{x\in\omega} \sum_{t\in\theta_\tau^+} v(x,t) w(x,t) = \tau \sum_{t\in\theta_\tau^+} (v,w)_\omega\,,$$

and the discrete Sobolev norms:

$$\|v\|_\omega^2 = (v,v)_\omega\,, \quad \|v\|_{Q_{h\tau}}^2 = (v,v)_{Q_{h\tau}}\,,$$

$$\|v\|_{W_2^{2,1}(Q_{h\tau})}^2 = \|v\|_{Q_{h\tau}}^2 + \sum_{i=1}^{2} \|v_{x_i}\|_{Q_{h\tau}}^2 + \sum_{i,j=1}^{2} \|v_{x_i x_j}\|_{Q_{h\tau}}^2 + \|v_{\bar t}\|_{Q_{h\tau}}^2\,.$$

The following assertion holds true :

Lemma 2. *Finite-difference scheme (5) satisfies a priori estimate*

$$\|z\|_{W_2^{2,1}(Q_{h\tau})} \le \sum_{i,j=1}^{2} \|\eta_{ij}\|_{Q_{h\tau}} + \|\varphi\|_{Q_{h\tau}}\,. \tag{6}$$

In such a way, the problem of deriving the convergence rate estimate for finite-difference scheme (4) is now reduced to estimating the right-hand side terms in (6).

We decompose term η_{ij} in the following manner: $\eta_{ij}=\sum_{k=1}^{7}\eta_{ijk}$, where

$$\begin{aligned}
\eta_{ij1} &= T_1^2T_2^2(a_{ij}T_t^-D_iD_ju)-(T_1^2T_2^2a_{ij})(T_1^2T_2^2T_t^-D_iD_ju)\,,\\
\eta_{ij2} &= (T_1^2T_2^2a_{ij}-a_{ij})(T_1^2T_2^2T_t^-D_iD_ju)\,,\\
\eta_{ij3} &= a_{ij}(T_1^2T_2^2T_t^-D_iD_ju-0.5(u_{\bar{x}_ix_j}+u_{x_i\bar{x}_j}))\,,\\
\eta_{ij4} &= T_1^2T_2^2(D_ia_{ij}T_t^-D_ju)-(T_1^2T_2^2D_ia_{ij})(T_1^2T_2^2T_t^-D_ju)\,,\\
\eta_{ij5} &= (T_1^2T_2^2D_ia_{ij}-0.5(a_{ij,x_i}+a_{ij,\bar{x}_i}))(T_1^2T_2^2T_t^-D_ju)\,,\\
\eta_{ij6} &= 0.5(a_{ij,x_i}+a_{ij,\bar{x}_i})(T_1^2T_2^2T_t^-D_ju-0.5(u_{x_j}^{-i}+u_{\bar{x}_j}^{+i}))\,,\\
\eta_{ij7} &= 0.25(a_{ij,x_i}-a_{ij,\bar{x}_i})(u_{x_j}^{-i}-u_{\bar{x}_j}^{+i})\,.
\end{aligned}$$

Let us derive the estimate (2) for $s=2\,,\ r=2$.

The value η_{ij1} in the node $(\cdot,t)\in\omega\times\{t\}$ can be represented in the form

$$\begin{aligned}
\eta_{ij1}(\cdot,t) = {} & \frac{1}{h^2}\iint_e k(\xi_1,\xi_2)a_{ij}(\xi_1,\xi_2)T_t^-D_iD_ju(\xi_1,\xi_2,t)d\xi_1d\xi_2-\\
&-\frac{1}{h^2}\iint_e k(\sigma_1,\sigma_2)a_{ij}(\sigma_1,\sigma_2)d\sigma_1d\sigma_2\times\frac{1}{h^2}\iint_e k(\xi_1,\xi_2)T_t^-D_iD_ju(\xi_1,\xi_2,t)d\xi_1d\xi_2
\end{aligned}\tag{7}$$

where $e=(x_1-h,x_1+h)\times(x_2-h,x_2+h)$ and

$$k(\xi_1,\xi_2)=\left(1-\frac{|\xi_1-x_1|}{h}\right)\left(1-\frac{|\xi_2-x_2|}{h}\right)\,.$$

From (7) immediately follows:

$$|\eta_{ij1}(\cdot,t)|\le\frac{C}{h}\|a_{ij}\|_{C(\bar{e})}\|T_t^-u(\cdot,t)\|_{W_2^2(e)}$$

Summation over the mesh ω yields:

$$\|\eta_{ij1}(\cdot,t)\|_\omega\le C\|a_{ij}\|_{C(\overline{\Omega})}\|T_t^-u(\cdot,t)\|_{W_2^2(\Omega)}\le C\|a_{ij}\|_{W_2^{1+\delta}(\Omega)}\|T_t^-u(\cdot,t)\|_{W_2^2(\Omega)}$$

From here, summing over the mesh θ_τ^+ we obtain

$$\|\eta_{ij1}\|_{Q_{h\tau}}\le C\|a_{ij}\|_{W_2^{1+\delta}(\Omega)}\|u\|_{W_2^{2,1}(Q)}\,.$$

Analogous estimates hold true also for the other terms η_{ijk} and for term φ. In these estimates we assume that $u\in W_2^{2+\varepsilon,1+\varepsilon/2}(Q)$, $\varepsilon>0$. In such a way we obtain the estimates:

$$\|\eta_{ij}\|_{Q_{h\tau}}\le C\|a_{ij}\|_{W_2^{1+\delta}(\Omega)}\|u\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q)}\,,\quad\text{and}\tag{8}$$

$$\|\varphi\|_{Q_{h\tau}}\le C\|u\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q)}\,.\tag{9}$$

From (6), (8) and (9) we obtain estimate (2) for $s=2\,,\ r=2$.

Let us derive estimate (2) for $s = 3\,,\ r = 2$.

The value η_{ij1} in the node $(\cdot, t) \in \omega \times \{t\}$ can be represented in the form

$$\eta_{ij1}(\cdot,t) = \frac{1}{2h^4} \iiiint_{e\times e} k(\xi_1,\xi_2)k(\sigma_1,\sigma_2)\left(\int_{\sigma_1}^{\xi_1} D_1 a_{ij}(\tau_1,\sigma_2)d\tau_1 + \int_{\sigma_2}^{\xi_2} D_2 a_{ij}(\xi_1,\tau_2)d\tau_2\right)$$
$$\times\, (T_t^- D_i D_j u(\xi_1,\xi_2,t) - T_t^- D_i D_j u(\sigma_1,\sigma_2,t))d\xi_1 d\xi_2 d\sigma_1 d\sigma_2\,, \tag{10}$$

From here, using Cauchy–Schwarz's and Hölder's inequality we obtain

$$|\eta_{ij1}(\cdot,t)| \le C\|a_{ij}\|_{W_p^1(e)} \|T_t^- u(\cdot,t)\|_{W^2_{\frac{2p}{p-2}}(e)}\,, \quad p > 2\,.$$

Summing over the meshes ω and θ_τ^+, using the imbeddings $W_2^2(\Omega) \subset W_p^1(\Omega)$, we simply obtain

$$\|\eta_{ij1}\|_{Q_{h\tau}} \le Ch\|a_{ij}\|_{W_2^2(\Omega)} \|u\|_{W_2^{3,3/2}(Q)}\,.$$

Analogous estimates hold true also for the other terms η_{ijk} and for term φ. In such a way we obtain the estimates:

$$\|\eta_{ij}\|_{Q_{h\tau}} \le Ch\|a_{ij}\|_{W_2^2(\Omega)} \|u\|_{W_2^{3,3/2}(Q)}\,, \quad \text{and} \tag{11}$$

$$\|\varphi\|_{Q_{h\tau}} \le Ch\|u\|_{W_2^{3,3/2}(Q)}\,. \tag{12}$$

From (6), (11) and (12) we obtain estimate (2) for $s = 3\,,\ r = 2$.

Let us derive estimate (2) for $s = 4,\ r = 2$.

From (10), using the representation

$$T_t^- D_i D_j u(\xi_1,\xi_2,t) - T_t^- D_i D_j u(\sigma_1,\sigma_2,t) =$$
$$= \int_{\sigma_1}^{\xi_1} T_t^- D_1 D_i D_j u(\rho_1,\sigma_2,t)d\rho_1 \ + \int_{\sigma_2}^{\xi_2} T_t^- D_2 D_i D_j u(\xi_1,\rho_2,t)d\rho_2\,,$$

and Cauchy-Schwarz's and Hölder's inequality we obtain

$$|\eta_{ij1}(\cdot,t)| \le Ch\|a_{ij}\|_{W_p^1(e)} \|T_t^- u(\cdot,t)\|_{W^3_{\frac{2p}{p-2}}(e)}\,, \quad p > 2\,.$$

Summing over the meshes ω and θ_τ^+, using the imbeddings $W_2^3(\Omega) \subset W_p^1(\Omega)$ and $W_2^4(\Omega) \subset W^3_{2p/(p-2)}(\Omega)$, we simply obtain

$$\|\eta_{ij1}\|_{Q_{h\tau}} \le Ch^2\|a_{ij}\|_{W_2^3(\Omega)} \|u\|_{W_2^{4,2}(Q)}\,.$$

Analogous estimates hold true also for the other terms η_{ijk} and for term φ. In such a way we obtain the estimates:

$$\|\eta_{ij}\|_{Q_{h\tau}} \le Ch^2\|a_{ij}\|_{W_2^3(\Omega)} \|u\|_{W_2^{4,2}(Q)}\,, \quad \text{and} \tag{13}$$

$$\|\varphi\|_{Q_{h\tau}} \le Ch^2\|u\|_{W_2^{4,2}(Q)}\,. \tag{14}$$

From (6), (13) and (14) we obtain estimate (2) for $s = 4\,,\ r = 2$.

Let us define the operators A_{ij} and B as follows:

$$\eta_{ij} = A_{ij}(a_{ij}, u) \quad \text{and} \quad \varphi = B(u)\,.$$

The operator A_{ij} is, obviously, bilinear. From (8), (11) and (13) it follows that it is a bounded bilinear operator from $W_2^{1+\delta}(\Omega) \times W_2^{2+\varepsilon,1+\varepsilon/2}(Q)$ to $L_2(Q_{h\tau})$, from $W_2^2(\Omega) \times W_2^{3,3/2}(Q)$ to $L_2(Q_{h\tau})$ and from $W_2^3(\Omega) \times W_2^{4,2}(Q)$ to $L_2(Q_{h\tau})$ with the norm:

$$\|A_{ij}\|_{W_2^{1+\delta}(\Omega)\times W_2^{2+\varepsilon,1+\varepsilon/2}(Q)\to L_2(Q_{h\tau})} \le C\,, \tag{15}$$

$$\|A_{ij}\|_{W_2^{2}(\Omega)\times W_2^{3,3/2}(Q)\to L_2(Q_{h\tau})} \le Ch\,. \tag{16}$$

$$\|A_{ij}\|_{W_2^{3}(\Omega)\times W_2^{4,2}(Q)\to L_2(Q_{h\tau})} \le Ch^2\,. \tag{17}$$

Applying Lemma 1, from (16) and (17) it follows that A_{ij} is a bounded bilinear operator from

$$\big(W_2^3(\Omega), W_2^2(\Omega)\big)_{\theta,2} \times \big(W_2^{4,2}(Q), W_2^{3,3/2}(Q)\big)_{\theta,2} = W_2^{3-\theta}(\Omega) \times W_2^{4-\theta,2-\theta/2}(Q)$$

to

$$(L_2(Q_{h\tau}), L_2(Q_{h\tau}))_{\theta,\infty} = L_2(Q_{h\tau})\,,$$

and

$$\|A_{ij}\|_{W_2^{3-\theta}(\Omega)\times W_2^{4-\theta,2-\theta/2}(Q)\to L_2(Q_{h\tau})} \le Ch^{2-\theta}\,, \quad 0<\theta<1\,.$$

Finally, we obtain the estimate:

$$\|\eta_{ij}\|_{Q_{h\tau}} \le Ch^{2-\theta}\|a_{ij}\|_{W_2^{3-\theta}(\Omega)}\|u\|_{W_2^{4-\theta,2-\theta/2}(Q)}\,, \quad 0<\theta<1\,.$$

Setting $4-\theta = s$, we obtain the estimate:

$$\|\eta_{ij}\|_{Q_{h\tau}} \le Ch^{s-2}\|a_{ij}\|_{W_2^{s-1}(\Omega)}\|u\|_{W_2^{s,s/2}(Q)}\,, \quad 3<s<4\,. \tag{18}$$

Similarly, from (15) and (16), by interpolation, we obtain the estimate:

$$\|\eta_{ij}\|_{Q_{h\tau}} \le Ch^{s-2}\|a_{ij}\|_{W_2^{s-1+\delta(3-s)}(\Omega)}\|u\|_{W_2^{s+\varepsilon(3-s),(s+\varepsilon(3-s))/2}(Q)}\,, \quad 2<s<3\,. \tag{19}$$

Analogously, we obtain the estimate of term φ:

$$\|\varphi\|_{Q_{h\tau}} \le Ch^{s-2}\|u\|_{W_2^{s,s/2}(Q)}\,, \quad 3<s<4\,, \tag{20}$$

$$\|\varphi\|_{Q_{h\tau}} \le Ch^{s-2}\|u\|_{W_2^{s+\varepsilon(3-s),(s+\varepsilon(3-s))/2}(Q)}\,, \quad 2<s<3. \tag{21}$$

Finally, from (8)–(14), (18)–(21) and (6) we obtain the main result of this paper:

Theorem. *Finite-difference scheme (4) converges in the norm of the space $W_2^{2,1}(Q_{h\tau})$ and, with condition $k_1h^2 \le \tau \le k_2h^2$, the following estimate holds true:*

$$\|u-v\|_{W_2^{2,1}(Q_{h\tau})} \le Ch^{s-2}(\max_{i,j}\|a_{ij}\|_{W_2^{s-1+\delta(3-s)}(\Omega)}+1)\|u\|_{W_2^{s+\varepsilon(3-s),(s+\varepsilon(3-s))/2}(Q)}\,, \quad 2\le s\le 3\,,$$

$$\|u-v\|_{W_2^{2,1}(Q_{h\tau})} \le Ch^{s-2}(\max_{i,j}\|a_{ij}\|_{W_2^{s-1}(\Omega)}+1)\|u\|_{W_2^{s,s/2}(Q)}\,, \ 3\le s\le 4\,.$$

The second estimate is consistent with the smoothness of data, while the first estimate is "almost" consistent with the smoothness of data.

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