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SOME CONVERGENCE RATE ESTIMATES FOR FINITE DIFFERENCE SCHEMES

Boško S. Jovanović and Branislav Z. Popović

Abstract. In this work we use function space interpolation to prove some convergence rate estimates for finite difference schemes. We concentrate on a Dirichlet boundary value problem for a second-order linear elliptic equation with variable coefficients in the unit 3-dimensional cube. We assume that the solution to the problem and the coefficients of the equation belong to corresponding Sobolev spaces.

1. Introduction

In this work we use interpolation theory to prove some convergence rate estimates for FDS. Our model problem will be a Dirichlet BVP for a second-order linear elliptic equation with variable coefficients in the unit 3-dimensional cube $\Omega=(0,1)^3$:

$$-\sum_{i,j=1}^{3} D_i(a_{ij}D_ju)=f \quad \text{in} \quad \Omega, \qquad u=0 \quad \text{on} \quad \Gamma=\partial\Omega. \tag{1}$$

We shall assume that the generalized solution of the BVP belongs to the Sobolev space$W_2^s(\Omega)$, $2\le s\le 4$, with the right-hand side $f(x)$ belonging to $W_2^{s-2}(\Omega)$. Initially we assume that the coefficients $a_{ij}(x)$ belong to the space of multipliers $M(W_2^{s-1}(\Omega))$; for this it is sufficient that [10]:

$$a_{ij}\in W_2^{s-1}(\Omega), \quad \text{for } \frac{5}{2}<s\le 4,$$
$$a_{ij}\in W_{3/(s-1)}^{s-1+\delta}(\Omega), \quad \delta>0, \quad \text{for } 2\le s\le \frac{5}{2}.$$

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We also assume that the corresponding differential operator is symmetric and strongly elliptic, i.e.

$$a_{ij} = a_{ji}, \quad \sum_{i,j=1}^{3} a_{ij} y_i y_j \geq c_0 \sum_{i=1}^{3} y_i^2, \quad x \in \Omega, \quad c_0 = \text{const} > 0.$$

Let $\overline{\omega}$ be a uniform mesh in $\overline{\Omega}$ with the step size h, $\omega = \overline{\omega} \cap \Omega$, $\gamma = \overline{\omega} \cap \Gamma$, etc. We define finite differences v_{x_i} and $v_{\overline{x}_i}$ in the usual manner [11]:

$$v_{x_i} = (v^{+i} - v)/h, \qquad v_{\overline{x}_i} = (v - v^{-i})/h,$$

where $v^{\pm i}(x) = v(x \pm hr_i)$, and r_i is the unit vector along the x_i axis.

We approximate the BVP with following FDS:

$$L_h v = T_1^2 T_2^2 T_3^2 f \quad \text{in} \quad \omega, \quad v = 0 \quad \text{on} \quad \gamma, \tag{2}$$

where T_i is Steklov smoothing operator on x_i, i.e.

$$T_i^+ f(x) = \int_0^1 f(x + htr_i)\, dt = T_i^- f(x + hr_i) = T_i f(x + 0.5hr_i)$$

(Hence, $T_i T_j f = T_j T_i f$ and $T_i^+ D_i u = u_{x_i}$, $T_i^- D_i u = u_{\overline{x}_i}$), and

$$L_h v = -\frac{1}{2} \sum_{i,j=1}^{3} \left[(a_{ij} v_{\overline{x}_j})_{x_i} + (a_{ij} v_{x_j})_{\overline{x}_i} \right].$$

Let u be the solution of the BVP and v the solution of the FDS. We define the error as $z = u - v$. Pur aim is to show that

$$\|u - v\|_{W_2^2(\omega)} \leq C \cdot h^{s-2} \|u\|_{W_2^s(\Omega)}, \quad 2 \leq s \leq 4, \tag{3}$$

where C is positive constant depending of the coefficients, but independent of h and u.

The finite-difference scheme (2) is the standard symmetric FDS [11] with averaged right-hand side. Note that for $s \leq 7/2$ the right-hand side is a discontinuous function, so without averaging the FDS is not well-defined.

Estimates of the type

$$\|u - v\|_{W_p^k(\omega)} \leq C \cdot h^{s-k} \|u\|_{W_p^s(\Omega)} \tag{4}$$

are said to be consistent with the smoothness of the solution of the BVP [9].

The same technique is used in papers of B.S. Jovanovic [6] (constant coeficient case) and [7], [14] ($n = 2$).

Estimates of type (4) have been obtained for a broad class of elliptic problems by Lazarov, Makarov, Samarski, Jovanović, Süli, Ivanović etc (see [5, 8, 9, 12]). As a rule the Bramble-Hilbert lemma [3] and results of Dupont and Scott [4] are used for proving those results.

2. Interpolation of Banach Spaces

Let A_0 and A_1 be two Banach spaces, linearly and continuosly embedded in a topological linear space $\mathcal{A}$. Two such spaces are called *interpolation pair* $\{A_0, A_1\}$. Consider also the spaces $A_0 \cap A_1$ and $A_0 + A_1$ with corresponding norms (see [2, 13]).

Let us introduce a category $\mathcal{C}_0$, whose objects $A, B, C, \ldots$ are Banach spaces, and whose morphisms are bounded linear operators $L \in \mathcal{L}(A, B)$, and a category $\mathcal{C}_1$, whose objects are interpolations pairs $\{A_0, A_1\}, \{B_0, B_1\}, \ldots$ and whose morphisms are $L \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$, where $\mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ denotes the set of bounded linear operators from $A_0 + A_1$ into $B_0 + B_1$, whose restrictions on A_i belong to the set $\mathcal{L}(A_i, B_i)$, $i = 1, 2$.

A functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_0$ is called an *interpolation functor* if $A_0 \cap A_1 \subset \mathcal{F}(\{A_0, A_1\}) \subset A_0 + A_1$ for every interpolation pair $\{A_0, A_1\}$, while for every morphism $L \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$, $\mathcal{F}(L)$ is the restriction of the operator L on $\mathcal{F}(\{A_0, A_1\})$.

The corresponding Banach space $A = \mathcal{F}(\{A_0, A_1\})$ is called an *interpolation space*.Obviously $A_0 \cap A_1$ and $A_0 + A_1$ are interpolation spaces.

If the inequality

$$\|L\|_{\mathcal{F}(\{A_0,A_1\})\to\mathcal{F}(\{B_0,B_1\})} \le C\|L\|_{A_0\to B_0}^{1-\theta}\|L\|_{A_1\to B_1}^{\theta},$$

where $0 < \theta < 1$ and $C = \text{const} \ge 1$, is satisfied for every morphism L of category $\mathcal{C}_1$, the interpolation functor $\mathcal{F}$ is said to be of the *type* θ.

Let us consider the so called complex interpolation method [13]. We define the following sets of complex numbers: $S = \{z \in \mathbb{C} : 0 < \Re z < 1\}$ and $\overline{S} = \{z \in \mathbb{C} : 0 \le \Re z \le 1\}$. For a given interpolation pair $\{A_0, A_1\}$ we introduce the set $\mathcal{M}(A_0, A_1)$ of continuous functions $f\colon \overline{S} \to A_0 + A_1$, analytic in S, which satisfy the following conditions:

(i) $\sup\limits_{z\in\overline{S}} \|f(z)\|_{A_0+A_1} < \infty$,

(ii) $f(j + it) \in A_j, \quad j = 0, 1, \quad t \in \mathbb{R}$,

(iii) the mapings $t \to f(j + it), \quad j = 0, 1$, are continuous on t, and

(iv) $\|f\|_{\mathcal{M}(A_0,A_1)} = \max\Big\{\sup\limits_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \sup\limits_{t\in\mathbb{R}} \|f(1+it)\|_{A_1}\Big\} < \infty.$

For $0 < \theta < 1$ with $[A_0, A_1]_\theta$ we denote the set of elements $a \in A_0 + A_1$ which satisfy the conditions:

(i) there exists a function $f \in \mathcal{M}(A_0, A_1)$ such that $f(\theta) = a$, and

(ii) $\|a\|_{[A_0,A_1]_\theta} = \inf\limits_{\substack{f\in\mathcal{M}(A_0,A_1)\\ f(\theta)=a}} \|f\|_{\mathcal{M}(A_0,A_1)} < \infty.$

The space $[A_0, A_1]_\theta$ defined in that way is an interpolation space. The corresponding interpolation functor $\mathcal{F}(\{A_0, A_1\}) = [A_0, A_1]_\theta$ is of the type θ, with constant $C = 1$. Analogous assertion holds true for bilinear operators [13]:

LEMMA 1. *Let* $A_0 \subset A_1$, $B_0 \subset B_1$, $C_0 \subset C_1$ *and let* $L\colon A_1 \times B_1 \to C_1$ *be a continuous bilinear form whose restriction on* $A_0 \times B_0$ *is a continuous maping with values in* C_0. *Then* L *is continuous maping from* $[A_0, A_1]_\theta \times [B_0, B_1]_\theta$ *into* $[C_0, C_1]_\theta$, *and*

$$\|L\|_{[A_0,A_1]_\theta \times [B_0,B_1]_\theta \to [C_0,C_1]_\theta} \le \|L\|^{1-\theta}_{A_0 \times B_0 \to C_0} \|L\|^{\theta}_{A_1 \times B_1 \to C_1}.$$

3. Spaces H_p^s, B_{pq}^s and W_p^s

As examples of interpolation function spaces we consider the spaces of Bessel potentials H_p^s, the Besov spaces B_{pq}^s and the Sobolev spaces W_p^s (see [1], [2] and [13]). The spaces H_p^s and B_{pq}^s are spaces of distributions. We know that $\mathcal{D}(\mathbb{R}^n) \subset H_p^s(\mathbb{R}^n)$ and $B_{pq}^s \subset \mathcal{D}'(\mathbb{R}^n)$ where $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ is the set of infinitely differentiable functions with compact support, and $\mathcal{D}'(\mathbb{R}^n)$ is the set of distributions. For $s = 0$, $H_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, where L_p is the Lebesgue space of integrable functions. For $1 < p < \infty$ the Sobolev spaces W_p^s are defined in the following manner:

$$W_p^s(\mathbb{R}^n) = \begin{cases} H_p^s(\mathbb{R}^n), & s = 0, 1, 2, \dots \\ pp^s(\mathbb{R}^n), & 0 < s \ne \text{integer} \end{cases} \tag{5}$$

with the norm defined as

$$\|f\|_{W_p^s} = \Big(\sum_{k<s} |f|^p_{W_p^k} + |f|^p_{W_p^s} \Big)^{1/p},$$

where

$$|f|_{W_p^r} = \begin{cases} \Big(\sum\limits_{|\alpha|=r} \int\limits_{\mathbb{R}^n} |D^\alpha f(x)|^p dx \Big)^{1/p}, & r = 0, 1, 2, \dots \\ \Big(\sum\limits_{|\alpha|=[r]} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+p(r-[r])}} dx dy \Big)^{1/p}, & 0 < r \ne \text{integer}. \end{cases}$$

Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x| = (x_1 + \dots + x_n)^{1/2}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ and $[r]$ is the integer part of r. Obviously, $W_p^s(\mathbb{R}^n) \subset L_p(\mathbb{R}^n)$, $s \ge 0$.

For $-\infty < s < \infty$, $1 < p < \infty$, $\varepsilon > 0$ and $1 \le q_0 \le q_1 \le \infty$ the folloving imbeddings hold true [13]:

$$B^{s+\varepsilon}_{p,\infty}(\mathbb{R}^n) \subset B^s_{p1}(\mathbb{R}^n) \subset B^s_{pq_0}(\mathbb{R}^n) \subset B^s_{pq_1}(\mathbb{R}^n) \subset B^s_{p,\infty}(\mathbb{R}^n) \subset B^{s-\varepsilon}_{p1}(\mathbb{R}^n),$$

$$H_p^{s+\varepsilon}(\mathbb{R}^n) \subset H_p^s(\mathbb{R}^n) \quad \text{and}$$

$$B^s_{p,\min\{p,2\}}(\mathbb{R}^n) \subset H_p^s(\mathbb{R}^n) \subset B^s_{p,\max\{p,2\}}(\mathbb{R}^n). \tag{6}$$

For $-\infty < t \le s < \infty$, $1 < p \le q < \infty$, $1 \le r \le \infty$ and $s - n/p \ge t - n/q$ we also have

$$B^s_{pr}(\mathbb{R}^n) \subset B^t_{qr}(\mathbb{R}^n) \quad \text{and} \quad H_p^s(\mathbb{R}^n) \subset H_q^t(\mathbb{R}^n).$$

The following assertion holds true [13]:

LEMMA 2. *For* $-\infty < s_0, s_1 < \infty,\ 1 < p_0, p_1 < \infty,\ 1 \le q_0 < \infty,\ 1 \le q_1 \le \infty$ *and* $0 < \theta < 1$ *we have*

$$\left[H^{s_0}_{p_0}(\mathbb{R}^n), H^{s_1}_{p_1}(\mathbb{R}^n)\right]_\theta = H^s_p(\mathbb{R}^n) \quad \text{and} \tag{7}$$

$$\left[B^{s_0}_{p_0 q_0}(\mathbb{R}^n), B^{s_1}_{p_1 q_1}(\mathbb{R}^n)\right]_\theta = B^s_{pq}(\mathbb{R}^n), \tag{8}$$

where

$$s = (1-\theta)s_0 + s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

From (7), (8) and (5), for $s_0, s_1 \ge 0$, it follows that

$$\left[W^{s_0}_p(\mathbb{R}^n), W^{s_1}_p(\mathbb{R}^n)\right]_\theta = W^s_p(\mathbb{R}^n), \quad s = (1-\theta)s_0 + s_1, \tag{9}$$

if $s_0,\ s_1$ and s are all integer, or fractional numbers. For $p = 2$ from (6) it follows that $W^s_2(\mathbb{R}^n) = H^s_2(\mathbb{R}^n) = B^s_{22}(\mathbb{R}^n)$ and (9) holds without the restriction that s_0, s_1 and s are of the same kind.

The previous results hold for the spaces H^s_p, B^s_{pq} and W^s_p in a bounded domain $\Omega \subset \mathbb{R}^n$ which satisfies the cone condition. Here we assume that $s \ge 0$ for H^s_p spaces, and $s > 0$ for B^s_{pq} spaces.

4. Convergence of Finite Difference Schemes

Let u be the solution of the BVP (1) and let v be the solution of the FDS (2). The error $z = u - v$ satisfies the conditions

$$L_h z = \sum_{i,j=1}^{3} \psi_{ij}, \quad \text{in} \ \ \omega, \quad z = 0 \ \ \text{on} \ \ \gamma, \tag{10}$$

where

$$\psi_{ij} = TD_i(a_{ij}D_j u) - \frac{1}{2}\left[(a_{ij}u_{\overline{x}_j})_{x_i} + (a_{ij}u_{x_j})_{\overline{x}_i}\right], \quad Tu = T_1^2 T_2^2 T_3^2 u, \quad i,j = 1,2,3.$$

Let $(v,w)_\omega = (v,w)_{L_2(\omega)} = h^3 \sum_{x\in\omega} v(x)w(x)$ and $\|v\|^2_\omega = (v,v)_\omega$ denote the discrete inner product and the discrete L_2-norm on ω. We also define the discrete Sobolev norm

$$\|v\|^2_{W^2_2(\omega)} = \|v\|^2_\omega + \sum_{i=1}^{3} \|v_{x_i}\|^2_{\omega_i} + \sum_{i=1}^{3} \|v_{x_i\overline{x}_i}\|^2_\omega + \sum_{i<j}^{3} \|v_{x_i x_j}\|^2_{\omega_{ij}},$$

where ω_i and ω_{ij} are subsets of $\overline{\omega}$ where corresponding finite differences are well defined.

The following assertion holds true [5]:

LEMMA 3. *The FDS (10) satisfies the a priori estimate*

$$\|z\|_{W^2_2(\omega)} \le C \cdot \sum_{i,j=1}^{3} \|\psi_{ij}\|_\omega. \tag{11}$$

The problem of deriving the convergence rate estimates for thr FDS (2) is reduced to estimating the right-hand side terms in (11). Let us decompose ψ_{ij} in the following manner: $\psi_{ij}=\sum_{k=1}^{7}\psi_{ijk}$, where

$$\begin{aligned}
\psi_{ij1}&=T(a_{ij}D_iD_ju)-(Ta_{ij})(TD_iD_ju),\\
\psi_{ij2}&=(Ta_{ij}-a_{ij})(TD_iD_ju),\\
\psi_{ij3}&=a_{ij}\left[TD_iD_ju-0.5(u_{\overline{x}_ix_j}+u_{x_i\overline{x}_j})\right],\\
\psi_{ij4}&=T(D_ia_{ij}D_ju)-(TD_ia_{ij})(TD_ju),\\
\psi_{ij5}&=\left[TD_ia_{ij}-0.5(a_{ij,x_i}+a_{ij,\overline{x}_i})\right](TD_ju),\\
\psi_{ij6}&=0.5(a_{ij,x_i}+a_{ij,\overline{x}_i})\left[TD_ju-0.5(u_{x_j}^{-i}+u_{\overline{x}_j}^{+i})\right],\\
\psi_{ij7}&=0.25(a_{ij,x_i}+a_{ij,\overline{x}_i})(u_{x_j}^{-i}+u_{\overline{x}_j}^{+i}).
\end{aligned}$$

The value ψ_{ij1} in the node $x\in\omega$ can be represented in the form

$$\begin{aligned}
\psi_{ij1}=&\frac{1}{h^6}\int\cdots\int_{e\times e}\Phi(\xi_1,\xi_2,\xi_3)\Phi(\sigma_1,\sigma_2,\sigma_3)[a_{ij}(\xi_1,\xi_2,\xi_3)-a_{ij}(\sigma_1,\sigma_2,\sigma_3)]\times\\
&\times D_iD_ju(\xi_1,\xi_2,\xi_3)\,d\xi_1d\xi_2d\xi_3d\sigma_1d\sigma_2d\sigma_3,
\end{aligned}\tag{12}$$

where $e=(x_1-h,x_1+h)\times(x_2-h,x_2+h)\times(x_3-h,x_3+h)$ and

$$\Phi(\xi_1,\xi_2,\xi_3)=\left(1-\frac{|\xi_1-x_1|}{h}\right)\left(1-\frac{|\xi_2-x_2|}{h}\right)\left(1-\frac{|\xi_3-x_3|}{h}\right).$$

Now, from (12) it follows that:

$$|\psi_{ij1}|\le\frac{C}{h^{3/2}}\|a_{ij}\|_{C(\bar e)}\|D_iD_ju\|_{L_2(e)}\le\frac{C}{h^{3/2}}\|a_{ij}\|_{C(\overline{\Omega})}\|u\|_{W_2^2(e)}.$$

From here, summing over the mesh ω we obtain

$$\|\psi_{ij1}\|_\omega\le C\cdot\|a_{ij}\|_{C(\overline{\Omega})}\|u\|_{W_2^2(\Omega)}\le C\cdot\|a_{ij}\|_{W_p^{1+\varepsilon}(\Omega)}\|u\|_{W_2^2(\Omega)},\ \varepsilon>0,\ p\ge3.\tag{13}$$

Transforming $a_{ij}(\xi_1,\xi_2,\xi_3)-a_{ij}(\sigma_1,\sigma_2,\sigma_3)$ in (12) to integral form

$$\begin{aligned}
a_{ij}(\xi_1,\xi_2,\xi_3)-a_{ij}(\sigma_1,\sigma_2,\sigma_3)=&\int_{\sigma_1}^{\xi_1}D_1a_{ij}(\tau_1,\sigma_2,\sigma_3)\,d\tau_1+\\
&+\int_{\sigma_2}^{\xi_2}D_2a_{ij}(\xi_1,\tau_2,\sigma_3)\,d\tau_2+\int_{\sigma_3}^{\xi_3}D_3a_{ij}(\xi_1,\xi_2,\tau_3)\,d\tau_3
\end{aligned}\tag{14}$$

and exchanging ξ_i and σ_i, we obtain

$$\begin{aligned}
\psi_{ij1}=&\frac{1}{2h^6}\int\cdots\int_{e\times e}\Phi(\xi_1,\xi_2,\xi_3)\Phi(\sigma_1,\sigma_2,\sigma_3)\times\left[\int_{\sigma_1}^{\xi_1}D_1a_{ij}(\tau_1,\xi_2,\xi_3)\,d\tau_1+\right.\\
&\left.+\int_{\sigma_2}^{\xi_2}D_2a_{ij}(\sigma_1,\tau_2,\xi_3)\,d\tau_2+\int_{\sigma_3}^{\xi_3}D_3a_{ij}(\sigma_1,\sigma_2,\tau_3)\,d\tau_3\right]\times\\
&\times[D_iD_ju(\xi_1,\xi_2,\xi_3)-D_iD_ju(\sigma_1,\sigma_2,\sigma_3)]\ d\xi_1d\xi_2d\xi_3d\sigma_1d\sigma_2d\sigma_3.
\end{aligned}\tag{15}$$

Finally, transforming $D_iD_ju(\xi_1,\xi_2,\xi_3)-D_iD_ju(\sigma_1,\sigma_2,\sigma_3)$ in (15) to integral form (like in (14)) and applying Hölder's inequality, we obtain

$$|\psi_{ij1}|\le C\cdot h^{1/2}\|a_{ij}\|_{W_p^1(e)}\|u\|_{W^3_{2p/(p-2)}(e)},\quad p>2.$$

From here, summing over the mesh ω, and using the imbeddings $W_p^3\subset W_p^1$ and $W_2^4\subset W^3_{2p/(p-2)}$ for $p\ge 3$, we obtain

$$\begin{aligned}\|\psi_{ij1}\|_\omega &\le C\cdot h^2\|a_{ij}\|_{W_p^1(\Omega)}\|u\|_{W^3_{2p/(p-2)}(\Omega)}\\ &\le C\cdot h^2\|a_{ij}\|_{W_p^3(\Omega)}\|u\|_{W_2^4(\Omega)},\quad p\ge 3.\end{aligned}\tag{16}$$

Estimates analogous to (13) and (16) hold true for the other terms ψ_{ijk} [14] and so we obtain

$$\|\psi_{ij}\|_\omega\le C\cdot\|a_{ij}\|_{W_p^{1+\varepsilon}(\Omega)}\|u\|_{W_2^2(\Omega)},\quad \varepsilon>0,\ p\ge 3\tag{17}$$

$$\|\psi_{ij}\|_\omega\le C\cdot h^2\|a_{ij}\|_{W_p^3(\Omega)}\|u\|_{W_2^4(\Omega)},\quad p\ge 3.\tag{18}$$

The mapping $(a_{ij},u)\to\psi_{ij}$ is bilinear. From (17) and (18) it follows that it is a bounded bilinear operator from $W_p^{1+\varepsilon}(\Omega)\times W_2^2(\Omega)$ to $L_2(\omega)$ and from $W_p^3(\Omega)\times W_2^4(\Omega)$ to $L_2(\omega)$. Applying Lemma 1, from (17) and (18) it follows that ψ_{ij} is a bounded bilinear operator from $\left[W_p^{1+\varepsilon}(\Omega),W_p^3(\Omega)\right]_\theta\times\left[W_2^2(\Omega),W_2^4(\Omega)\right]_\theta$ to $L_2(\omega)$, with the norm $M\le C\cdot h^{2\theta}$. According to Lemma 2, (8) and (9)

$$\left[W_p^{1+\varepsilon}(\Omega),W_p^3(\Omega)\right]_\theta=B_{pp}^{1+\varepsilon+\theta(2-\varepsilon)}(\Omega)\quad\text{and}\quad\left[W_2^2(\Omega),W_2^4(\Omega)\right]_\theta=W_2^{2+2\theta}(\Omega).$$

Setting $2+2\theta=s$, we obtain

$$\|\psi_{ij}\|_\omega\le C\cdot h^{s-2}\|a_{ij}\|_{B_{pp}^{s-1+\varepsilon(2-s/2)}(\Omega)}\|u\|_{W_2^s(\Omega)},\quad p\ge 3,\quad 2<s<4.\tag{19}$$

Combining (11) and (17)–(19) we have thus proved the following result:

THEOREM. *The FDS (2) converges in the norm of the space $W_2^2(\omega)$ and following estimate, which is consistent with the smoothness of the data, holds true*

$$\|u-v\|_{W_2^2(\omega)}\le C\cdot h^{s-2}\cdot\max_{ij}\|a_{ij}\|_{B_{pp}^{s-1+\varepsilon(2-s/2)}(\Omega)}\cdot\|u\|_{W_2^s(\Omega)},\quad p\ge 3,\quad 2\le s\le 4.$$

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University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Yugoslavia

University of Kragujevac, Faculty of Science, Radoja Domanovića 12, 34000 Kragujevac, Yugoslavia