МАТЕМАТИЧКИ ВЕСНИК
49 (1997), 229–233

UDK 517.55
оригинални научни рад
research paper

BESOV SPACES ON BOUNDED SYMMETRIC DOMAINS

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Abstract. We define and study a class of holomorphic Besov type spaces B^p, $0 < p < 1$, on bounded symmetric domains Ω. We show that the dual of holomorphic Besov space B^p, $0 < p < 1$, on bounded symmetric domain Ω can be identified with the Bloch space $\mathcal{B}^\infty$.

1. Introduction

Let Ω be an irreducible symmetric domain in C^n in its Harish-Chandra realization. In [11] and [13] K. Zhu defined and studied a class of holomorphic Besov-type spaces B^p on Ω for $1 \le p \le \infty$. In [4] analogous holomorphic Besov spaces B^p are defined for $0 < p < 1$ and some of the results presented in [11] and [13] are extended to the case $0 < p < 1$. The main purpose of this paper is to show that the dual of B^p, $0 < p < 1$, can be indentified with the Bloch space $\mathcal{B}^\infty$.

It is well known [5] that the domain Ω is uniquely determined (up to a biholomorphic mapping among standard irreducible bounded symmetric domains) by three analytic invariants; r, a and b, all of which are nonnegative integers. The invariant r is called the rank of Ω, which is of course always positive. See [5] for the definition of a and b. We shall make extensive use of the following invariant of Ω: $N = a(r-1) + b + 2$.

Let ν be Lebesgue measure on Ω normalized so that $\nu(\Omega) = 1$. For $0 < p < \infty$ the Bergman space $L^p_a(\Omega)$ is the closed subspace of $L^p(\Omega, d\nu)$ consisting of holomorphic functions. The Bergman projection P (namely, the orthogonal projection from $L^2(\Omega, d\nu)$ onto $L^2_a(\Omega)$) is an integral operator

$$Pf(z) = \int_\Omega K(z,w) f(w)\, d\nu(w), \qquad z \in \Omega, \quad f \in L^2(\Omega, d\nu).$$

By [3] there exists a polynomial $h(z,w)$ in z and $\bar{w}$ such that the Bergman kernel of Ω is given by

$$K(z,w) = h(z,w)^{-N}, \qquad z, w \in \Omega.$$

AMS Subject Classification: 32 A 37

Communicated at the 4th Symposium on Mathematical Analysis and Its Applications, Aranđelovac 1997

Throughout this paper we assume α is a real number satisfying $\alpha > -1$. Let c_α be a positive normalizing constant such that the measure $d\nu_\alpha(z) = c_\alpha h(z,z)^\alpha d\nu(z)$ has total mass 1 on Ω.

Let $H(\Omega)$ be the space of all holomorphic functions in Ω. We equip $H(\Omega)$ with the topology of uniform convergence on compact sets. In [13] it is shown that the operator

$$D^{m,\alpha}\colon H(\Omega) \to H(\Omega), \qquad m \geq 0, \quad \alpha > -1,$$

defined by

$$D^{m,\alpha} f(z) = \lim_{r\to 1} \int_\Omega \frac{f(rw)\, d\nu_\alpha(w)}{h(z,w)^{N+\alpha+m}}, \quad f \in H(\Omega),$$

is continuous and invertible on $H(\Omega)$. The inverse of $D^{m,\alpha}$ admits the following integral representation:

$$D_{m,\alpha} f(z) = c_{m+\alpha} \lim_{r\to 1} \int_\Omega \frac{h(w,w)^{m+\alpha} f(rw)\, d\nu(w)}{h(z,w)^{N+\alpha}}, \; f \in H(\Omega), \; z \in \Omega.$$

We note that if

$$f \in L_a^{1,\alpha}(\Omega) = L^1(\Omega, d\nu_\alpha) \cap H(\Omega)$$

then

$$D^{m,\alpha} f(z) = \int_\Omega \frac{f(w)\, d\nu_\alpha(w)}{h(z,w)^{N+\alpha+m}}.$$

The above formula extends the domain of $D^{m,\alpha}$ to $L^1(\Omega, d\nu_\alpha)$. We write

$$V_{m,\alpha} f(z) = h(z,z)^m D^{m,\alpha} f(z), \text{ for } f \in L^1(\Omega, d\nu_\alpha), \quad m \geq 0, \quad \alpha > -1,$$

and

$$E_{m,\alpha} f(z) = h(z,z)^m D^{m,\alpha} f(z), \quad m \geq 0, \quad \alpha > -1, \quad \text{for} \quad f \in H(\Omega).$$

Thus, if $f \in L_a^{1,\alpha}(\Omega)$, then $E_{m,\alpha} f = V_{m,\alpha} f$.

We begin with a result from [4].

Theorem 1.1. *Let $k, m > \frac{N-1}{p}$ and $\alpha, \beta > \max\{\frac{a(r-1)}{2p} - N, -1\}$. If $0 < p \leq 1$ and $f \in H(\Omega)$ then $\int_\Omega |E_{m,\alpha} f(z)|^p\, d\tau(z) < \infty$ if and only if $\int_\Omega |E_{k,\beta} f(z)|^p\, d\tau(z) < \infty$, where $d\tau(z) = h(z,z)^{-N}\, d\nu(z)$ is the Möbious invariant measure on Ω.*

Recall that the holomorphic Besov space B^p, $1 \leq p \leq \infty$ consists of functions in $H(\Omega)$ such that $E_{N,0} f$ is in $L^p(\Omega, d\tau)$ (see [11] and [13]). For any irreducible bounded symmetric domain Ω we have $\frac{a(r-1)}{2} - N \leq -2$. Thus, a holomorphic function f in Ω belongs to B^1 if and only if $E_{m,\alpha} f$ is in $L^1(\Omega, d\tau)$ for some (any) $m > N-1$ and some (any) $\alpha > -1$. This is also proved in [13], Theorem 4, by a different method.

Definition 1.2. For $0 < p \leq 1$ the holomorphic Besov space $B^p = B^p(\Omega)$ consists of holomorphic functions $f \in H(\Omega)$ such that

$$\|f\|_{B^p} = |f(0)| + ||E_{m,\alpha} f||_{L^p(\Omega, d\tau)} < \infty,$$

for some (any) $m > \frac{N-1}{p}$ and $\alpha > \max\{\frac{a(r-1)}{2p} - N, -1\}$.

For every z in Ω let $E_r(z)$ be the closed Bergman metric ball with center z and radius $r > 0$, i.e.,

$$E_r(z) = \{w : \beta(z, w) \leq r\},$$

where $\beta(\cdot, \cdot)$ is the Bergman metric on Ω.

For a complex measurable function f on B we define

$$M_{\infty,r} = \operatorname{esssup}\{\, |f(w)| : \ w \in E_r(z)\,\}$$

and

$$M_{p,r}f(z) = \left[\frac{1}{\tau(r)} \int_{E_r(z)} |f(w)|^p \, d\tau(w)\right]^{\frac{1}{p}}, \qquad 0 < p < \infty,$$

where $\tau(r) = \tau(E_r(z))$.

For $0 < p, q \leq \infty$, we define $L_r^{p,q}(\Omega, d\tau)$ to be the space of all measurable functions f on Ω for which

$$\|f\|_{L_r^{p,q}(\Omega, d\tau)} = \|M_{p,r}f\|_{L^q(\Omega, d\tau)} < \infty.$$

Since the definition is independent of r, $0 < r < 1$, we will write $L^{p,q}(\Omega, d\tau)$ instead of $L_r^{p,q}(\Omega, d\tau)$ (see [1]).

We let P_α denote the orthogonal projection from $L^2(\Omega, d\nu_\alpha)$ onto $L_a^{2,\alpha}(\Omega) = L^2(\Omega, d\nu_\alpha) \cap H(\Omega)$. It can be shown that (see [9], for instance)

$$P_\alpha f(z) = \int_\Omega \frac{f(w)\, d\nu_\alpha(w)}{h(z, w)^{N+\alpha}}, \quad z \in \Omega, \ f \in L^2(\Omega, d\nu_\alpha).$$

The above formula extends the domain of P_α to $L^1(\Omega, d\nu_\alpha)$. Note that $P_\alpha f = D^{0,\alpha} f$ for $f \in L^1(\Omega, d\nu_\alpha)$.

If $1 \leq p \leq \infty$ and $\alpha > -1$ then $B^p = P_\alpha L^p(\Omega, d\tau)$, see [13]. In [4] it is shown that the analytic Besov space B^p, $0 < p < 1$, can be naturally embedded as a complemented subspace of $L^{1,p}(\Omega, d\tau)$ by a topological embedding

$$E_{m,\alpha} \colon B^p \to L^{1,p}(\Omega, d\tau).$$

It is also shown that $E_{m,\alpha} \circ P_\alpha$ is projection on this embedded copy and that $B^p = P_\alpha L^{1,p}(\Omega, d\tau)$.

More precicely the following theorem is proven.

Theorem 1.3. *Let $0 < p < 1$. Then for any $\alpha > \max\{\frac{a(r-1)}{2p} - N, -1\}$,*

$$P_\alpha \colon L^{1,p}(\Omega, d\tau) \to B^p$$

is a continuous linear map. Moreover if $m > \frac{N-1}{p}$ and $\alpha > \max\{\frac{a(r-1)}{2p} - N, -1\}$ then

$$E_{m,\alpha} \colon B^p \to L^{1,p}(\Omega, d\tau)$$

is a topological embedding.

Now we apply Theorem 1.3. to obtain a result about duality.

THEOREM 1.4. *Let* $0 < p < 1$, $m > \frac{N-1}{p}$ *and* $\alpha = m - N$. *The integral pairing*

$$\langle f, g\rangle_\tau = \int_\Omega E_{m,\alpha} f(z)\, \overline{E_{m,\alpha} g(z)}\, d\tau(z)$$

induces the following duality $(B^p)^\star = B^\infty$.

2. Duality

A linear functional λ on B^p, $0 < p < 1$, is said to be bounded if

$$\|\lambda\| = \sup\{|\lambda(f)| : \|f\|_{B^p} \le 1\} < \infty.$$

The dual space of B^p, denoted $(B^p)^\star$, is then the space of all bounded linear functionals on B^p. In [13] it is shown that each $(L_a^{p,\alpha})^\star$, $0 < p \le 1$, can be identified with B^∞ via volume integral pairing

$$\langle f, g\rangle_\beta = \lim_{r\to 1} \int_\Omega f(rz) g(z)\, d\nu_\beta(z), \quad \text{where} \quad \beta = \frac{N+\alpha}{p} - N.$$

Our Theorem 1.4 shows that $(B^p)^\star$ can also be identified with B^∞, but via a different integral pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\tau$.

Proof of Theorem 1.4. First, assume that g is a function in B^∞. Then

$$\|g\|_{B^\infty} \ge \sup_{z\in\Omega} h(z,z)^m |D^{m,\alpha} g(z)| = \sup_{z\in\Omega} |E_{m,\alpha} g(z)|, \quad \text{(see [13])}.$$

We show that g gives rise to a bounded linear functional on B^p under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\tau$. By Theorem 1.3 if $f \in B^p$ then

$$\|f\|_{B^p} \ge \|E_{m,\alpha} f\|_{L^{1,p}(\Omega, d\tau)} \ge C \|E_{m,\alpha} f\|_{L^1(\Omega, d\tau)}.$$

Thus if $f \in B^p$, then we have

$$|\langle f, g\rangle_\tau| \le \sup_{z\in\Omega} |E_{m,\alpha} g(z)| \|E_{m,\alpha} f\|_{L^1(\Omega,d\tau)} \le C \|f\|_{B^p} \|g\|_{B^\infty}.$$

Conversely, assume that λ is a bounded linear functional on B^p; we show that λ arises from a function in B^∞. Since $E_{m,\alpha}$ is a topological embedding of B^p into $L^{1,p}(\Omega, d\tau)$, $\lambda \circ E_{m,\alpha}^{-1}$ extends to a bounded linear functional on $L^1(\Omega, d\tau)$. Thus there exists a function $\varphi \in L^\infty(\Omega, d\tau)$ such that

$$\lambda \circ E_{m,\alpha}^{-1}(\psi) = \int_\Omega \psi(z) \overline{\varphi(z)}\, d\tau(z), \qquad \psi \in L^1(\Omega, d\tau).$$

When $f \in B^p$, $E_{m,\alpha} f \in L^{1,p}(\Omega, d\tau) \subset L^1(\Omega, d\tau)$. Therefore

$$\lambda(f) = \int_\Omega E_{m,\alpha} f(z) \overline{\varphi(z)}\, d\tau(z) = \int_\Omega V_{m,\alpha} f(z) \overline{\varphi(z)}\, d\tau(z), \quad f \in B^p.$$

As was noted in Introduction $E_{m,\alpha} f = V_{m,\alpha} f$ if $f \in L^{1,\alpha}_a(\Omega)$. Since $B^p \subset B^\infty$, we have that $\int_\Omega |f(\xi)| h(\xi,\xi)^\alpha \, d\nu(\xi) < \infty$. (See the remark following Lemma 7 in [11]). Let $h = P_\alpha \varphi$. Then $h \in B^\infty$ by Theorem 6 ([13])

$$\begin{aligned} E_{m,\alpha} h(z) = V_{m,\alpha} h(z) &= h(z,z)^m D^{m,\alpha} P_\alpha \varphi(z) = h(z,z)^m D^{m,\alpha}(D^{0,\alpha}\varphi)(z) \\ &= h(z,z)^m D^{m,\alpha} \varphi(z) = V_{m,\alpha}\varphi(z). \end{aligned}$$

To finish the proof of Theorem 1.4 it remains to show that

$$\langle V_{m,\alpha} f, V_{m,\alpha}\varphi\rangle_\tau = \langle V^2_{m,\alpha} f, \varphi\rangle_\tau \quad \text{and that } \; c_{m+\alpha} V^2_{m,\alpha} f = c_\alpha V_{m,\alpha} f.$$

This follows easily from Fubini's theorem and the reproducing property of P_α. Note that $\alpha = m - N$. We leave the details to the interested reader. Thus,

$$\lambda(f) = \int_\Omega V_{m,\alpha} f(z) \overline{V_{m,\alpha} g(z)} \, d\tau(z) = \int_\Omega E_{m,\alpha} f(z) \overline{E_{m,\alpha} g(z)} \, d\tau(z)$$

for all $f \in B^p$, where $g = c_{m+\alpha} c_\alpha^{-1} h \in B^\infty$. ■

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(received 04.09.1997.)

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