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SOME PROPERTIES OF HAUSDORFF MEASURE OF NONCOMPACTNESS ON LOCALLY BOUNDED TOPOLOGICAL VECTOR SPACES

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Abstract. In this note we present some properties of Hausdorff measure of noncompactness on locally bounded topological vector space.

Introduction

Let X be a Hausdorff topological vector space. A set $A \subseteq X$ is *bounded* if for each neighborhood of zero U there is a scalar α such that $A \subseteq \alpha U$. The space X is *locally bounded* if it contains a bounded neighborhood of zero.

Examples. Normed spaces, ℓ^p $(p > 0)$ and L^p $(p > 0)$ are locally bounded spaces.

Proposition 0.1 (see Rolewicz [4], Wilansky [5]) *Hausdorff topological vector space X is locally bounded if and only if there exist a real number p $(0 < p \leq 1)$ and a function $|\cdot\|: X \to [0, +\infty)$ such that:*

1) $|x\| = 0$ if and only if $x = 0$;

2) $|\alpha x\| = |\alpha|^p |x\|$;

3) $|x + y\| \leq |x\| + |y\|$;

4) function $d' : X^2 \to [0, \infty)$ defined by $d'(x, y) = |x - y\|$ is a metric on X;

5) original topology on X is equivalent with the topology of metric space (X, d).

The theory of measures of noncompactness has many applications in Functional analaysis and Operator theory (see [1],[3]). If Q is a bounded subset of a metric space X, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

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In the proofs of our results we need the following well known properties of Hausdorff measure of noncompactness.

Proposition 0.2 (see Banás and Goebel [1], Rakočević [3]) *If Q, Q_1 and Q_2 are bounded subsets of a metric spaces (X,d) then*

1) $\chi(Q)=0$ if and only if Q is a totally bounded set;

2) $Q_1 \subseteq Q_2$ implies $\chi(Q_1) \leq \chi(Q_2)$;

3) $\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$.

In this paper we investigate some properties of measure of noncompactness on arbitrary locally bounded Hausdorff topological vector space. Corresponding results for ℓ^p $(0<p)$ spaces were obtained by I. Jovanović and V. Rakočević [2].

Results

Let (X,d) be a metric space, $x \in X$ and $r>0$. By $B(x,r)$ we denote $\{y \in X : d(x,y) \leq r\}$.

Proposition 1. *If Q, Q_1 and Q_2 are bounded subsets of an arbitrary metric linear space X and $x \in X$, then:*

1) $\chi(Q_1+Q_2) \leq \chi(Q_1)+\chi(Q_2)$;

2) $\chi(x+Q)=\chi(Q)$.

Proof. Let $\delta>0$ be an arbitrary positive real number. From

$$Q_1+Q_2 \subseteq \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}[B(x_i,\chi(Q_1)+\delta)+B(y_j,\chi(Q_2)+\delta)]$$

it follows that $z \in Q_1+Q_2$ implies that there exist $z_1 \in Q_1$ and $z_2 \in Q_2$ such that $z=z_1+z_2$, $d(z_1,x_i)<\chi(Q_1)+\delta$ and $d(z_2,y_j)<\chi(Q_2)+\delta$, for some i,j $(1 \leq i \leq n;\ 1 \leq j \leq m)$. Since

$$\begin{aligned} d(z,x_i+y_j) &= d(z_1+z_2,x_i+y_j) = d(z_1-x_i,y_j-z_2) \\ &\leq d(z_1-x_i,0)+d(z_2-y_j,0) = d(z_1,x_i)+d(z_2,y_j) \\ &\leq \chi(Q_1)+\chi(Q_2)+2\delta, \end{aligned}$$

when $\delta \to 0^+$, we have $\chi(Q_1+Q_2) \leq \chi(Q_1)+\chi(Q_2)$.

From 1) we have $\chi(x+Q) \leq \chi(\{x\})+\chi(Q)=\chi(Q)$, which implies $\chi(Q)=\chi(-x+x+Q) \leq \chi(x+Q)$. So $\chi(x+Q)=\chi(Q)$. ∎

For proofs of the next proposition we need the following lemma.

Lemma. *If X is a locally bounded Hausdorff topological vector space, and $r,s>0$ then $B(0,rs)=r^{1/p}B(0,s)$, for some p, $(0<p\leq 1)$.*

Proof. From $x \in X$ and $|x\| \leq s$ it follows $r|x\| \leq rs$ which implies $|r^{\frac{1}{p}}x\| \leq rs$, for some p, $(0<p\leq 1)$. ∎

PROPOSITION 2. *If X is a locally bounded Hausdorff topological vector space, $Q \subseteq X$ its bounded subset and α arbitrary scalar then*

$$\chi(\alpha Q) = |\alpha|^p \chi(Q)$$

for some p, $0 < p \leq 1$.

Proof. Let $\alpha \neq 0$. From $Q \subseteq \bigcup_{i=1}^n \{x_i + B(0, \chi(Q))\}$ it follows

$$\alpha Q \subseteq \bigcup_{i=1}^n \{\alpha x_i + \alpha B(0, \chi(Q))\} = \bigcup_{i=1}^n \{\alpha x_i + B(0, |\alpha|^p \chi(Q))\}$$

which implies $\chi(\alpha Q) \leq |\alpha|^p \chi(Q)$. Since $\alpha^{-1}\alpha Q = Q$, we have $\chi(Q) \leq |\alpha|^{-p}\chi(\alpha Q)$. So $|\alpha|^p \chi(Q) \leq \chi(\alpha Q)$. It follows $\chi(\alpha Q) = |\alpha|^p \chi(Q)$. ∎

PROPOSITION 3. *If X is an infinite-dimensional locally bounded Hausdorff topological vector space, and $B(0,1)$ is its closed unit ball then*

$$\chi(B(0,1)) = 1.$$

Proof. Let us remark that clearly $\chi(K) \leq 1$. If $\chi(K) = s < 1$ then we can find $\varepsilon > 0$ such that $s + \varepsilon < 1$. Now, there is an $(s+\varepsilon)$-net of $B(0,1)$, say $x_1, \dots, x_n$. Hence

$$B(0,1) \subseteq \bigcup_{i=1}^n \{x_i + B(0, s+\varepsilon)\} = \bigcup_{i=1}^n \{x_i + (s+\varepsilon)^{\frac{1}{p}} B(0,1)\},$$

and

$$\begin{aligned} s = \chi(B(0,1)) &\leq \max_{1 \leq i \leq n} \chi(x_i + (s+\varepsilon)^{\frac{1}{p}} B(0,1)) \\ &= \chi((s+\varepsilon)^{\frac{1}{p}} B(0,1)) = (s+\varepsilon)\chi(B(0,1)) = (s+\varepsilon)s. \end{aligned}$$

From $s + \varepsilon < 1$, it follows that $s = \chi(B(0,1)) = 0$ i.e. $B(0,1)$ is totally bounded, which implies that X is a finite-dimensional space. Hence we get a contradiction, and the proof is complete. ∎

COROLLARY. *If X is an infinite-dimensional locally bounded Hausdorff topological vector space, $x_0 \in X$ and $r > 0$ then $\chi(B(x_0, r)) = r$.*

Proof. $\chi(B(x_0,r)) = \chi(x_0 + B(0,r)) = \chi(B(0,r)) = \chi(r^{\frac{1}{p}} B(0,1)) = r\chi(B(0,1)) = r$. ∎

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