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ON OPERATORS IN BOCHNER SPACES

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Abstract. Estimates for the measure of noncompactness of bounded subsets of spaces of (Bochner-) integrable functions are obtained, a new class of condensing operators is discussed, and the solvability of a certain operator equation in a Hilbert space is proved.

In this paper we discuss a new class of condensing operators, and we prove the solvability of a certain operator equation. An extension of some results from [8] is obtained.

Let us recall some definitions. The *measure of noncompactness* $\beta(U) = \beta_E(U)$ [1] of a bounded set U in a normed space E is defined as the supremum of all numbers $r > 0$ such that there exists a sequence $\{u_n\}$ in U with $\|u_n - u_m\| \geq r$ for every $n \neq m$. Given two Banach spaces G and E, a continuous operator $S\colon G \to E$ is called *β-condensing* if

$$\beta_E(SU) < \beta_G(U)$$

for every bounded $U \subset G$ with noncompact closure.

There exists a large amount of literature devoted to measure of noncompactness and condensing operators (see, for example, [1,2,4, 6–8]).

Let Ω be a domain in R^n. Let E be a *regular* space of μ-measurable functions on a domain Ω; here regularity means that every element in E has an absolutely continuous norm. Let P_D denote the operator of multiplication by the characteristic function χ_D of a measurable subset $D \subset \Omega$, i.e. $P_D u = \chi_D u$. For bounded $U \subset E$ put

$$\nu(U) = \nu_E(U) = \overline{\lim_{\mu(D)\to 0}} \sup_{u \in U} \|P_D u\|_E,$$

for $U \subset E$.

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The measure ν has been studied in [2,6]. In particular, it was shown in [6] (see also [1]) that

$$\beta(U) = 2^{1/2}\nu(U) \tag{1}$$

for every μ-compact (i.e., compact in measure) subset U of a separable Hilbert space E.

Let Δ be a bounded interval on the real axis and E some Banach space. For $1 \le p < \infty$, we denote by $L_p(0,T;E)$ the set of all Bochner-measurable functions with the property that the function $t \mapsto \|u(t)\|_E$ belongs to $L_p(0,T)$.

For any partition $\Delta = D_1 \cup \cdots \cup D_l$ of Δ into Lebesgue-measurable disjoint subsets D_i, we denote by $\tilde{V}$ the set of all functions

$$\tilde{u}(t) = \sum_{i=1}^{l} b_i \chi_{D_i}(t),$$

where χ_{D_i} is the characteristic function of D_i as above, and b_i are elements from E $(1 \le i \le l)$.

Lemma 1. *Let $\widetilde{U} \subseteq \widetilde{V}$ be bounded in $L_p(\Delta;E)$. For arbitrary $t_0 \in \Delta$, let*

$$\widetilde{U}(t_0) \stackrel{\text{def}}{=} \{\widetilde{u}(t_0) : \widetilde{u} \in \widetilde{U}\}.$$

Then the function $\beta_E(\widetilde{U}(t))$ is simple, i.e.

$$\beta_E(\widetilde{U}(t)) = \sum_{i=1}^{l} a_i \chi_{D_i}(t),$$

and

$$\beta_{L_p(\Delta;E)}(\widetilde{U}) \le \left(\int_\Delta \beta_E^p(\widetilde{U}(t))\,dt \right)^{1/p}.$$

Proof. The proof of this assertion is analogous to the proof of Lemma 2.1 from [8]. ■

We denote by U some subset of $L_p(\Delta;E)$ which allows an ϵ-approximation, for every $\epsilon > 0$, through a set

$$\widetilde{U}_\epsilon \stackrel{\text{def}}{=} \{\widetilde{u} : \widetilde{u}(t) = \sum_{i=1}^{l_\epsilon} b_i \chi_{D_i}(t) \quad (b_i \in E)\}.$$

More precisely, we require that

$$\rho_E(U(t), \widetilde{U}_\epsilon(t)) \le k_1 \epsilon \tag{2}$$

for almost all $t \in \Delta$, where ρ_E denotes the Hausdorff distance in E, the constant $k_1 > 0$ is independent of ϵ, but the integer $l_\epsilon < \infty$ may depend on ϵ.

THEOREM 1. *Let U be a bounded set in $L_p(\Delta; E)$ which allows an ϵ-approximation (2) for every ϵ. Then*

$$\beta_{L_p(\Delta;E)}(U) \le \|\beta_E(U)\|_{L_p(\Delta)}.$$

Proof. The proof of this assertion is analogous to the proof of Theorem 2.1 from [8]. ∎

Let H be some Hilbert space. As usual, we identify H with its conjugate space H^*. Let $W^{1,2}(\Delta; H) = W^{1,2}(b, d; H)$ ($\Delta = (b, d)$) for some $-\infty < b < d < \infty$ denote the space of all functions $u\colon \Delta \to H$ such that both u and u'_t belong to $L_2(\Delta; H)$, equipped with the norm

$$\|u\|_{W^{1,2}(\Delta;H)} = \|u\|_{L_2(\Delta;H)} + \|u'_t\|_{L_2(\Delta;H)}.$$

By Lemma 1.11 from [3], $W^{1,2}(\Delta; H)$ is embedded in $C(\overline{\Delta}; H)$, i.e.,

$$\|u\|_{C(\overline{\Delta};H)} \le c\|u\|_{W^{1,2}(\Delta;H)}. \tag{3}$$

Let $W_0^{1,2}(b, d; H)$ be the subspace of all functions $u \in W^{1,2}(b, d; H)$ such that $u(b) = u(d) = \mathbf{0}$ ($\mathbf{0}$ is zero of H). In $W_0^{1,2}(b, d; H)$ we have

$$\|u\|_{L_2(b,d;H)} \le k\left(\int_b^d \|u'(t)\|_H^2\, dt\right)^{1/2} = k\|u'_t\|_{L_2(b,d;H)}. \tag{4}$$

Finally, for $u, v \in W_0^{1,2}(b, d; H)$ we put

$$\|u\|_{1,2,0} = \left(\int_b^d \|u'_t\|_H^2\, dt\right)^{1/2}.$$

Some notations are in order.

Let $\Delta \subseteq (b, d)$ be an interval, $\overline{\Delta}$ its closure, $|\Delta|$ its length, Δ_δ the δ-neighbourhood of Δ, u_Δ an approximation of a function u on Δ, $u_\delta \in W_0^{1,2}(\Delta_\delta; H)$ an extension of $u \in W^{1,2}(\Delta; H)$ preserving the norm in $C(\overline{\Delta}; H)$ for $\delta = |\Delta|/2$, and U_δ the set of all extensions u_δ of functions $u \in U \subset W^{1,2}(\Delta; H)$.

Let $L_2(\Omega)$ denote the Lebesgue space and $W^{1,2}(\Omega)$ the Sobolev space. We shall now consider two particular cases of H, namely $H_1 = L_2(\Omega)$ and $H_2 = W^{1,2}(\Omega)$, here Ω is a domain in R^n of finite measure but, in general, with irregular boundary. In both cases the space $W^{1,2}(0, T; H_i)$ ($i = 1, 2$) consists of all functions $(t, x) \mapsto u(t, x)$ such that $u(t, .)$, $u'_t(t, .) \in H_i$ for each $t \in \Delta$.

LEMMA 2. *Let $f \in L_1(\Delta; H_2)$, and let $\Psi\colon H_i \mapsto H_i$ ($i = 1, 2$) be an operator satisfying the inequality*

$$\|\Psi(\phi)\|_{H_i} \le c_1 + \sum_{j=1}^{J} \check{c}_j \|\phi\|_{H_i}^{\alpha_{i,j}} \tag{5}$$

for all $\phi \in H_i$, where $c_1 \geq 0$, $\check{c}_j \geq 0$, and $\alpha_{i,j} > 1$ are real constants which may depended on H_i. Then there exist operators $F_{\Delta,i}\colon W^{1,2}(\Delta; H_i) \to W^{1,2}(\Delta; H_i)$, such that the equality

$$\int_\Delta \langle (F_{\Delta,i}u)'_t(t), v'_t(t)\rangle_{H_i}\, dt = \int_\Delta \langle f(t) - \Psi(u(t)), v(t)\rangle_{H_i}\, dt$$

is true for arbitrary functions $v \in W_0^{1,2}(\Delta; H_i)$. Moreover,

$$\|F_{\Delta,i}u\|_{C(\overline{\Delta}; H_i)} \leq c_0\|f\|_{L_1(\Delta;H_2)} + \check{c}_1|\Delta|^{1/2} + |\Delta|^{1/2}\sum_{j=1}^{J} \widetilde{c}_j \|u\|^{\alpha_{i,j}}_{C(\overline{\Delta};H_i)}, \tag{6}$$

$$\|F_{\Delta,i}u\|_{W^{1,2}(\Delta;H_i)} \leq c_0\|f\|_{L_1(\Delta;H_2)} + \check{c}_1|\Delta|^{1/2} + |\Delta|^{1/2}\sum_{j=1}^{J} \widetilde{\widetilde{c}}_j \|u\|^{\alpha_{i,j}}_{W^{1,2}(\Delta;H_i)}, \tag{7}$$

and

$$\beta_{W^{1,2}(\Delta;H_i)}(F_{\Delta,i}U) \leq c_3\beta_{L_2(\Delta_\delta;H_i)}(\Psi(U_\delta))(U \subset W^{1,2}(\Delta; H_i)), \tag{8}$$

where the constants c_0, $\check{c}_1$, $\widetilde{c}_j$, $\widetilde{\widetilde{c}}_j$, and c_3 are independent of u, U, Δ and δ.

Proof. Let u_δ, $v_\delta \in W_0^{1,2}(\Delta_\delta; H_i)$. The estimates

$$\begin{aligned}\|\Psi(u_\delta)\|_{L_2(\Delta_\delta;H_i)} &\leq \left\| c_1 + \sum_{j=1}^{J} \check{c}_j\, \|u_\delta\|^{\alpha_{i,j}}_{H_i} \right\|_{L_2(\Delta_\delta;H_i)} \leq \\ &\leq (c_1 + \sum_{j=1}^{J} \check{c}_j\, \|u\|^{\alpha_{i,j}}_{C(\overline{\Delta};H_i)})(2|\Delta|)^{1/2} \leq (c_1 + c^{\alpha_{i,j}} \sum_{j=1}^{J} \check{c}_j\, \|u\|^{\alpha_{i,j}}_{W^{1,2}(\Delta;H_i)})\,(2|\Delta|)^{1/2}\end{aligned} \tag{9}$$

can easily be deduced from assumptions (5) on Ψ and the embedding (3). Putting $f_\delta(t) = P_\Delta f(t)$ we have

$$\begin{aligned}\int_{\Delta_\delta} |\langle f_\delta(t), v_\delta(t)\rangle_{H_i}|\, dt &= \int_\Delta |\langle f(t), v_\delta(t)\rangle_{H_i}|\, dt \leq \|v_\delta\|_{C(\overline{\Delta};H_i)}\|f\|_{L_1(\Delta;H_2)} \\ &\leq c\|v_\delta\|_{W^{1,2}(\Delta;H_i)}\|f\|_{L_1(\Delta;H_2)} \leq c(k+1)\|v_\delta\|_{W_0^{1,2}(\Delta_\delta;H_i)}\|f\|_{L_1(\Delta;H_2)}.\end{aligned}$$

This shows that the linear functional

$$R\,(v_\delta) = \int_{\Delta_\delta} \langle f_\delta(t) - \Psi(u(t)), v_\delta(t)\rangle_{H_i}\, dt$$

is bounded in module for all $v_\delta \in W_0^{1,2}(\Delta_\delta; H_i)$. By the Riesz representation theorem, there exists a bounded (generally speaking, nonlinear) operator $F_{\delta,i}\colon W_0^{1,2}(\Delta_\delta; H_i) \to W_0^{1,2}(\Delta_\delta; H_i)$ such that

$$\begin{aligned}\langle F_{\delta,i}u_\delta, v_\delta\rangle_{1,2,0} &= \int_{\Delta_\delta} \langle (F_{\delta,i}u_\delta)'_t(t), (v_\delta)'_t(t)\rangle_{H_i}\, dt \\ &= \int_{\Delta_\delta} \langle f_\delta(t) - \Psi(u_\delta(t)), v_\delta(t)\rangle_{H_i}\, dt \\ &\leq (c(k+1)\|f\|_{L_1(\Delta;H_2)} + k\|\Psi(u_\delta)\|_{L_2(\Delta_\delta;H_i)})\|v_\delta\|_{W_0^{1,2}(\Delta_\delta;H_i)}.\end{aligned}$$

Putting in the last equality $v_\delta = F_{\delta,i}u_\delta$, we conclude that

$$\|F_{\delta,i}u_\delta\|_{W_0^{1,2}(\Delta_\delta;H_i)} \le c(k+1)\|f(t)\|_{L_1(\Delta;H_2)} + k\|\Psi(u_\delta)\|_{L_2(\Delta_\delta;H_i)}. \tag{10}$$

We define an operator $(F_{\Delta,i}u)$ as approximation of $F_{\delta,i}u_\delta$ on Δ. Taking into consideration (3), (4), (9), and (10), we obtain then (6) and (7), since

$$\begin{aligned}
\|F_{\Delta,i}u\|_{C(\overline{\Delta};H_i)} &\le c\|F_{\Delta,i}u\|_{W^{1,2}(\Delta;H_i)} \le c(k+1)\|F_{\delta,i}u_\delta\|_{W_0^{1,2}(\Delta_\delta;H_i)} \\
&\le c^2(k+1)^2\|f\|_{L_1(\Delta;H_2)} + ck(k+1)\|\Psi(u_\delta)\|_{L_2(\Delta_\delta;H_i)} \\
&\le c^2(k+1)^2\|f\|_{L_1(\Delta;H_2)} + ck(k+1)(c_1 + \sum_{j=1}^{J} \breve{c}_j \|u\|^{\alpha_{i,j}}_{C(\overline{\Delta};H_i)})(2|\Delta|)^{1/2} \\
&\le c^2(k+1)^2\|f\|_{L_1(\Delta;H_2)} + ck(k+1)(c_1 + c^{\alpha_{i,j}} \sum_{j=1}^{J} \breve{c}_j \|u\|^{\alpha_{i,j}}_{W^{1,2}(\Delta;H_i)})\,(2|\Delta|)^{1/2}.
\end{aligned}$$

The inequality

$$\|F_{\delta,i}u_\delta - F_{\delta,i}v_\delta\|_{W_0^{1,2}(\Delta_\delta;H_i)} \le k\|\Psi(u_\delta) - \Psi(v_\delta)\|_{L_2(\Delta_\delta;H_i)}$$

for arbitrary $u_\delta, v_\delta \in W_0^{1,2}(\Delta_\delta;H_i)$ is proved analogously to (10). Therefore, by the definition of β and (4) we have for any subset U of $W^{1,2}(\Delta;H_i)$

$$\beta_{W^{1,2}(\Delta;H_i)}(F_{\Delta,i}U) \le (k+1)\beta_{W_0^{1,2}(\Delta_\delta;H_i)}(F_{\delta,i}U_\delta) \le k(k+1)\beta_{L_2(\Delta_\delta;H_i)}(\Psi U_\delta),$$

as claimed. ■

Corollary 1. *Let $\Delta \subseteq \Delta_1 \subseteq (b,d)$, $\widetilde{u}$ be some fixed function from $W^{1,2}(\Delta_1;H_i)$, $\widetilde{u}_\Delta$ its approximation on Δ, and $\widetilde{U}$ the set of all functions u from $W^{1,2}(\Delta_1;H_i)$ which coincide on Δ with $\widetilde{u}$. Then for arbitrary $u \in \widetilde{U}$ the approximation $F_{\Delta_1,i}u$ differs on interval Δ from $F_{\Delta,i}\widetilde{u}_\Delta$ only by a constant depending on u.*

Corollary 2. *Let the assumptions of Corollary 1 be satisfied. Suppose that, for every ϕ, $\phi_1 \in H_1$, we have*

$$\Psi(\phi + \phi_1) = \Psi(\phi) \tag{11}$$

if and only if $\phi_1 = \mathbf{0}$. Let

$$\int_\Delta \langle (\widetilde{u})'_t(t), v'(t)\rangle_{H_1}\, dt = \int_\Delta \langle f(t) - \Psi(\widetilde{u}(t)), v(t)\rangle_{H_1}\, dt$$

and for some $u \in \widetilde{U}$ and $\phi \in H_1$

$$\int_\Delta \langle (F_{\Delta_1,2}(u+\phi))'_t(t), v'(t)\rangle_{H_1}\, dt = \int_\Delta \langle f(t) - \Psi(u(t)+\phi), v(t)\rangle_{H_1}\, dt$$

for all $v \in W_0^{1,2}(\Delta;H_1)$. Then $\phi = \mathbf{0}$.

LEMMA 3. *Let $f(t,.) \in H_2$ for all $t \in \Delta$, and let $\Psi : H_i \to H_i$ be an operator which satisfies (7) $(i = 1,2)$. Let $F_{\Delta,i}\colon W^{1,2}(\Delta; H_i) \to W^{1,2}(\Delta; H_i)$ be the corresponding operators, defined in Lemma 2. Then for all $u \in W^{1,2}(\Delta; H_2)$ we have $F_{\Delta,1}(u(t,x)) \equiv F_{\Delta,2}(u(t,x))$ on Δ.*

Proof. The proof of this assertion is analogous to the proof of Lemma 3.2 from [8]. ■

We shall show now that in a particular case of Lemma 3 we are led to condensing operators. Let us denote by $\overline{B(0,r)}$ the closure of the set $\phi \in H_2$, with $\|\phi\|_{H_2} \le r$ in the norm H_1.

THEOREM 2. *Let the assumption (5) be satisfied. Given r_0 assume that, for each $r \le r_0$ and all functions ϕ, ϕ_1 from $\overline{B(0,r)}$ for the operator $\Psi\colon H_1 \to H_1$ the next inequalities are true:*

$$|\Psi(\phi)| \le |\phi_0 + \widetilde{c}\|\phi\|_{H_1}^{\alpha-1}\phi|, \tag{12}$$

$$\|\Psi(\phi) - \Psi(\phi_1)\|_{H_1} \le \widetilde{k}(r)\|\phi - \phi_1\|_{H_1}. \tag{13}$$

are true. Then there exists $r > 0$ sufficiently small such that, for every bounded set $U \subset W^{1,2}(\Delta; H_1)$ with values in $\overline{B(0,r)}$, the inequality

$$\beta_{W^{1,2}(\Delta;H_1)}(F_{\Delta,1}U) \le a\beta_{W^{1,2}(\Delta;H_1)}(U), \tag{14}$$

holds with some $0 < a < 1$, i.e. $F_{\Delta,1}$ is a condensing map.

Proof. Let U be any bounded subset $W^{1,2}(\Delta; H_1)$; in particular, U satisfies the inequality (2). From (13) it follows that the set $\Psi(U)$ satisfies the inequality (2), too. By [5, Theorem 4.8.4], every subset of $W^{1,2}(\Omega)$ is μ-compact (μ being the Lebesgue measure). Consequently, by our assumptions on Ψ and our choice of the set U, the set $\Psi(U(t_0)) = \{\Psi(u(t_0,.)) : u \in U\}$ is also μ-compact for fixed $t_0 \in \Delta$. Thus from (1), (2), (8), (12) and Theorem 1 we obtain

$$\begin{aligned}
\beta_{W^{1,2}(\Delta;H_1)}(F_{\Delta,1}U) &\le c_3\beta_{L_2(\Delta_\delta;H_1)}(\Psi(U_\delta)) \le c_3\left(\int_{\Delta_\delta} \beta_{H_1}^2(\Psi(U_\delta(t)))\,dt\right)^{1/2} \\
&\le \sqrt{2}c_3\left(\int_{\Delta_\delta} \nu_{H_1}^2\left\{\phi_0(x) + \widetilde{c}\|u_\delta(t,x)\|_{H_1}^{\alpha_1-1}u_\delta(t,x) : u_\delta \in U_\delta\right\}\,dt\right)^{1/2} \\
&\le c_3 r^{\alpha_1-1}\widetilde{c}\left(\int_{\Delta_\delta} \beta_{H_1}^2(U_\delta(t))\,dt\right)^{1/2} \le cc_3 r^{\alpha_1-1}\widetilde{c}(2|\Delta|)^{1/2}\beta_{W^{1,2}(\Delta;H_1)}(U).
\end{aligned}$$

Taking r sufficiently small we arrive at the inequality (14). ■

As example of an application of our results we study now the existence of solutions $u \in W_0^{1,2}(0,T;H_1)$ of the ordinary operator differential equation

$$-u_{tt}''(t) + \Psi(u(t)) = f(t), \quad t \in (0,T), \quad u(0) = u(T) = 0, \tag{15}$$

where $0 < T < \infty$ and $f \in L_1(0,T;H_2)$ are given. We say that $\widetilde{u} \in W_0^{1,2}(0,T;H_1)$ is a *generalized solution* of (15) if

$$\int_0^T \langle \widetilde{u}'(t), v'(t)\rangle_{H_1}\, dt + \int_0^T \langle \Psi(\widetilde{u}(t)), v(t)\rangle_{H_1}\, dt = \int_0^T \langle f(t), v(t)\rangle_{H_1}\, dt \tag{16}$$

for any $v \in W_0^{1,2}(0,T;H_1)$.

THEOREM 3. *Let the assumptions (5), (11), (12) and (13) be satisfied. Then the equation (15) has a generalized solution in the space* $W_0^{1,2}(0,T;H_1)$ *for each* $f \in L_1(0,T;H_2)$.

Proof. Let $F_{\Delta,i}u$ be the operators defined in Lemma 2 for H_i $(i = 1,2)$. By (7) we have

$$\|F_{\Delta,i}u\|_{W^{1,2}(\Delta;H_i)} \le c_0\|f\|_{L_1(\Delta;H_2)} + \check{c}_1|\Delta|^{1/2} + |\Delta|^{1/2}\sum_{j=1}^{J} \widetilde{\widetilde{c}}_j \|u\|^{\alpha_{i,j}}_{W^{1,2}(\Delta;H_i)} \le r_0$$

if $\|u\|_{W^{1,2}(\Delta;H_i)} \le r_0$ for some $0 < r_0 < 1$, and $|\Delta| \le \tau$ for τ sufficiently small. We take as Δ the interval $(0,\tau)$ and consider the Hilbert space $W^{1,2}(\Delta;H_1)$ of functions satisfying $u(0,x) = 0$ for all $x \in \Omega$. From Lemma 3 it follows that $F_{\Delta,1}u(t,x) \equiv F_{\Delta,2}u(t,x)$ if $u \in W^{1,2}(\Delta;H_2)$. By (6) there exist $\tau_1 \le \tau$ and $r > 0$ sufficiently small such that for $\Delta = (0,\tau_1)$ we have

$$\|F_{\Delta,i}u\|_{C(\overline{\Delta};H_i)} \le c_0\|f\|_{L_1(\Delta;H_2)} + \check{c}_1|\Delta|^{1/2} + |\Delta|^{1/2}\sum_{j=1}^{J} \widetilde{c}_j \|u\|^{\alpha_{i,j}}_{C(\overline{\Delta};H_i)} \le r$$

if $\|u\|_{C(\overline{\Delta};H_2)} \le r$. By Theorem 2 we may choose $r > 0$ such that the inequality

$$\beta_{W^{1,2}(\Delta;H_1)}(F_{\Delta,1}U) < \beta_{W^{1,2}(\Delta;H_1)}(U)$$

is true for the operator $F_{\Delta,1}$ and for every bounded not precompact subset $U \subset W^{1,2}(\Delta;H_1)$ with values in $\overline{B(0,r)}$. This shows that $F_{\Delta,1}$ is a condensing map with respect to the measure of noncompactness β. Moreover, the set of all functions $u \in W^{1,2}(\Delta;\overline{B(0,r)})$, with $\|u\|_{W^{1,2}(\Delta;H_1)} \le r_0$ is closed, convex, nonempty and invariant with respect to $F_{\Delta,1}$. Thus, by an analogue to Schauder's fixed point principle for β-condensing maps [1], the operator $F_{\Delta_1,1}$ has a fixed point $u_1 \in W^{1,2}(\Delta;H_1)$. By Corollary 2, applied to $\Delta_1 = (\tau_1/2, 3/2\tau_1)$ the set $\tilde{U}$ of all functions $u \in W^{1,2}(\Delta_1;\overline{B(0,r)})$, $\|u\|_{W^{1,2}(\Delta_1;H_1)} \le r_0$, which coincide on $\Delta \cap \Delta_1$ with $u_1 + \phi$ for some $\phi \in H_2$ depending on u, is invariant with respect to the operator $F_{\Delta_1,1}$. Consequently, the operator $F_{\Delta_1,1}$ has also a fixed point $u_2 \in \tilde{U}$ which, by Corollary 2, coincides with u_1 on $\Delta \cap \Delta_1$. Now let

$$\tilde{u}(t,x) = \begin{cases} u_1(t,x), & \text{by } t \in (0,\tau), \\ u_2(t,x), & \text{by } t \in (\tau, 3\tau/2). \end{cases}$$

Then the equality (16) is true for all $v \in W_0^{1,2}(\Delta;H_1)$ with $\operatorname{supp} v \subseteq (0,3\tau/2)$, since every function $v \in W_0^{1,2}(\Delta;H_1)$ with $\operatorname{supp} v \subseteq (0,3\tau/2)$ can be decomposed into a sum $v_1 + v_2$, where $v_1 \in W_0^{1,2}(\Delta; H_1)$ with $\operatorname{supp} v_1 \subseteq (0,\tau)$, and $v_2 \in W_0^{1,2}(\Delta; H_1)$ with $\operatorname{supp} v_2 \subseteq (\tau/2, 3\tau/2)$. Applying this procedure a finite number of times, we obtain the solution on the whole $(0,T)$. ∎

Theorem 3 is illustrated in the next example.

EXAMPLE. Let $H_1 = L_2(\Omega)$. Let $\Psi\colon H_1 \to H_1$ be given by

$$\Psi(\phi) = \phi \sum_{j=1}^{J} \check{c}_j \, \|\phi\|_{H_1}^{\alpha_j - 1} \, (\phi \in H_1),$$

where $\check{c}_j \geq 0$, and $\alpha_j > 1$ are real constants. Let $\alpha_0 = \min\{\alpha_1, \ldots, \alpha_J\}$, and $\check{c}_0 = \max\{\check{c}_1, \ldots, \check{c}_J\}$. Then the condition (12) is true. Since there exists $0 < r_0 < 1$ such that

$$|\Psi(\phi)| \leq J\check{c}_0 \|\phi\|_{H_1}^{\alpha_0 - 1} |\phi|$$

if ϕ from $\overline{B(0, r_0)}$. It can easily be verified that (5), (11), (13) are satisfied too.

Theorem 3 ensures the the existence of a generalized solution of the boundary value problem (15) in the Bochner space $W_0^{1,2}(0,T; H_1)$ for each $f \in L_1(0,T; H_2)$.

REMARK. The operator $F_{\Delta,1}$ with the function Ψ, considered in Example is, in general, not compact [8].

REFERENCES

[**1**] Akhmerov, R. R., Kamenskij, M. I., Potapov, A. S., Rodkina, A. E., Sadovskij, B. N., *Measures of Noncompactness and Condensing Operators* (in Russian), Novosibirsk, Nauka 1986; Engl. transl.: Basel, Birkhäuser 1992.

[**2**] Appell, J., Zabrejko, P. P., *Nonlinear Superposition Operators*, London, Cambridge University Press, 1990.

[**3**] Gajewski, H., Gröger, K., Zacharias, K., *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Berlin, Akademie-Verlag 1974.

[**4**] Jovanović, I., Rakočević, V., *Multipliers of mixed-norm sequence spaces and measures of noncompactness*, Publ. de l'Inst. Math. **56(70)** (1994), 61–68.

[**5**] Maz'ja, V. G., *Sobolev Spaces*, Berlin, Springer 1985.

[**6**] Yerzakova, N. A., *On properties of the β-measure of noncompactness* (in Russian), Deposited in VINITI No. 6132-82, Voronezh 1982.

[**7**] Yerzakova, N. A., *On measures of non-compactness in regular spaces*, Zeitschr. Anal. Anw. **15**, 2 (1996), 299–307.

[**8**] Yerzakova, N. A., *Investigation of some operators and operater equations in a space of integrable functions* (in Russian), Khabarovsk, Inst. Appl. Math. FEB RAS. Preprint No. 3 (1997), 1–23.

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