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WEAK TOPOLOGY IN LOCALLY CONVEX SPACES WITH A FUNDAMENTAL SEQUENCE OF BOUNDED SUBSETS

Stojan Radenović and Zoran Kadelburg

Abstract. The result of Krassowska and Sliwa [6] about weak topologies in (DF)-spaces is extended to variuos other classes of locally convex spaces with a fundamental sequence of bounded subsets.

Among many known classes of locally convex linear topological spaces (*lcs*), the spaces with a fundamental sequence of the family of all bounded subsets or some of its subfamilies (such as precompact or compact subsets), play an important role. The following five classes of spaces

$$\begin{tikzcd}
& (dF) \arrow[d] \\
(DF) \arrow[r] \arrow[d] & D_b \arrow[d] \\
\text{dual-metric} \arrow[r] & (df)
\end{tikzcd}$$

have been studied from various aspects (inheritance properties, three-space-problem, etc). Let us recall the definitions.

An *lcs* (E,t) is called *countably-quasibarrelled* (resp. *σ-quasibarrelled*; *sequentially quasibarrelled*) if each $\beta(E',E)$-bounded subset which is a countable union of t-equicontinuous subsets (resp. $\beta(E',E)$-bounded sequence; $\beta(E',E)$-convergent sequence) is a t-equicontinuous subset. If, besides that, (E,t) possesses a fundamental sequence of t-bounded subsets, it is said to be of the type (DF) (resp. *dual-metric*; (df)). A barrel T of the space (E,t) is a *p-barrel* (resp. *b-barrel*) if

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its intersection with each t-precompact (resp. t-bounded) absolutely convex subset is a relative t-neighbourhood of the origin. The space (E,t) is *p-barrelled* (resp. *b-barrelled*) if each p-barrel (resp. b-barrel) in it is a t-neighbourhood of the origin. We say that the space (E,t) is of the type D_b (or $g(DF)$ by some authors) if it is b-barrelled with a fundamental sequence of t-bounded subsets. The *lcs* (E,t) is a (dF)*-space* if it is p-reflexive (i.e. p-barrelled and p-complete [1]) with a fundamental sequence of compact subsets. Remark that the space (E,t) is p-barrelled if and only if each E'_p-precompact subset is t-equicontinuous [2].

It is well known that the main fact that led A. Grothendieck to introduce the class of (DF)-spaces in [3] was that strong duals of Fréchet spaces are of that type. An interesting question may be the following: if (E,t) is a Hausdorff *lcs*, do the spaces $(E,\sigma(E,E'))$ and $(E',\sigma(E',E))$ with the weak topologies belong to some of the mentioned five classes of spaces? The first answer to such kind of question was given by S. O. Iyahen in [4] when he proved that the space $(c_0,\sigma(c_0,l^1))$ is not of the type (DF). In [8] it was proved that neither of the spaces $(E,\sigma(E,E'))$ and $(E',\sigma(E',E))$ is of the type (DF) if E is a Banach space of infinite dimension. D. Krassowska and W. Sliwa showed in [6] the most general result: for each dual pair $\langle E,E'\rangle$, the space $(E,\sigma(E,E'))$ (resp. $(E',\sigma(E',E))$) is of the type (DF) if and only if $\dim E<\infty$.

In this short note we show that the result of D. Krassowska and W. Sliwa is true also for the other four classes of spaces from the previous diagram.

THEOREM. *If (E,t) is a Hausdorff lcs, then the space $(E,\sigma(E,E'))$ (resp. $(E',\sigma(E',E))$) is of the type (df) (D_b; dual-metric; (dF)) if and only if* $\dim E<\infty$.

Proof. Taking into account the diagram, it is enough to consider the case of (*df*)-spaces. The 'only if' part is nontrivial. So, let $(E,\sigma(E,E'))$ be a (*df*)-space. Then each $E'_p=\beta(E',E)$-precompact subset is $\sigma(E,E')$-equicontinuous, and so $(E,\sigma(E,E'))$ is p-barrelled and also b-barrelled space. As the space $(E,\sigma(E,E'))$ possesses a fundamental sequence of bounded subsets, it is of the type D_b, and so by [7, Cor. 3.2.2] the space $(E'^*,\sigma(E'^*,E'))$, as its completion, is a D_b-space, too. Hence, the *lcs* $(E'^*,\sigma(E'^*,E'))$ has a fundamental sequence of bounded subsets, wherefrom it follows that the space E' equipped with the finest locally convex topology $\tau(E',E'^*)=\beta(E',E'^*)$ is metrizable. But, the only case when this is possible ([9; Ex. II,7]) is when E', and so also E, is of finite dimension. ∎

REMARK. As for each (*df*)-space (E,t), $\beta(E',E)$-precompact subsets are t-equicontinuous, the proof of the theorem could be derived in the similar way as in [6]. But we proceeded in the other way—using the fact that for each infinite-dimensional vector space E, the *lcs* $(E^*,\sigma(E^*,E)=\beta(E^*,E))$ has no fundamental sequence of bounded subsets (E^* is the algebraic dual).

CORROLARY. *If (E,t) is a metrizable and barrelled lcs of infinite dimension, then $(E',\sigma(E',E))$ is not a sequentially quasibarrelled space.*

Proof. The space $(E',\sigma(E',E))$ has a fundamental sequence of bounded subsets, but by the Theorem it is not of the type (*df*). ∎

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Faculty of Sciences, Radoja Domanovića 12, 34000 Kragujevac, Yugoslavia

Faculty of Mathematics, Studentski trg 16, P.O.Box 550, 11000 Beograd, Yugoslavia

e-mail: kadelbur@matf.bg.ac.yu