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MULTIPLIERS OF MIXED-NORM SEQUENCE SPACES AND MEASURES OF NONCOMPACTNESS. II

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Abstract. Let $l^{p,q}$, $0 < p$, $q \le \infty$, be the mixed norm sequence space, and $T_\lambda : l^{r,s} \to l^{u,v}$ the operator defined by the multiplier $T_\lambda(a) = \{\lambda_n a_n\}$, $\lambda = \{\lambda_n\} \in l^\infty$, $a = \{a_n\} \in l^{r,s}$. In this paper, we investigate the Hausdorff measure of noncompactness of the operator T_λ in the cases when $0 \le r, u, s, v \le \infty$, and prove necessary and sufficient conditions for T_λ to be compact. The paper is a continuation of [8] where we considered the cases $1 \le r, u, s, v \le \infty$.

1. Introduction and preliminaries

A complex sequence $\{\lambda_n\}$ is of class $l^{p,q}$, $0 < p, q \le \infty$, if

$$\sum_{m=0}^{\infty}\Big(\sum_{n\in I(m)} |\lambda_n|^p\Big)^{q/p} < \infty, \tag{1.1}$$

where $I(0) = \{0\}$ and $I(m) = \{n \in \mathbb{N} : 2^{m-1} \le n < 2^m\}$, for $m > 0$. In the case where p or q is infinite, replace the corresponding sum by a supremum.

It is known that $l^{p,q}$ with norm

$$\|\lambda\| = \bigg(\sum_{m=0}^{\infty}\Big(\sum_{n\in I(m)} |\lambda_n|^p\Big)^{q/p}\bigg)^{1/q}, \quad (1 \le p, q < \infty), \tag{1.2}$$

is a Banach space. Note that $l^{p,p} = l^p$, and that if p or q is infinite then the corresponding sum should be replaced by a supremum; thus

$$\|\lambda\| = \sup_m \Big(\sum_{n\in I(m)} |\lambda_n|^p\Big)^{1/p}, \quad (1 \le p < \infty,\ q = \infty). \tag{1.3}$$

Define

$$\|\lambda\| = \sum_{m=0}^{\infty}\Big(\sum_{n\in I(m)} |\lambda_n|^p\Big)^{q/p}, \quad (1 \le p < \infty,\ q < 1), \tag{1.4}$$

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$$\|\lambda\| = \sum_{m=0}^{\infty} \left(\sup_{n \in I(m)} |\lambda_n| \right)^q, \quad (p = \infty,\, q < 1). \tag{1.5}$$

$$\|\lambda\| = \sum_{m=0}^{\infty} \left(\sum_{n \in I(m)} |\lambda_n|^p \right)^{q/p}, \quad (p < 1,\, q \le p), \tag{1.6}$$

$$\|\lambda\| = \left(\sum_{m=0}^{\infty} \left(\sum_{n \in I(m)} |\lambda_n|^p \right)^{q/p} \right)^{q/p}, \quad (p < 1,\, p \le q < \infty), \tag{1.7}$$

$$\|\lambda\| = \sup_m \sum_{n \in I(m)} |\lambda_n|^p, \quad (p < 1,\, q = \infty). \tag{1.8}$$

For economy the dependence of $\|\lambda\|$ on p and q has not been indicated but it should be borne in mind. Thus in the case $1 \le p, q \le \infty$, $(l^{p,q}, \|\cdot\|)$ is a Banach space, usually called the mixed-norm space $l^{p,q}$, in the case $1 \le p \le \infty$, $0 < q < 1$, and in the case $0 < p < 1$, $q \le p$, it is a complete q-normed space; finally, in the case $0 < p < 1$, $p \le q$, it is a complete p-normed space (see e.g. [10], [11]).

If L is a subset of the set of all integers and $x = (x_i)$ is a sequence, we set $L(x)$ for the sequence $L(x) = (L(x)_i)$, where $L(x)_i = x_i$ if $i \in L$, and $L(x)_i = 0$ if $i \notin L$.

For any two subsets E and F of l^∞, the set of multipliers from E to F (denoted by (E, F)) is the set of all $\lambda = \{\lambda_n\} \in l^\infty$ such that $\lambda a = \{\lambda_n a_n\}$ is an element of F for all $a = \{a_n\} \in E$. Let $T_\lambda : E \to F$ be the operator defined by $T_\lambda(a) = \lambda a$, $(a \in E)$. For the convenience of a reader, recall the following well-known theorem of Kellog [9, Theorem 1].

Theorem (Kellog) 1.1. *Let $1 \le r, s, u, v \le \infty$, and define p and q by*

$$1/p = 1/u - 1/r \quad \text{if} \quad r > u, \qquad p = \infty \quad \text{if} \quad r \le u,$$
$$1/q = 1/v - 1/s \quad \text{if} \quad s > v, \qquad q = \infty \quad \text{if} \quad s \le v.$$

Then $(l^{r,s}, l^{u,v}) = l^{p,q}$.

Recall that Kellog (in [9, Theorem 1]) proved that the operator $T_\lambda : l^{r,s} \to l^{u,v}$, defined by $T_\lambda(x) = \lambda x$, $(x \in l^{r,s})$, is a bounded linear operator and that its operator norm $\|T_\lambda\|$ is equal to $\|\lambda\|$.

Remark. Let us remark that it was observed (see e.g. [3, Lemma 2], [4, Theorem 7.1, Theorem 8.1], [6, Lemma 2.4] or [7, Lemma 1.1.2]) that Kellog's theorem is true for $0 < r, s, u, v \le \infty$.

If X and Y are metric spaces, then $f : X \to Y$ is a compact map if $f(Q)$ is relatively compact (i.e., if the closure of $f(Q)$ is a compact subset of Y) subset of Y for each bounded subset Q of X. Recall that if Q is a bounded subset of a metric space X, then the *Hausdorff measure of noncompactness* of Q is denoted by $\chi(Q)$, and

$$\chi(Q) = \inf\{\epsilon > 0 : Q \quad \text{has a finite } \epsilon\text{-net in } X\}.$$

The function χ is called the *Hausdorff measure of noncompactness*, and for its properties and applications see e.g., ([1], [2], [5], [12], [13], [15]). Denote by $\overline{Q}$ the closure of Q. For the convenience of the reader, let us mention that: If Q, Q_1 and Q_2 are bounded subsets of a metric space (X,d), then

$$\begin{aligned}\chi(Q) = 0 \Longleftrightarrow Q &\quad \text{is a totally bounded set,}\\ \chi(Q) &= \chi(\overline{Q}),\\ Q_1 \subset Q_2 &\Longrightarrow \chi(Q_1) \le \chi(Q_2),\\ \chi(Q_1 \cup Q_2) &= \max\{\chi(Q_1), \chi(Q_2)\},\\ \chi(Q_1 \cap Q_2) &\le \min\{\chi(Q_1), \chi(Q_2)\}.\end{aligned}$$

If our space X is a normed space, then the function $\chi(Q)$ has some additional properties connected with the linear structure. We have e.g.

$$\begin{aligned}\chi(Q_1 + Q_2) &\le \chi(Q_1) + \chi(Q_2),\\ \chi(\lambda Q) &= |\lambda|\chi(Q) \quad \text{for each} \quad \lambda \in \mathbb{C}.\end{aligned}$$

If X (Y) is a p-normed space (resp. q-normed space), then let us denote by $B(X,Y)$ the set of all continuous linear operators from X into Y. For $A \in B(X,Y)$ the Hausdorff measure of noncompactness of A, denoted by $\|A\|_\chi$, is defined by $\|A\|_\chi = \chi(AK)$, where $K = \{x \in X : \|x\| \le 1\}$ is the unit ball in X. Further, A is compact if and only if $\|A\|_\chi = 0$.

In this paper, we investigate the Hausdorff measure of noncompactness of the operator T_λ in the cases when $0 < r, u, s, v \le \infty$, and prove necessary and sufficient conditions for T_λ to be compact. The paper is a continuation of [8] where we considered the cases $1 \le r, u, s, v \le \infty$.

2. Results

The following lemma extends the results of [8, Lemma 2.1].

Lemma 2.1. *Let Q be a bounded subset of $l^{u,v}$, $u \in (0,\infty]$, $v \in (0,\infty)$. Then*

$$\chi(Q) = \inf_{n\in N}\left[\sup_{(x_k)\in Q}\left(\sum_{m=n}^{\infty}\left(\sum_{k\in I(m)}|x_k|^u\right)^{v/u}\right)^{1/v}\right],\ (1 \le u,\ v < \infty) \tag{2.1}$$

$$\chi(Q) = \inf_{n\in N}\left[\sup_{(x_k)\in Q}\left(\sum_{m=n}^{\infty}\left(\sup_{k\in I(m)}|x_k|\right)^{v}\right)^{1/v}\right],\ (u = \infty,\ 1 \le v) \tag{2.2}$$

$$\chi(Q) = \inf_{n\in N}\left[\sup_{(x_k)\in Q}\left(\sum_{m=n}^{\infty}\left(\sum_{k\in I(m)}|x_k|^u\right)^{v/u}\right)\right],\ (1 \le u < \infty,\ v < 1) \tag{2.3}$$

$$\chi(Q) = \inf_{n\in N}\left[\sup_{(x_k)\in Q}\left(\sum_{m=n}^{\infty}\left(\sup_{k\in I(m)}|x_k|\right)^{v}\right)\right],\ (u = \infty,\ v < 1), \tag{2.4}$$

$$\chi(Q) = \inf_{n\in N}\left[\sup_{(x_k)\in Q}\left(\sum_{m=n}^{\infty}\left(\sum_{k\in I(m)}|x_k|^u\right)^{v/u}\right)\right],\ (u < 1,\ v \le u) \tag{2.5}$$

$$\chi(Q) = \inf_{n\in N}\Big[\sup_{(x_k)\in Q}\Big(\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)}|x_k|^u\Big)^{v/u}\Big)^{u/v}\Big],\ (u<1,\ u\le v). \tag{2.6}$$

Proof. The case $Q \subset l^{u,v}$, $u \in [1,\infty]$, $v \in [1,\infty)$ was proved in [8, Lemma 2.1]. The other cases $Q \subset l^{u,v}$, follow by the proofs of the previous case (we bear in mind that now $l^{u,v}$ is a v-normed space, or u-normed space); see also [8, Lemma 2.4]. ■

Now we prove the main result of this paper. Let us mention that in the proof we use the following result (see [14, p.7]) known as *Jansen's inequality*: Let $\{u_n\}$ be arbitrary sequence of complex numbers. Then

$$\sum\big(|u_n|^p\big)^{1/p} \text{ is a decreasing function of } p \text{ for } p>0.$$

THEOREM 2.2. *Let $0<r,u,s,v\le\infty$, and define p and q by*

$$1/p = 1/u - 1/r \quad \textit{if} \quad r>u, \qquad p=\infty \quad \textit{if} \quad r\le u,$$
$$1/q = 1/v - 1/s \quad \textit{if} \quad s>v, \qquad q=\infty \quad \textit{if} \quad s\le v.$$

Then $(l^{r,s}, l^{u,v}) = l^{p,q}$, and the operator $T_\lambda : l^{r,s}\to l^{u,v}$, defined by the multiplier $T_\lambda(a) = \{\lambda_n a_n\}$, $\lambda=\{\lambda_n\}\in l^{p,q}$, $a=\{a_n\}\in l^{r,s}$, is well defined (see Remark following Theorem 1.1). Now we have:

$$\|T_\lambda\|_\chi = 0,\ (v<s), \tag{2.7}$$

$$\|T_\lambda\|_\chi = \limsup_{n\to\infty}|\lambda_n|^v,\ (s\le v<1,\ r\le u), \tag{2.8}$$

$$\|T_\lambda\|_\chi = \limsup_{m\to\infty}\Big(\sum_{n\in I(m)}|\lambda_n|^p\Big)^{v/p},\ (s\le v<1,\ r>u), \tag{2.9}$$

$$\|T_\lambda\|_\chi = \limsup_{n\to\infty}|\lambda_n|,\ (1\le v<\infty,\ s\le v,\ r\le u,\ 1\le u), \tag{2.10}$$

$$\|T_\lambda\|_\chi = \limsup_{m\to\infty}\Big(\sum_{n\in I(m)}|\lambda_n|^p\Big)^{1/p},\ (1\le v<\infty,\ s\le v,\ r>u,\ 1\le u), \tag{2.11}$$

$$\|T_\lambda\|_\chi = \limsup_{n\to\infty}|\lambda_n|^u,\ (1\le v<\infty,\ s\le v,\ r\le u,\ u<1), \tag{2.12}$$

$$\|T_\lambda\|_\chi = \limsup_{m\to\infty}\Big(\sum_{n\in I(m)}|\lambda_n|^p\Big)^{u/p},\ (1\le v<\infty,\ s\le v,\ r>u,\ u<1), \tag{2.13}$$

$$\frac{1}{2}\cdot\limsup_{n\to\infty}|\lambda_n| \le \|T_\lambda\|_\chi \le \limsup_{n\to\infty}|\lambda_n|,\ (v=\infty,\ r\le u,\ 1\le u), \tag{2.14}$$

$$\frac{1}{2}\cdot\limsup_{m\to\infty}\Big(\sum_{n\in I(m)}|\lambda_n|^p\Big)^{1/p} \le \|T_\lambda\|_\chi \le$$
$$\le \limsup_{m\to\infty}\Big(\sum_{n\in I(m)}|\lambda_n|^p\Big)^{1/p},\ (v=\infty,\ r>u,\ 1\le u). \tag{2.15}$$

$$\frac{1}{2}\cdot \limsup_{n\to\infty} |\lambda_n|^u \leq \|T_\lambda\|_\chi \leq \limsup_{n\to\infty} |\lambda_n|^u, \ (v=\infty,\, r\leq u,\, u<1), \tag{2.16}$$

$$\frac{1}{2}\cdot \limsup_{m\to\infty}\Big(\sum_{n\in I(m)} |\lambda_n|^p\Big)^{u/p} \leq \|T_\lambda\|_\chi \leq$$
$$\leq \limsup_{m\to\infty}\Big(\sum_{n\in I(m)} |\lambda_n|^p\Big)^{u/p}, \quad (v=\infty,\, r>u,\, u<1). \tag{2.17}$$

Proof. Set $K=\{x\in l^{r,s} : \|x\|\leq 1\}$. Then $\|T_\lambda\|=\chi(K)$, and the proof in the cases (2.7)–(2.13) begins from the corresponding formulae for $\chi(K)$ (see Lemma 2.1). In the remaining cases (2.14)–(2.17) our results are not as sharp as in the previous cases, but still precise enough to get necessary and sufficient conditions for the compactness of T_λ (se Corollary 2.3). As a general remark let us mention that in the proof of Theorem 2.2 we use the techniques from the proofs of [8, Theorem 2.2] and [9, Theorem 1].

To prove (2.7), let us remark that in this case q is a real number. Then by (2.1)–(2.6), using the techniques from the proof of [8, (2.2.1)] and [9, Theorem 1] we can get the result. For example in the case $v<s$, $v<1$ and $1\leq u<r<\infty$, we know that p and q are real numbers. Now for $\lambda\in l^{p,q}$, by Lemma 2.1, we have

$$\|T_\lambda\|_\chi = \lim_{n\to\infty}\ \sup_{x\in K}\ \Big(\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)} |\lambda_k x_k|^u\Big)^{v/u}\Big), \tag{2.18}$$

where $x=(x_1,\,x_2,\,\dots)\in K$. By applying Hölder's inequality first to the inner sum in (2.18) with $\alpha=r/(r-u)$, and to the outer sum with $\beta=s/(s-v)$ (see [9]) we get (we note that the condition $v<1$ is not essential for the proof)

$$\Big(\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)} |\lambda_k x_k|^u\Big)^{v/u}\Big)^{1/v} \leq$$
$$\leq \Big(\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)} |\lambda_k|^p\Big)^{q/p}\Big)^{1/q}\Big(\sum_{m=0}^{\infty}\Big(\sum_{n\in I(m)} |x_n|^r\Big)^{s/r}\Big)^{1/s}. \tag{2.19}$$

Now, (2.7) follows by (2.18) and (2.19).

Now, suppose that $1\leq u<r<\infty$, $v<1$, $v<s=\infty$. Hence, $q=v$, and again by [9, Theorem 1] we get

$$\Big(\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)} |\lambda_k x_k|^u\Big)^{v/u}\Big)^{1/v} \leq$$
$$\leq \Big(\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)} |\lambda_k|^p\Big)^{q/p}\Big)^{1/q}\Big(\sup_{m}\Big(\sum_{k\in I(m)} |x_k|^r\Big)^{1/r}\Big). \tag{2.20}$$

Now, (2.7) follows by (2.18) and (2.20).

Further, in the case $v < s$, $u < 1$, $v \le u$ and $r \le u$, we have (let us remark that $p = \infty$)

$$\begin{aligned}
&\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)}|\lambda_k x_k|^u\Big)^{v/u} \le \sum_{m=n}^{\infty}\Big(\sup_{k\in I(m)}|\lambda_k|^v\Big)\Big(\sum_{k\in I(m)}|x_k|^u\Big)^{v/u}\\
&\le \Big(\sum_{m=n}^{\infty}\Big(\sup_{k\in I(m)}|\lambda_k|^v\Big)^{s/(s-v)}\Big)^{(s-v)/s}\Big(\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)}|x_k|^u\Big)^{(v/u)\cdot(s/v)}\Big)^{v/s}\\
&\le \Big(\sum_{m=n}^{\infty}\Big(\sup_{k\in I(m)}|\lambda_k|^q\Big)\Big)^{v/q}\Big(\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)}|x_k|^r\Big)^{s/r}\Big)^{v/s}. \qquad (2.21)
\end{aligned}$$

Now, (2.7) follows by (2.5) and (2.21). Let us remark that all the other possibilities for r, s, u, v in the case $v < s$ can be proved in a similar way, and we omit the proof.

Now suppose that $s \le v < \infty$. Hence $q = \infty$. Further suppose that $r \le u$. Thus $p = \infty$. Hence we have to consider the cases: (2.8), (2.10) and (2.12). All these cases can be proved (with natural changes for u, v) as in the case [8, (2.2.2)].

Let $\epsilon > 0$. Then there is a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that

$$|\lambda_{n_k}| > \limsup_{n\to\infty}|\lambda_n| - \epsilon. \qquad (2.22)$$

Set $M = \{n_k : k = 1, 2, \dots\}$, and let $e_i = \{\delta_{ij}\} \in l^\infty$, $i = 1, 2, \dots$. Now

$$\|T_\lambda\|_\chi = \chi(T_\lambda K) \ge \chi(T_{M(\lambda)}K) \ge \chi(\{M(\lambda)e_i : i \in \mathbb{N}\}), \qquad (2.23)$$

and by Lemma 2.1

$$\|T_\lambda\|_\chi \ge \begin{cases} \limsup\limits_{n\to\infty}|\lambda_n|^v, & \text{for } v < 1\\ \limsup\limits_{n\to\infty}|\lambda_n|, & \text{for } 1 \le v, 1 \le u\\ \limsup\limits_{n\to\infty}|\lambda_n|^u, & \text{for } 1 \le v, u < 1.\end{cases} \qquad (2.24)$$

To prove "$\le$" in (2.24), we could use the method of proof of [8, (2.2.11)], or by direct calculation, in the case, say, $s \le v < 1$, $v \le u < \infty$ and $r \le u$, for each n we have

$$\begin{aligned}
\|T_\lambda\|_\chi &\le \sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)}|\lambda_k x_k|^u\Big)^{v/u}\\
&\le \sum_{m=n}^{\infty}\Big(\sup_{k\in I(m)}|\lambda_k|^v\Big)\Big(\sum_{k\in I(m)}|x_k|^u\Big)^{v/u}\\
&\le \sup_{m\ge n}\Big(\sup_{k\in I(m)}|\lambda_k|^v\Big)\cdot\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)}|x_k|^u\Big)^{v/u}\\
&\le \sup_{m\ge n}\Big(\sup_{k\in I(m)}|\lambda_k|^v\Big)\cdot\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)}|x_k|^r\Big)^{v/r}
\end{aligned}$$

$$\leq \sup_{m\geq n}\Big(\sup_{k\in I(m)}|\lambda_k|^v\Big)\cdot\Bigg[\Big(\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)}|x_k|^r\Big)^{s/r}\Big)^{1/s}\Bigg]^v. \tag{2.25}$$

Now, by (2.25) we get

$$\|T_\lambda\|_\chi \leq \limsup_{n\to\infty}|\lambda_n|^v, \quad \text{for } s\leq v<1,\ v\leq u<\infty \text{ and } r\leq u. \tag{2.26}$$

To prove "$\leq$" in the case $s \leq v < 1$, $u = \infty$, suppose that $\epsilon > 0$. Set $L = \{n : |\lambda_n| > \limsup_{n\to\infty}|\lambda_n| + \epsilon\}$ and $W = \mathbb{N}\setminus L$. Hence

$$T_\lambda(K) = T_{W(\lambda)}(K) + T_{L(\lambda)}(K).$$

Clearly, L is a finite set, and the multiplier $T_{L(\lambda)}$ is a compact operator. Hence

$$\chi(T_\lambda(K)) \leq \chi(T_{W(\lambda)}(K)) + \chi(T_{L(\lambda)}(K)) = \chi(T_{W(\lambda)}(K)). \tag{2.27}$$

and by (2.4) we have

$$\begin{aligned}\|T_{W(\lambda)}\|_\chi = \chi(T_{W(\lambda)}(K)) &= \inf_{n\in N}\Bigg[\sup_{(x_k)\in K}\Bigg(\sum_{m=n}^{\infty}\Big(\sup_{k\in I(m)\cap W}|\lambda_k x_k|\Big)^v\Bigg)\Bigg]\\ &\leq \Big[\limsup_{n\to\infty}(|\lambda_n|+\epsilon)\Big]^v\cdot\inf_{n\in N}\Bigg[\sup_{(x_k)\in K}\Bigg(\sum_{m=n}^{\infty}\Big(\sup_{k\in I(m)\cap W}|x_k|\Big)^v\Bigg)\Bigg]\\ &\leq \Big[\limsup_{n\to\infty}(|\lambda_n|+\epsilon)\Big]^v.\end{aligned}$$

Hence

$$\|T_\lambda\|_\chi \leq \limsup_{n\to\infty}|\lambda_n|^v \quad \text{for } s\leq v<1,\ u=\infty. \tag{2.28}$$

Now suppose that $s \leq v < \infty$ and $r > u$. Hence $q = \infty$ and p is a real number. Thus we have to consider the cases (2.9), (2.11) and (2.13). All these cases can be proved (with natural changes for u, v) as in the case [8, (2.2.3)] (let us remark that in the proof of [8, (2.2.3)] in the line following [8, (2.2.12)] M should be $M = \bigcup_k I(m_k)$, and $\mathbb{N}\setminus L(\lambda_i)$ should be $\bigcup\{I(m) : m\in\mathbb{N}\setminus L(\lambda_i)\}$).

Let us prove (2.9). Let $\epsilon > 0$. Then there is a subsequence $\{I(m_k)\}$ of $\{I(m)\}$ such that

$$\Big(\sum_{n\in I(m_k)}|\lambda_n|^p\Big)^{1/p} > \limsup_{m\to\infty}\Big(\sum_{n\in I(m)}|\lambda_n|^p\Big)^{1/p} - \epsilon,\ k\in N. \tag{2.29}$$

Set $M = \cup_k I(m_k)$, and $c_k = \left(\sum_{n\in I(m_k)}|\lambda_n|^p\right)^{-1/r}$, $k = 1, 2, \ldots$. For each k, define the sequence $x^{(k)} = (x_n^{(k)})$, by

$$x_n^{(k)} = \begin{cases} c_k|\lambda_n|^{p/r}, & \text{if } n\in I(m_k)\\ 0, & \text{otherwise.}\end{cases} \tag{2.30}$$

Now $x^{(k)} \in l^{r,s}$ and $\|x^{(k)}\| = 1$, $k = 1, 2, \ldots$. Further, in the case $1 \leq u < \infty$,

$v < 1$, by (2.3) we get

$$\begin{aligned}\|T_\lambda\|_\chi &= \chi(T_\lambda K) \ge \chi(T_{M(\lambda)}K) \ge \chi(\{M(\lambda)x^{(k)} : k \in N\})\\ &= \inf_{n\in N}\left[\sup_{k\in N}\left(\sum_{i\in I(m_k)} |(M(\lambda)\,x^{(k)})_i|^u\right)^{v/u}\right]\\ &= \inf_{n\in N}\left[\sup_{k\in N}\left(\sum_{i\in I(m_k)} |\lambda_i x_i^{(k)}|^u\right)^{v/u}\right]\\ &= \sup_{k\in N}\left(\sum_{i\in I(m_k)} |\lambda_i|^p\right)^{v/p} \ge \left[\limsup_{k\to\infty}\left(\sum_{n\in I(m_k)} |\lambda_n|^p\right)^{1/p} - \epsilon\right]^v.\end{aligned} \tag{2.31}$$

Since the inequalities for the remaining two cases are quite similar, we omit the proofs. Hence

$$\|T_\lambda\|_\chi \ge \begin{cases} \limsup\limits_{n\to\infty}\left(\sum\limits_{n\in I(m)} |\lambda_n|^p\right)^{v/p}, & \text{for } s \le v < 1,\ r > u\\ \limsup\limits_{n\to\infty}\left(\sum\limits_{n\in I(m)} |\lambda_n|^p\right)^{v/p}, & \text{for } s \le v,\ 1 \le v < \infty,\ r > u \ge 1\\ \limsup\limits_{n\to\infty}\left(\sum\limits_{n\in I(m)} |\lambda_n|^p\right)^{v/p}, & \text{for } s \le v,\ 1 \le v < \infty,\ u < 1. \end{cases} \tag{2.32}$$

To prove "$\le$" in (2.32), suppose that $\epsilon > 0$. Then

$$L \equiv \left\{m : \left(\sum_{n\in I(m)} |\lambda_n|^p\right)^{1/p} > \limsup_{m\to\infty}\left(\sum_{n\in I(m)} |\lambda_n|^p\right)^{1/p} + \epsilon\right\}$$

is a finite set. Set $W = \cup\{I(m) : m \in \mathbb{N} \setminus L\}$, and $U = \mathbb{N} \setminus W$. Now

$$T_\lambda(K) = T_{W(\lambda)}(K) + T_{U(\lambda)}(K),$$

and

$$\|T_\lambda\|_\chi = \chi(T_\lambda(K)) \le \chi(T_{W(\lambda)}(K)) + \chi(T_{U(\lambda)}(K)) = \chi(T_{W(\lambda)}(K)). \tag{2.33}$$

In the case $1 \le u < \infty$, $0 < v < 1$, by (2.3) we have

$$\chi(T_{W(\lambda)}(K)) = \inf_{n\in N}\left\{\sup_{(x_k)\in K}\left[\sum_{m=n}^{\infty}\left(\sum_{k\in I(m)} |W(\lambda)_k x_k|^u\right)^{v/u}\right]\right\}. \tag{2.34}$$

Applying Holder's inequality to the inner sum with $r/(r-u)$, we get

$$\begin{aligned}&\chi(T_{W(\lambda)}(K)) \le\\ &\le \inf_{n\in N}\left\{\sup_{(x_k)\in K}\left[\sum_{m=n}^{\infty}\left(\Big(\sum_{k\in I(m)} |W(\lambda)_k|^p\Big)^{v/p}\Big(\sum_{k\in I(m)} |x_k|^r\Big)^{v/r}\right)\right]\right\}\\ &\le \inf_{n\in N}\left\{\sup_{(x_k)\in K}\left[\left(\sup_{m\ge n}\Big(\sum_{k\in I(m)} |W(\lambda)_k|^p\Big)^{v/p}\right)\cdot\sum_{m=n}^{\infty}\Big(\sum_{k\in I(m)} |x_k|^r\Big)^{v/r}\right]\right\}.\end{aligned} \tag{2.35}$$

Now, by (2.33) we have

$$\chi(T_\lambda(K)) \le \chi(T_{W(\lambda)}(K)) \le \limsup_{m\to\infty}\Big(\big(\sum_{n\in I(m)}|\lambda_n|^p\big)^{1/p}+\epsilon\Big)^v,$$

and finally

$$\|T_\lambda\|_\chi \le \limsup_{m\to\infty}\Big(\sum_{n\in I(m)}|\lambda_n|^p\Big)^{v/p}. \tag{2.36}$$

Hence, now (2.9) follows from (2.32) and (2.36). Since the inequalities for the remaining cases are quite similar, we omit the proofs. Let us remark that in these cases the proof could be given by a direct calculation as in (2.25)

Finally let us consider the case $v=\infty$. We have to prove (2.14), (2.15), (2.16) and (2.17). The right inequalities in all these cases follows by the proof of the corresponding cases for the parameters r and u when v is real. To prove the left inequalities let us recall (see e.g. [1, Theorem 1.1.7 and Remark 1.3.2]) that if Q is a bounded subset of a metric space (X,d), $\alpha>0$ and u_n is a sequence in Q such that

$$d(u_n,u_m)>\alpha,\quad n\ne m,\quad \text{then}\quad \alpha<2\chi(Q). \tag{2.37}$$

Hence, for example the left inequality in (2.15) (similarly (2.17)) follows from the fact that (using the notations of the proof of (2.9), more precisely we use M and $x^{(k)}$, $k=1,2,3,\dots$), for $i\ne j$

$$\|M(\lambda)x^{(i)}-M(\lambda)x^{(j)}\| \ge \begin{cases} \limsup\limits_{m\to\infty}\Big(\Big(\sum\limits_{n\in I(m)}|\lambda_n|^p\Big)^{1/p}-\epsilon\Big), & \text{for } 1\le u\\ \limsup\limits_{m\to\infty}\Big(\Big(\sum\limits_{n\in I(m)}|\lambda_n|^p\Big)^{1/p}-\epsilon\Big)^u, & \text{for } u<1\end{cases} \tag{2.38}$$

Finally let us remark that the left inequalities in (2.14) and (2.15) can be proved by the argument in the proof of [8, (2.2.4)]. This completes the proof of Theorem 2.2. ∎

Now as a corollary of the above theorem we have

Corollary 2.3. *Let $1\le r,u\le\infty$, $0<s,v\le\infty$, and define p and q by*

$$\begin{aligned} 1/p&=1/u-1/r \quad \text{if}\quad r>u, \qquad p=\infty \quad \text{if}\quad r\le u,\\ 1/q&=1/v-1/s \quad \text{if}\quad s>v, \qquad q=\infty \quad \text{if}\quad s\le v.\end{aligned}$$

Then, for $\lambda\in(l^{r,s},l^{u,v})=l^{p,q}$, we have:

T_λ is a compact, if $v<s$, (2.39)

T_λ is a compact $\iff \limsup\limits_{n\to\infty}|\lambda_n|=0$, *if $s\le v$ and $r\le u$,* (2.40)

$$T_\lambda \text{ is a compact } \iff \limsup_{m\to\infty}\Big(\sum_{n\in I(m)}|\lambda_n|^p\Big)^{1/p}=0,\ \text{if } s\le v \text{ and } r>u. \tag{2.41}$$

Let us mention that the results of Theorem 2.1 and Corollary 2.3 can be extended from dyadic to general blocks without problem (see [4]).

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REFERENCES

[**1**] Р. Р. Ахмеров, М. И. Каменский, А. С. Потапов и др., *Меры некомпактности и уплотняющие операторы*, Наука, Новосибирск, 1986.

[**2**] J. Banás and K. Goebl, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics 60, Marcel Dekker, New York and Basel, 1980.

[**3**] O. Blasco, *Multipliers on weighted Besov spaces of analytic functions*, Contemporary Mathematics, **144** (1993), 23–33.

[**4**] K.-G. Grosse-Erdmann, *The blocking technique, weighted mean operators and Hardy's inequality*, preprint.

[**5**] O. Hadžić, *Fixed Point Theory in Topological Vector Spaces*, University of Novi Sad, Institute of Mathematics, Novi Sad, 1984.

[**6**] M. Jevtić and M. Pavlović, *On multipliers from* H^p *to* l^q, $0 < q < p < 1$, Arch. Math. **56** (1991), 174–180.

[**7**] I. Jovanović, *Množitelji i dekompozicije prostora analitičkih funkcija sa mešovitim normama*, Doktorska disertacija, Univerzitet u Beogradu, Beograd, 1992.

[**8**] I. Jovanović and V. Rakočević, *Multipliers of mixed-norm sequence spaces and measures of noncompactness*, Publ. Inst. Math. **56(70)** (1994), 61–68.

[**9**] C. N. Kellog, *An extension of the Hausdorff-Young theorem*, Michigan Math. J., **18** (1971), 121–127.

[**10**] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, Cambridge, 1970.

[**11**] E. Malkowsky, *Matrix transformations in a new class of sequence spaces that includes spaces of absolutely and strongly summable sequences*, Habilitationsschrift, Giessen, 1988.

[**12**] V. Rakočević, *Funkcionalna analiza*, Naučna knjiga, Beograd, 1994.

[**13**] M. R. Tasković, *Osnovi teorija fiksne tačke*, Zavod za udžbenike i nastavna sredstva, Beograd, 1986.

[**14**] A. Wilansky, *Functional Analysis*, Blaisdell Publ. Com., New York-Toronto-London, 1964.

[**15**] N. A. Yerzakova, *Non-linear superposition operators on space* $C([0,1],E)$, J. Math. Anal. Appl. **181** (1994), 385–391.

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