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NEW VERSIONS OF GRÖTZSCH PRINCIPLE AND REICH-STREBEL INEQUALITY

V. Marković and M. Mateljević

Abstract. In this note we will state a new version of Grötzsch's principle and using this principle we will sketch the proof of a generalization of the main inequality. Also, we will announce some related results and briefly explain that one can use new version of the main inequality to study uniqueness property of harmonic mapping in general. More details will be given in a forthcoming paper.

A. The Problem of Grötzsch

If Q is a square and R is a rectangle, not a square, there is no conformal mapping of Q on R which maps vertices on vertices. Instead, Grötzsch asks for the most nearly conformal mapping of this kind and took the first step toward the creation of a theory of q.c. mappings.

Let $w = f(z)$ be a mapping from one region to another. It is convenient to use notations $df = p\,dz + q\,d\bar z$, where $p = \partial f$ and $q = \bar\partial f$. Also, we introduce the complex dilatation $\mu_f = \dfrac{\bar\partial f}{\partial f}$ and dilatation

$$D_f = \frac{|\partial f| + |\bar\partial f|}{|\partial f| - |\bar\partial f|}$$

We pass to Grötzsch problem and give it a precise meaning by saying that f is most nearly conformal if $\sup D_f$ is as small as possible.

It is convenient to suppose that we work in this paper with mapping which are sense preserving and whose Jacobian is positive a. e. on the domain of definition.

Let R, R' be two rectangles with sides a, b and a', b'. We may assume that $b : a \le b' : a'$.

The mapping f is supposed to be C^1-homeomorphism from $\overline{R}$ onto $\overline{R'}$, which takes a-sides into a-sides and b-sides into b-sides. Next, let Γ_x be vertical segment

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which is intersection of the line $\operatorname{Re} z = x$ with $\overline{R}$ and Γ'_x the curve which is image of Γ_x under f.

The starting point of Grötzsch's approach has obvious geometric meaning

$$b' \leq \operatorname{length}(\Gamma'_x) = \int_0^a |p - q| \, dy. \tag{A1}$$

It is interesting here that using that

$$\iint_R J_f dx \, dy = a'b' \tag{A2}$$

and Cauchy-Schwarz inequality one can get

$$\frac{b'}{a'} : \frac{b}{a} \leq \sup D_f. \tag{A3}$$

The minimum is attained for the affine mapping.

B. A version of Grötzsch principle

The restriction to C^1-mapping is not essential. The inequality $(A3)$ holds for *qc* mapping (see, for example [**Ah**]).

Before we give further extension of Grötzsch's principle, let us consider the following example when $(A1)$ and $(A3)$ do not hold.

Example 1. Let $\alpha \colon I \to I$, where $I = [0,1]$, be Cantor function and let $f(z) = x + i(y + \alpha(y))$.

Note that this function does not satisfy ACL property and the known formula for length of curve by means of first partial derivatives does not hold.

In order to provide that the formula for the length of the curve (by means of the first partial derivatives) holds it looks resonable to assume that f has the first generalized derivatives on R (or, more generally, that f satisfies ACL property on R).

Further, it will be convenient to use notation $T_\mu = \dfrac{|1-\mu|^2}{1-|\mu|^2}$.

Recall that we suppose that Jacobian $J_f > 0$ a. e. on the domain of definition.

Theorem 1. *Suppose that f is a homeomorphism of closed rectangle $\overline{R}$ onto closed rectangle $\overline{R'}$, which maps a-sides onto a'-sides and b-sides onto b'-sides. If we, in addition, suppose that $K = \frac{b'}{a'} : \frac{b}{a} \geq 1$ and that f has the first generalized derivatives on R, then*

$$K \leq \frac{1}{ab} \iint_R T_\mu dx \, dy.$$

Outline of proof. Since f satisfies ACL property,

$$b' \leq \int_{\Gamma_x} |df| = \int_{\Gamma_x} |p| \, |1 - \mu| \, dy$$

a.e. $x \in [0, a]$.

Next, we integrate this w.r.t. dx over $[0,a]$ and use the fact that Jacobian of f equals

$$J_f = |p|^2(1-|\mu|^2) \quad \text{a.e. on } R.$$

To finish the proof we need to use Caushy-Schwarz inequality and

$$\int_R J_f dx\, dy \le m(R') = a'b'. \tag{B1}$$

Note that this inequality is a substitution for the equality $(A2)$. ■

To illustrate that the above method is usefull, let us consider the following corollaries.

Since $T_\mu(z) \le D_f(z)$ a.e. in R, we get (imediatelly from Theorem 1)

COROLLARY 1. *Under the conditions of Theorem 1,* $K \le \sup D_f$.

COROLLARY 2. *Let f be a conformal mapping from R onto R', which maps vertices onto vertices. Then* $\frac{a'}{b'} = \frac{a}{b}$.

Further developments of the ideas outlined above leads us to Theorem 2 (see below), which will be used in the proof of a new version of the main inequality.

Let D be a vertically convex domain of finite area in the complex plane; and let F be a homemomorphism from the domain D onto domain G in $\mathbb{C}$.

Suppose that we have a metric $ds = \varrho(w)|dw|$ on G. Denote by Θ_x the interval which is intersection of D by the straight line $\operatorname{Re} z = x$ and let γ_x be the curve which is the image of Θ_x under f.

THEOREM 2. *With notation and hypotheses just stated, suppose, (in addition) that mapping F has the first generalized derivatives and that*

$$\operatorname{length}(\Theta_x) \le \int_{\gamma_x} \varrho(w)\,|dw| \quad a.e.$$

Then

$$area(D) \le \left[\iint_G \varrho^2(w) du\, dv\right]^{1/2} \left[\iint_D T_\vartheta d\xi d\eta\right]^{1/2}, \tag{B2}$$

where $\vartheta = \operatorname{Belt}[F]$.

C. New version of main inequality

Let Δ denote the unit disk in the complex plane. Recall that in the proof of main inequality in the case of unit disk the following results play important role.

C1. Suppose that φ is an analytic function in $\overline{\Delta}$ and let φ has only simple zeros. The following decomposition is possible [**S**].

Up to a set of Lebesque 2-dimesional measure 0, $\Delta = \bigcup_{k=1}^n \Sigma_k$, where $\{\Sigma_k\}$ are disjoint simple connected "strip" domains. Each Σ_k is swept out by a family of trajectories of holomorphic quadratic differential $\varphi(z)dz^2$ and in each Σ_k there exists a single-valued schlicht branch $\zeta = \Phi_k(z)$ of $\int \sqrt{\varphi(z)}\,dz$. Each region $\Sigma_k' = \Phi_k(\Sigma_k)$ is verticaly convex.

C2. The trajectories of holomorphic quadratic differential are globaly geodesics in Teichmüller metric $ds^2 = |\varphi(z)|\,|dz|^2$. Suppose that:

(a) f is a homeomorphism of $\overline{\Delta}$ onto itself;

(b) f has the first generalized derivatives on Δ;

(c) f is the identity on $\partial\Delta$.

We will use the following notation: $\mu = \mathrm{Belt}\,[f]$ and

$$T_\mu\varphi(z) = \frac{\left|1 - \mu(z)\frac{\varphi(z)}{|\varphi(z)|}\right|^2}{1 - |\mu(z)|^2}.$$

Applying Theorem 2 to the function $f \circ \Phi_k^{-1}$ and then changing the variables $z = \Phi_k^{-1}(\zeta)$ and summing the obtained inequalities leads us to the following result.

THEOREM 3. *Under above assumptions,*

$$\iint_\Delta |\varphi|\,dx\,dy \le \iint_\Delta |\varphi(z)| T_\mu\varphi(z)\,dx\,dy$$

Thus, we have a version of main inequality which is applicable to mappings which are not q.c. mappings.

Let us briefly describe further results and applications.

C3. Note that we can get the corresponding inequality if we consider the Riemann surface R of finite analytic type instead of disk Δ and if we suppose that h is self-mapping of R which is homotopic to the identity (instead of (c) in **C2**).

C4. Using as motivation the standard definition of harmonic mapping (see, for example [**J**]) we can give more general definition.

Let Ω be a domain in $\mathbb{C}$ and $f\colon \Omega \to C$, which has the first generalized derivatives on Ω; and let $ds = \varrho(w)\,|dw|$ be a metric on $f(\Omega)$.

We say that f is a harmonic mapping in general sense (w.r.t. ϱ) if there is an analytic function φ on Ω such that

$$(\varrho \circ f)p\overline{q} = \varphi \qquad \text{a.e. in} \quad \Omega.$$

EXAMPLE 2. Let $\Omega = \{z : |x| < 1\}$ and $f(z) = 2(\operatorname{sgn} x)\sqrt{|x|} + iy$ and $\varrho(w) = \frac{u^2}{4-u^2}$. Function f is a harmonic mapping in general sense, but it is not qc on Ω.

New version of main inequality enables us to study uniqueness property of harmonic mapping in general sense.

THEOREM 4. *Suppose that f and g are harmonic diffeomorphisms (or, more generally, harmonic in general homeomorphisms) from the unit disc onto itself, continuous on the closure of the unit disc and suppose that $f = g$ on the boundary of the unit disk. If, in addition, we suppose that the energy integrals of f and g are finite, then they are identical.*

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Matematički fakultet, Studentski trg 16, 11000 Beograd, Yugoslavia
E-mail: miodrag@matf.bg.ac.yu, vmarkov@matf.bg.ac.yu