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RECENT RESULTS IN THE THEORY OF MATRIX TRANSFORMATIONS IN SEQUENCE SPACES

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Abstract. In this paper we give a survey of recent results in the theory of matrix transfomrations between sequence spaces. We shall deal with sequence spaces that are closely related to various concepts of summability, study their topological structures, find their Schauder-bases and determine their β-duals. Further we give necessary and sufficient conditions for matrix transformations between them.

1. Introduction and well-known results

We shall write ω for the set of all complex sequences $x=(x_k)_{k=0}^\infty$ and ϕ, l_∞, c and c_0 for the sets of all finite, bounded, convergent sequences and sequences convergent to naught, respectively; and finally, for $1 \leqslant p < \infty$, $l_p = \{ x \in \omega : \sum_{k=0}^\infty |x_k|^p < \infty \}$.

By e and $e^{(n)}$ $(n = 0, 1, \dots)$ we denote the sequences such that $e_k = 1$ for $k = 0, 1, \dots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

A *BK space* is a Banach sequence space with continuous coordinates. A sequence $(b_n)_{n=0}^\infty$ in a linear metric space X is called a (*Schauder-*) *basis* if for each $x \in X$ there exists a unique sequence $(\lambda_n)_{n=0}^\infty$ of scalars such that $x = \sum_{n=0}^\infty \lambda_n b_n$. A BK spaces $X \supset \phi$ is said to have AK if every $x = (x_k)_{k=0}^\infty \in X$ has a unique representation $x = \sum_{n=0}^\infty x_n e^{(n)}$.

Let $A = (a_{nk})_{n,k=0}^\infty$ be a infinite matrix of complex numbers, $x \in \omega$ and $1 \leqslant p < \infty$. Then we shall write

$$A_n(x) = \sum_{k=0}^\infty a_{nk}x_k, \qquad A_n(|x|^p) = \sum_{k=0}^\infty a_{nk}|x_k|^p \quad (n = 0, 1, \dots);$$
$$A(x) = (A_n(x))_{n=0}^\infty \quad \text{and} \quad A(|x|^p) = (A_n(|x|^p))_{n=0}^\infty.$$

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For any subset X of ω, we define the sets

$$X_A = \{\, x \in \omega : A(x) \in X \,\} \quad \text{and} \quad X_{[A]^p} = \{\, x \in \omega : A(|x|^p) \in X \,\}.$$

If $p = 1$, then we omit the index p, i.e. we write $X_{[A]} = A_{[A]^1}$ for short. For instance, if E is the matrix defined by $e_{nk} = 1$ $(0 \leqslant k \leqslant n)$ and $e_{nk} = 0$ $(k > n)$ for all $n = 0, 1, \ldots$, then $cs = c_E$ and $bs = (l_{bb})_E$ are the sets of convergent and bounded series.

We shall be interested in sequence spaces that are closely related to the concepts of ordinary, strong and absolute summability with index $p \geqslant 1$ [5, pp. 185, 189, 190]. Further, we shall study the topological properties of these sequence spaces, give their β-duals and characterize matrix transformations between them.

The reader is referred to [2, 13, 19] for the results in classical summability theory, and to [18, 14, 5, 15, 19] for the theory of sequence spaces.

2. The classical BK spaces

In this section, we shall state the fundamental results. It is well known [18, 5, 14] that the spaces l_p $(1 \leqslant p < \infty)$, c_0, c and l_∞ are BK spaces with their natural norms, l_p and c have AK, every sequence $x = (x_k)_{k=0}^\infty \in c$ has a unique representation $x = le + \sum_{k=0}^\infty (x_k - l)e^{(k)}$ where $l = \lim_{k\to\infty} x_k$ and l_∞ has no Schauder basis.

If X and Y are arbitrary subsets of ω and z any sequence, then we shall write

$$z^{-1} * X = \{x \in \omega : xz \in X\} \quad \text{and} \quad M(X, Y) = \bigcap_{x \in X} x^{-1} * Y.$$

In the special case, where $Y = cs$, the set

$$X^\beta = M(X, cs) = \left\{ a \in \omega : \sum_{k=0}^{\infty} a_k x_k \text{ converges for all } x \in X \right\}$$

is called the *β-dual of X*. If X is an arbitrary normed space, then we denote its *continuous dual* by X^*, i.e. X^* is the space of all continuous linear functionals on X, with the norm $\|\cdot\|^*$ defined by

$$\|f\|^* = \sup\{|f(x)| : \|x\| = 1\} \qquad (f \in X^*).$$

There is a close relation between the continuous dual and the β-dual of a BK space.

THEOREM 2.1. ([18, Theorems 4.3.15, p. 64 and 7.2.9, p. 107) *Let $(X, \|\cdot\|)$ be a BK space.*

(a) Then X^β is a BK space with $\|a\|_\beta = \sup\{\sup_n |\sum_{k=0}^n a_k x_k| : \|x\| = 1\}$.

(b) The inclusion $X^\beta \subset X^*$ holds in the following sense: Let the map $\hat{}\colon X^\beta \to X^*$ be defined by $\hat{}(a) = \hat{a}\colon X \to \mathbf{C}$ $(a \in X^\beta)$ where $\hat{a}(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in X$. Then $\hat{}$ is an isomorphism into X^*. Further, if X has AK, then the map $\hat{}$ is onto X^*.

The β-duals of the classical sequence spaces are well-known [16, 18, 14]:

$$\omega^\beta = \phi, \quad \phi^\beta = \omega, \quad l_1^\beta = l_\infty \quad \text{and} \quad c_0^\beta = c^\beta = l_\infty^\beta = l_1.$$

If we put $q = p/(p-1)$ for $1 < p < \infty$, then $l_p^\beta = l_q$. The spaces c^* and c_0^* are norm isomorphic with l_1. Further, we have $\|a\|_\beta = \|a\|_1$ for all $a \in l_\infty^\beta$.

Let A be an infinite matrix of complex numbers and $A_n = (a_{nk})_{k=0}^\infty$ be the sequence in the n^{th}-row of A. By (X,Y) we denote the class of all matrices A that map the set $X \subset \omega$ into the set $Y \subset \omega$. Thus

$$A \in (X,Y) \quad \text{if and only if} \quad \begin{cases} A_n \in X^\beta \text{ for all } n \\ \text{and} \\ A(x) = (A_n(x))_{n=0}^\infty \in Y \text{ for all } x \in X. \end{cases}$$

The most important result in the theory of matrix transformations is

Theorem 2.2 ([18, Theorem 4.2.8, p. 57) *Matrix transformations between BK spaces are continuous.*

All our results are obtained from the characterization of the class (X, l_∞):

Theorem 2.3. ([10, Lemma 4.1]) *Let X be a BK space.*

(a) Then $A \in (X, l_\infty)$ if and only if

$$\|A\|^* = \sup_n \|A_n\|^* < \infty \text{ where } \|A_n\|^* = \sup_n \{|A_n(x)| : \|x\| = 1\} \tag{2.1}$$

for $n = 0, 1, \ldots$.

(b) Further, if $(b^k)_{k=0}^\infty$ is a Schauder basis of X, Y and Y_1 are BK spaces with Y_1 a closed subspace of Y, then $A \in (X, Y_1)$ if and only if $A \in (X,Y)$ and $A(b^{(k)}) \in Y_1$ for all k.

Let T be a triangle, i.e. $t_{nk} = 0$ for all $k > n$ and $t_{nn} \neq 0$ $(n = 0, 1, \ldots)$. Further let B be a positive triangle. A subset X of ω is called *normal* if $x \in X$ and $|y_k| \leqslant |x_k|$ $(k = 0, 1, \ldots)$ together imply $y \in X$, and a norm $\|\cdot\|$ on X is called *monotonous* if $|y_k| \leqslant |x_k|$ $(k = 0, 1, \ldots)$ for $x, y \in X$ implies $\|y\| \leqslant \|x\|$.

Theorem 2.4. *(a) Let X be a BK space with the norm $\|\cdot\|$. Then X_T is a BK space with*

$$\|x\|_T = \|T(x)\| \quad \textit{for all } x \in X_T \quad \text{([18, Theorem 4.3.12, p. 63]).}$$

(b) Let X be a normal BK space with monotonous norm $\|\cdot\|$. We put

$$x_n = (B_n(|y|^p))^{1/p} \textit{ for } 1 \leqslant p < \infty \quad (n = 0, 1, \ldots)$$

and $Y = \{y \in \omega : x \in X\}$. Then Y is a BK space with $\|y\|_Y = \|x\|$ for all $y \in Y$.

The characterizations of the classes (X, Y_T) and $(X, Y_{[B]})$ can be reduced to those of (X,Y).

THEOREM 2.5. ([9, Theorem 1] and [10, Theorem 2.4]) *Let X and Y be arbitrary sets of sequences, T a triangle and B a positive triangle.*

(a) Then $A \in (X, Y_T)$ if and only if $TA \in (X, Y)$.

(b) For each $m = 0, 1, \dots,$ let N_m denote a subset of the set $\{0, \dots, m\}$, $N = (N_m)_{m=0}^{\infty}$ be the sequence of the sets N_m and $\mathcal{N}$ be the class of all such sequences. Further, if A is an infinite matrix, then for each sequence $N \in \mathcal{N}$ let $S^N(B, A)$ be the matrix defined by

$$S_m^N(B,A) = \sum_{n \in N_m} b_{mn} A_n, \ \text{i.e. } s_{mk}^N(B,A) = \sum_{n \in N_m} b_{mn} a_{nk} \quad (m, k = 0, 1, \dots).$$

Finally, let T be a normal set of sequences. Then $A \in (X, Y_{[B]})$ if and only if $S^N(B, A) \in (X, Y)$ for all $N \in \mathcal{N}$.

3. Applications

3.1. Sequences that are $(\overline{N}, q)$-summable or bounded. Let $(q_k)_{k=0}^{\infty}$ be a positive sequence and Q the sequence with $Q_n = \sum_{k=0}^{n} q_k$ $(n = 0, 1, \dots)$. Further, let the matrix $\overline{N}_q$ be defined by $(\overline{N}_q)_{n,k} = q_k/Q_n$ $(0 \leqslant k \leqslant n)$ and $(\overline{N}_q)_{n,k} = 0$ $(k > n)$ for all n. Then we define the sets

$$(\overline{N}, q)_0 = (c_0)_{\overline{N}_q}, \quad (\overline{N}, q) = c_{\overline{N}_q} \quad \text{and} \quad (\overline{N}, q)_{\infty} = (l_{\infty})_{\overline{N}_q}$$

of sequences that are $(\overline{N}, q)$ *summable to naught*, *summable* and *bounded*.

We shall write $\mathcal{U}$ for the set of all sequences u such that $u_k \neq 0$ $(k = 0, 1, \dots)$. For $u \in \mathcal{U}$, let $1/u = (1/u_k)_{k=0}^{\infty}$.

THEOREM 3.1. (cf. [1, Theorem 2]) *(a) Let X be a BK space with basis $(b^{(k)})_{k=0}^{\infty}$, $u \in \mathcal{U}$ and $c^{(k)} = (1/u)b^{(k)}$ $(k = 0, 1, \dots)$. Then $(c^{(k)})_{k=0}^{\infty}$ is a basis of $Y = u^{-1} * X$.*

(b) Let $u \in \mathcal{U}$ be a sequence such that

$$|u_0| \leqslant |u_1| \leqslant \cdots \quad \text{and} \quad |u_n| \to \infty \ (n \to \infty),$$

and T a triangle with $t_{nk} = 1/u_n$ $(0 \leqslant k \leqslant n)$ and $t_{nk} = 0$ $(k > n)$ for all n. Then $(c_0)_T$ has AK.

We have by Theorems 2.4 and 3.1:

THEOREM 3.2. (cf. [1, Corollary 1]) *Each of the sets $(\overline{N}, q)_0$, $(\overline{N}, q)$ and $(\overline{N}, q)_{\infty}$ is a BK space with*

$$\|x\|_{\overline{N}_q} = \sup_n \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k x_k \right|.$$

If $Q_n \to \infty$ $(n \to \infty)$, then $(\overline{N}, q)_0$ has AK, and every sequence $x = (x_k)_{k=0}^{\infty} \in (\overline{N}, q)$ has a unique representation

$$x = le + \sum_{k=0}^{\infty} (x_k - l)e^{(k)} \quad \text{where } l \in \mathbf{C} \text{ is such that } x - le \in (\overline{N}, q)_0.$$

Let the operator $\Delta^+ : \omega \to \omega$ be defined by

$$\Delta^+ x = ((\Delta^+ x)_k)_{k=0}^{\infty} = (x_k - x_{k+1})_{k=0}^{\infty}.$$

THEOREM 3.3. (cf. [1, Theorem 5]) *Let T be a triangle such that*

$$\|T\| = \sup_n \left(\sum_{k=0}^{n} |t_{nk}| \right) < \infty \quad \textit{and} \quad \lim_{n\to\infty} t_{nk} = 0 \quad (k = 0, 1, \dots),$$

$(c_0)_T$ have AK and $Y = (c_0)_T \oplus le$. Then $f \in Y^*$ if and only if

$$f(x) = l_{\chi_f} + \sum_{k=0}^{\infty} a_k x_k \textit{ with } a \in Y^{\beta}, \textit{ where } l \in \mathbf{C} \textit{ is such that}$$

$$x - le \in (c_0)_T \textit{ and } \chi_f = f(e) - l \sum_{k=0}^{\infty} a_k.$$

THEOREM 3.4. (cf. [1, Theorem 6]) *Let $\Delta^+ : \omega \to \omega$ be defined by $\Delta^* x = ((\Delta^+ x)_k)_{k=0}^{\infty}$. We put*

$$\mathcal{N}_0 = (1/q)^{-1} * ((Q^{-1} * l_1)_{\Delta^+} \cap (Q^{-1} * l_\infty)),$$
$$\mathcal{N} = (1/q)^{-1} * ((Q^{-1} * l_1)_{\Delta^+} \cap (Q^{-1} * c)),$$
$$\mathcal{N}_\infty = (1/q)^{-1} * ((Q^{-1} * l_1)_{\Delta^+} \cap (Q^{-1} * c_0)).$$

(a) Then $(\overline{N}, q)_0^{\beta} = \mathcal{N}_0$, $(\overline{N}, q)^{\beta} = \mathcal{N}$ and $(\overline{N}, q)_\infty^{\beta} = \mathcal{N}_\infty$.

(b) Let $Q_n \to \infty$ for $n \to \infty$. Then $f \in (\overline{N}, q)_0^*$ if and only if

$$f(x) = \sum_{k=0}^{\infty} a_k x_k \quad \textit{with } a \in \mathcal{N}_0 \textit{ and} \quad \|f\|^* = \sup_n \left(\sum_{k=0}^{n-1} Q_k \left| \Delta^+ \frac{a_k}{q_k} \right| + |a_n Q_n / q_n| \right).$$

Then $f \in (\overline{N}, q)^*$ if and only if

$$f(x) = l_{\chi_f} + \sum_{k=0}^{\infty} a_k x_k \textit{ with } a \in \mathcal{N} \textit{ where } l \textit{ is such that}$$

$$x - le \in (\overline{N}, q)_0 \textit{ and } \chi_f = f(e) - l \sum_{k=0}^{\infty} a_k$$

and $\|f\|^* = |\chi_f| + \sup_n \left(\sum_{k=0}^{n-1} Q_k \left| \Delta^+ \frac{a_k}{q_k} \right| + |a_n Q_n / q_n| \right).$

It follows from Theorems 2.3, 3.2 and 3.4:

THEOREM 3.5. *Let $Q_n \to \infty$ $(n \to \infty)$ and consider the conditions*

$$\sup_n \left(\sum_{k=0}^{n-1} Q_k |\Delta^+ (a_{nk}/q_k)| + |Q_n a_{nk}/q_n| \right) < \infty \tag{3.1}$$

(3.2) $A_n Q/q \in c$ *for all* n (3.3) $A_n Q/q \in c_0$ *for all* n

(3.4) $\lim\limits_{n\to\infty} a_{nk} = 0$ *for all* k (3.5) $\lim\limits_{n\to\infty} a_{nk} = l_k$ *for all* $k = 0$

(3.6) $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} a_{nk} = 0$ (3.7) $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} a_{nk} = l$

Then

$A \in ((\overline{N}, q)_0, l_\infty)$ *if and only if* (3.1);
$A \in ((\overline{N}, q), l_\infty)$ *if and only if* (3.1) *and* (3.2);
$A \in ((\overline{N}, q)_\infty, l_\infty)$ *if and only if* (3.1) *and* (3.3);
$A \in ((\overline{N}, q)_0, c_0)$ *if and only if* (3.1) *and* (3.4);
$A \in ((\overline{N}, q)_0, c)$ *if and only if* (3.1) *and* (3.5);
$A \in ((\overline{N}, q), c_0)$ *if and only if* (3.1), (3.2), (3.4) *and* (3.6);
$A \in ((\overline{N}, q), c)$ *if and only if* (3.1), (3.2), (3.5) *and* (3.7).

The conditions for $A \in ((\overline{N}, q)_\infty, (\overline{N}, p)_\infty)$ *etc. are obtained by replacing the entries of* A *above by the entries of* $C = TA$ *where* $t_{nl} = p_l/P_l$ $(0 \leqslant l \leqslant n)$ *and* $t_{nl} = 0$ $(l > n)$ *for all* n.

3.2. Spaces of sequences of m^{th} order differences. Let m be a positive integer. We define the operators $\Delta^{(m)}$, $\Sigma^{(m)} \colon \omega \to \omega$ by

$$(\Delta^{(1)}x)_k = \Delta^{(1)}x_k = x_k - x_{k-1}, \quad (\Sigma^{(1)}x)_k = \Sigma^{(1)}x_k = \sum_{j=0}^{k} x_j \quad (k = 0, 1, \dots),$$
$$\Delta^{(m)} = \Delta^{(1)} \circ \Delta^{(m-1)}, \qquad \Sigma^{(m)} = \Sigma^{(1)} \circ \Sigma^{(m-1)} \quad (m \geqslant 2).$$

We shall write $X(\Delta^{(m)}) = X_{\Delta^{(m)}} = \{x \in \omega : \Delta^{(m)}x \in X\}$ for $X \in \{l_\infty, c, c_0\}$.

Theorem 3.6. ([10, Proposition 1, Theorem 1]) *(a) The sets* $l_\infty(\Delta^{(m)})$, $c(\Delta^{(m)})$ *and* $c_0(\Delta^{(m)})$ *are BK spaces,*

$$\|x\|_{\Delta^{(m)}} = \sup_k |(\Delta^{(m)}x)_k| = \sup_k \left| \sum_{j=0}^{m} (-1)^j \tbinom{m}{j} x_{k-j} \right|.$$

(b) We define the sequences $b^k(m)$ *by*

$$b_n^{(-1)}(m) = \binom{m+n}{n}, \quad \textit{and, for } k \geqslant 0,\ b_n^{(k)}(m) = \begin{cases} 0, & (n \leqslant k-1), \\ \binom{m+n-k-1}{n-k}, & (n \geqslant k). \end{cases}$$

Then every sequence $x = (x_k)_{k=0}^{\infty} \in c_0(\Delta^{(m)})$ *has a unique representation*

$$x = \sum_{k=0}^{\infty} \lambda_k(m) b^{(k)}(m) \quad \textit{where} \quad \lambda_k(m) = (\Delta^{(m)}x)_k \ (k = 0, 1, \dots),$$

and every sequence $x = (x_k)_{k=0}^{\infty} \in c(\Delta^{(m)})$ *has a unique representation*

$$x = lb^{(-1)}(m) + \sum_{k=0}^{\infty} (\lambda_k(m) - l) b^{(k)}(m) \quad \textit{where} \quad l = \lim_{k\to\infty} (\Delta^{(m)} x)_k.$$

Given any sequence a we define the sequence $R^{(m)}(a)$ by

$$R_k^{(1)}(a) = \sum_{j=k}^{\infty} a_j \ \ (k = 0, 1, \dots), \quad R^{(m)}(a) = R^{(1)}(R^{(m-1)}(a)) \ \ (m \geqslant 2)$$

provided the series converge. Further we write

$$R^{(m)}(X) = \{\, x \in \omega : R^{(m)}(x) \in X \,\} \quad \text{for any } X \subset \omega.$$

THEOREM 3.7. ([10, Theorem 3, Lemma 4]) *We put*

$$M_\infty^\beta(m) = (((k^m))^{-1} * cs) \cap R^{(m)}(l_1),$$
$$M_0^\beta(m) = \left(\bigcap_{v \in c_0^+} (\Sigma^{(m)} v)^{-1} * cs \right) \cap R^{(m)}(l_1).$$

Then

$$(c(\Delta^{(m)}))^\beta = (l_\infty(\Delta^{(m)}))^\beta = M_\infty^\beta(m), \quad (c_0(\Delta^{(m)}))^\beta = M_0^\beta(m),$$
$$(l_\infty(\Delta^{(m)}))^\beta \neq (c_0(\Delta^{(m)}))^\beta;$$
$$\|a\|^* = \sum_{k=0}^{\infty} |R_k^{(m)}| \ \ \textit{on} \ \ (c_0(\Delta^{(m)}))^\beta, \ (c(\Delta^{(m)}))^\beta \ \textit{and} \ (l_\infty(\Delta^{(m)}))^\beta.$$

Since obviously $A_n(b^{(-1)}(m)) = \sum_{j=0}^{\infty} \binom{m+j}{j} a_{nj}$ and, for $k \geqslant 0$,

$$A_n(b^{(k)}(m)) = \sum_{j=k}^{\infty} \binom{m-1+j-k}{j-k} a_{nj} \quad \text{for all } n,$$

we conclude from Theorems 3.6, 2.3 and 3.7:

THEOREM 3.8. *We consider the conditions*

$$M(l_\infty(\Delta^{(m)}), l_\infty) = \sup_n \|R^{(n)}(A_n)\|_1 < \infty, \tag{3.8}$$

(3.9) $A_n \in (k^{(m)})^{-1} * cs$

(3.10) $A_n \in \bigcap_{v \in c_0^+} (\Sigma^{(m)} v)^{-1} * cs$

(3.11) $\lim_{n\to\infty} A_n(b^{(k)}(m)) = 0, \ k \geqslant 0$

(3.12) $\lim_{n\to\infty} A_n(b^{(k)}(m)) = l_k, \ k \geqslant 0$

(3.13) $\lim_{n\to\infty} A_n(b^{(k)}(-1)) = 0$

(3.14) $\lim_{n\to\infty} A_n(b^{(k)}(-1)) = l_{-1}.$

Obviuosly $(l_\infty(\Delta^{(m)}), l_\infty) = (c(\Delta^{(m)}), l_\infty)$, *and*

$$\begin{array}{lll} A \in (l_\infty(\Delta^{(m)}), l_\infty) & \textit{if and only if} & (3.8)\ \textit{and}\ (3.9); \\ A \in (c_0(\Delta^{(m)}), l_\infty) & \textit{if and only if} & (3.8)\ \textit{and}\ (3.10); \\ A \in (c_0(\Delta^{(m)}), c_0) & \textit{if and only if} & (3.8), (3.10)\ \textit{and}\ (3.11); \\ A \in (c_0(\Delta^{(m)}), c) & \textit{if and only if} & (3.8), (3.10)\ \textit{and}\ (3.12); \\ A \in (c(\Delta^{(m)}), c_0) & \textit{if and only if} & (3.8), (3.9), (3.11)\ \textit{and}\ (3.13); \\ A \in (c(\Delta^{(m)}), c) & \textit{if and only if} & (3.8), (3.9), (3.12)\ \textit{and}\ (3.14). \end{array}$$

3.3. Spaces of sequences that are Λ-strongly convergent or bounded. Let $\mu = (\mu_n)_{n=0}^\infty$ be a nondecreasing sequence of positive reals tending to infinity. If $(n(\nu))_{\nu=0}^\infty$ is a sequence such that $0 = n(0) < n(1) < n(2) < \cdots$, then we shall write $K^{\langle\nu\rangle} = \{k \in \mathbf{Z} : n(\nu) \leqslant k \leqslant n(\nu+1) - 1\}$, and Σ_ν and $\max_\nu$ for the sum and maximum taken over all k in $K^{\langle\nu\rangle}$. We define the matrices $B = (b_{nk})_{n,k=0}^\infty$, $\tilde{B} = (\tilde{b}_{\nu k})_{\nu,k=0}^\infty$ and $\Delta(\mu)$ by

$$b_{nk} = \begin{cases} \dfrac{1}{\lambda_n} & (0 \leqslant k \leqslant n) \\ 0 & (k < n) \end{cases} \quad \text{and} \quad \tilde{b}_{\nu k} = \begin{cases} \dfrac{1}{\lambda_{n(\nu+1)}} & (k \in K^{\langle\nu\rangle}) \\ 0 & (k \notin K^{\langle\nu\rangle}). \end{cases}$$

$$\Delta_{nk}(\mu) = \begin{cases} -\mu_{n-1} & (k = n-1) \\ \mu_n & (k = n) \quad (n = 0, 1, \dots) \text{ where } \mu_{-1} = 0 \\ 0 & (\text{otherwise}). \end{cases}$$

The following sets were defined in [12]:

$$\begin{array}{ll} c_0(\mu) = ((c_0)_{[B]})_{\Delta(\mu)}, & \tilde{c}_0(\mu) = ((c_0)_{[\tilde{B}]})_{\Delta(\mu)}, \\ c(\mu) = \{x \in \omega : x - le \in c_0(\mu)\}, & \tilde{c}(\mu) = \{x \in \omega : x - le \in \tilde{c}_0(\mu)\}, \\ c_\infty(\mu) = ((l_\infty)_{[B]})_{\Delta(\mu)}, & \tilde{c}_\infty(\mu) = ((l_\infty)_{[\tilde{B}]})_{\Delta(\mu)}. \end{array}$$

THEOREM 3.9. ([8, Theorem 2(c)]) *The spaces* $c_0(\mu)$, $c(\mu)$ *and* $c_\infty(\mu)$ *are BK spaces with*

$$\|x\|' = \|B(|\Delta(\mu)(x)|)\|_\infty = \sup_{n \geqslant 0} \left(\frac{1}{\mu_n} \sum_{k=0}^{n} |\mu_k x_k - \mu_{k-1} x_{k-1}| \right);$$

$c_0(\mu)$ *has AK; every sequence* $x = (x_k)_{k=0}^\infty \in c_\mu$ *has a unique representation*

$$x = le + \sum_{k=1}^{\infty} (x_k - l) e^{(k)} \quad \textit{where } l \in \mathbf{C} \textit{ is such that } x - le \in c_0(\mu).$$

A sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ of positive reals is called *exponentially bounded* if there is an integer $m \geqslant 2$ such that for all integers ν there is at least one λ_n in the interval $[m^{\nu}, m^{\nu+1})$. It is known (cf. [7, Lemma1]) that a nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ of positive reals is exponentially bounded if and only if the following condition holds:

$$\text{There are reals } s \leqslant t \text{ in the open unit interval such that for some}$$
$$\text{subsequence } (\lambda_{n(\nu+1)})_{\nu=0}^{\infty}, \quad s \leqslant \frac{\lambda_{n(\nu)}}{\lambda_{n(\nu+1)}} \leqslant t \text{ for all } \nu = 0, 1, \dots \tag{E}$$

A subsequence $(\lambda_{n(\nu+1)})_{\nu=0}^{\infty}$ of an exponentially bounded sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ that satisfies condition (E) will be called an *associated subsequence.* From now on, let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ always be a nondecreasing exponentially bounded sequence of positive reals and $(\lambda_{n(\nu+1)})_{\nu=0}^{\infty}$ an associated subsequence.

Theorem 3.10. ([8, Theorem 2]) *We have $c_0(\Lambda) = \tilde{c}_0(\Lambda)$, $c(\Lambda) = \tilde{c}(\Lambda)$ and $c_\infty(\Lambda) = \tilde{c}_\infty(\Lambda)$. The norms $\|x\|'$ and*

$$\|x\| = \|\tilde{B}(|\Delta(\Lambda)(x)|)\|_\infty = \sup_{\nu \geqslant 0} \left(\frac{1}{\lambda_{n(\nu+1)}} \Sigma_\nu |\lambda_k x_k - \lambda_{k-1} x_{k-1}| \right)$$

are equivalent on $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$. Thus each of the spaces $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ is a BK space with $\|\cdot\|$ (cf. [18, Corollary 4.2.4, p. 56).

Theorem 3.11. ([9, Lemma2]) *We put*

$$\mathcal{C}(\Lambda) = \left\{ a \in \omega : \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max\nolimits_\nu \left| \sum_{k=n}^{\infty} \frac{a_k}{\lambda_k} \right| < \infty \right\},$$
$$\|a\|_{\mathcal{C}(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max\nolimits_\nu \left| \sum_{k=n}^{\infty} \frac{a_k}{\lambda_k} \right| \quad \textit{foa all } a \in \mathcal{C}(\Lambda).$$

Then $(c_0(\Lambda))^\beta = (c(\Lambda))^\beta = (c_\infty(\Lambda))^\beta = \mathcal{C}(\Lambda)$ and $\|a\|^* = \|a\|_{\mathcal{C}(\Lambda)}$ on $\mathcal{C}(\Lambda)$.

As in the previous sections, it is now easy to characterize the classes (X, Y) where $X = c_\infty(\Lambda)$, $c(\Lambda)$, $c_0(\Lambda)$ and $Y = l_\infty, c, c_0, c_\infty(\mu), c(\mu), c_0(\mu)$. For instance, if we put $\Delta_n(\mu_n a_{nj}) = \mu_n a_{nj} - \mu_{n-1} a_{n-1,j}$ and

$$M(c_\infty(\Lambda), c_\infty(\mu)) = \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)} \max_\nu \left| \sum_{j=k}^{\infty} \frac{1}{\lambda_j} \left(\frac{1}{\mu_m} \sum_{n \in N_m} \Delta_n(\mu_n a_{nj}) \right) \right| \right) \right),$$

then $A \in (c_\infty(\Lambda), c_\infty(\mu))$ if and only if $M(c_\infty(\Lambda), c_\infty(\mu)) < \infty$ and $A \in (c(\Lambda), c(\mu))$ if and only if $M(c_\infty(\Lambda), c_\infty(\mu)) < \infty$,

$$\lim_{m \to \infty} \left(\frac{1}{\mu_m} \sum_{n=0}^{m} |\Delta_n(\mu_n(a_{nk} - l_k))| \right) = 0 \quad \text{for all } k,$$
$$\lim_{m \to \infty} \left(\frac{1}{\mu_m} \sum_{n=0}^{m} \left| \Delta_n \left(\mu_n \left(\sum_{k=0}^{\infty} a_{nk} - l_k \right) \right) \right| \right) = 0.$$

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