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QUASIHYPONORMAL OPERATORS AND THE CONTINUITY OF THE APPROXIMATE POINT SPECTRUM

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Abstract. Let H be a separable Hilbert space. We write $\sigma(A)$ for the spectrum of $A \in B(H)$, $\sigma_a(A)$ and $\sigma_{ea}(A)$ for the approximate point and the essential approximate point spectrum of A. Operator $A \in B(H)$ is quasihyponormal if $\|A^*Ax\| \le \|A^2x\|$ for all $x \in H$.

In this paper we show that the approximate point spectrum σ_a and the essential approximate point spectrum σ_{ea} are continuous in the set of all quasihyponormal operators.

1. Introduction

Let H be a complex infinite-dimensional separable Hilbert space and let $B(H)$ ($K(H)$) denote the Banach algebra of all bounded operators (the ideal of all compact operators) on H. If $A \in B(H)$, then $\sigma(A)$ denotes the spectrum of A and $\rho(A)$ denotes the resolvent set of A. It is well-known that the following sets are semigroups of operators on H:

$$\Phi_+(H) = \{A \in B(H) : \mathcal{R}(A) \text{ is closed and } \dim \mathcal{N}(A) < \infty\}$$
$$\Phi_-(H) = \{A \in B(H) : \mathcal{R}(A) \text{ is closed and } \dim H/\mathcal{R}(A) < \infty\}.$$

The semigroup of semi-Fredholm operators is $\Phi(H) = \Phi_+(H) \cup \Phi_-(H)$. If A is semi-Fredholm and $\alpha(A) = \dim \mathcal{N}(A)$ and $\beta(A) = \dim H/\mathcal{R}(A)$, then we define an index by $i(A) = \alpha(A) - \beta(A)$. We also consider set

$$\Phi_+^-(H) = \{A \in \Phi_+(H) : i(A) \le 0\}.$$

For $A \in B(H)$, the following definitions are well-known:

$$\sigma_a(A) = \{\lambda \in \mathbb{C} : \inf_{x \in H,\, \|x\|=1} \|(A-\lambda)x\| = 0\} - \text{the approximate point spectrum,}$$

$$\sigma_w(A) = \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_0(H)\} \ - \text{ the Weyl spectrum ,}$$

$$\sigma_{ea}(A) = \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_+^-(H)\} - \text{the essential approximate point spectrum}$$

$$\sigma_{ab}(A) = \cap\{\sigma_a(A+K) : AK = KA, K \in K(H)\} \ - \text{ the Browder essential approximate point spectrum}$$

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Let $\pi_{00}(A)$ be the set of all $\lambda \in \mathbf{C}$ such that λ is an isolated point of $\sigma(A)$ and $0 < \dim \mathcal{N}(A-\lambda) < \infty$, let $\pi_0(A)$ be the set of all normal eigenvalues of A, that is the set of all isolated points of $\sigma(A)$ for which the corresponding spectral projection has finite-dimensional range. We say that A obeys Weyl's theorem [3], [7], if

$$\sigma_w(A) = \sigma(A) \setminus \pi_{00}(A)\,.$$

If (τ_n) is a sequence of compact subsets of $\mathbf{C}$, then its limit inferior is

$$\liminf \tau_n = \{\lambda \in \mathbf{C} :\ \text{there are } \lambda_n \in \tau_n \text{ with } \lambda_n \to \lambda\}$$

and its limit superior is

$$\limsup \tau_n = \{\lambda \in \mathbf{C} :\ \text{there are } \lambda_{n_k} \in \tau_{n_k} \text{ with } \lambda_{n_k} \to \lambda\}\,.$$

If $\liminf \tau_n = \limsup \tau_n$, then $\lim \tau_n$ is defined by this common limit. A mapping p, defined on $B(H)$, whose values are compact subsets of $\mathbf{C}$, is said to be upper (lower) semi-continuous at A, provided that if $A_n \to A$ then $\limsup p(A_n) \subset p(A)$ ($p(A) \subset \liminf p(A_n)$). If p is both upper and lower semi-continuous at A, then it is said to be continuous at A and in this case $\lim p(A_n) = p(A)$.

We say that $A \in B(H)$ is quasihyponormal provided that $\|A^*Ax\| \le \|A^2x\|$ for all $x \in H$. In this paper we show that the essential approximate point spectrum, the Browder essential approximate point spectrum and the approximate point spectrum are continuous functions in the set of all quasihyponormal operators.

2. General results

We say that $A \in B(H)$ is quasihyponormal provided that $\|A^*Ax\| \le \|A^2x\|$ for all $x \in H$. In [6], V. A. Erovenko showed that the spectrum is a continuous function in the set of all quasihyponormal operators. In this paper we show that the essential approximate point spectrum, the Browder essential approximate point spectrum and the approximate point spectrum are continuous functions in the set of all quasihyponormal operators.

Theorem 2.1. *If A_n^* are quasihyponormal operators in $B(H)$ and $A_n \to A$, then* $\lim_{n\to\infty} \sigma_{ea}(A_n) = \sigma_{ea}(A)$.

Proof. Since σ_{ea} is upper semi-continuous [4, Theorem 2.1], we have to show that $\sigma_{ea}(A) \subset \liminf \sigma_{ea}(A_n)$. Suppose that σ_{ea} is not lower semi-continuous at A. Then exists $\epsilon > 0$ and $\lambda \in \sigma_{ea}(A)$ such that $\lambda \notin (\sigma_{ea}(A_n))_\epsilon$ for every $n \in \mathbb{N}$ such that $n > n_0$. Hence, for $n > n_0$ we have that $\lambda \notin \sigma_{ea}(A_n)$. By [9] we have that $A_n - \lambda \in \Phi_+^-(H)$, i.e. $i(A_n - \lambda) \le 0$ and $\alpha(A_n - \lambda) < \infty$.

Now we consider two cases:

Case I: Let $\lambda \neq 0$ and let $n \in \mathbb{N}$ such that $n > n_0$. If $x \in \mathcal{N}(A_n^* - \overline{\lambda})$, then $A_n^*x = \overline{\lambda}x$ and since A_n^* is quasihypinormal operator we have that

$$\|A_nx\| = \|A_n(\frac{1}{\lambda}A_n^*x)\| = \frac{1}{|\lambda|}\|A_nA_n^*x\| \le \frac{1}{|\lambda|}\|(A_n^*)^2x\| \le |\lambda|\|x\|\,.$$

This fact implies that $((A_n-\lambda)x,(A_n-\lambda)x)\le 0$, so $x\in\mathcal{N}(A_n-\lambda)$. It proves that

$$\beta(A_n-\lambda)=\alpha(A_n-\lambda)^*\le\alpha(A_n-\lambda)$$

i.e. $i(A_n-\lambda)\ge 0$. Then by $i(A_n-\lambda)\ge 0$ and $A_n-\lambda\in\Phi_+^-(H)$ we have that $i(A_n-\lambda)=0$. Since $A_n-\lambda\to A-\lambda$ we have that $i(A-\lambda)=0$, i.e. $\beta(A-\lambda)=\alpha(A-\lambda)<\infty$.

Hence, we have that $\alpha(A-\lambda)<\infty$ and $i(A-\lambda)\le 0$, i.e. $A-\lambda\in\Phi_+^-(H) \implies \lambda\notin\sigma_{ea}(A)$. This is a contradiction.

Case II: Let $\lambda=0$ and let $n\in\mathbb{N}$ such that $n>n_0$. If $x\in\mathcal{N}((A_n^*)^2)$, then

$$0=\|(A_n^*)^2x)\|\ge\|A_nA_n^*x\|\ge 0 \quad\implies\quad A_nA_n^*x=0$$

and now

$$(A_n^*x,A_n^*x)=(x,A_nA_n^*x)=(x,0)=0$$

i.e. $x\in\mathcal{N}(A_n^*)$. Hence $a(A_n^*)=1$ and $\alpha(A_n^*)<\infty$, by [1, p. 57] we have that

$$\beta(A_n)=\alpha(A_n^*)\le\beta(A_n^*)=\alpha(A_n)\,,$$

i.e. $i(A_n-\lambda)\ge 0$ and the proof continuous as previous case. ∎

THEOREM 2.2. *If A_n^*, A^* are quasihyponormal operators in $B(H)$ and $A_n\to A$, then* $\lim_{n\to\infty}\sigma_{ab}(A_n)=\sigma_{ab}(A)$.

Proof. Suppose that $\lambda\in\sigma_a(A)\setminus\sigma_{ea}(A)$. Then $A-\lambda\in\Phi_+^-(H)$ and $0<\alpha(A-\lambda)<\infty$ [9]. If $\lambda\ne 0$, since A^* is quasihyponormal, by the proof of Theorem 2.1 we get that $\alpha((A-\lambda)^*)\le\alpha(A-\lambda)<\infty$. If $\lambda=0$, then by the proof of Theorem 2.1. we get again $\alpha(A^*)\le\alpha(A)=\beta(A^*)<\infty$. Anyway, we get $\alpha((A-\lambda)^*)\le\alpha(A-\lambda)<\infty$. Obviously, $i(A-\lambda)=\alpha(A-\lambda)-\alpha((A-\lambda)^*)\ge 0$. Since $A-\lambda\in\Phi_+^-(H)$, we get that $0=i(A-\lambda)=i((A-\lambda)^*)$, so $\overline{\lambda}\notin\sigma_w(A^*)$. It is well-known that quasihyponormal operators obey the Weyl's theorem [3], [7], so $\overline{\lambda}\in\pi_{00}(A^*)$ and λ is an isolated point of $\sigma(A)$. Now, λ is isolated in $\sigma_a(A)$ and by [9] we get that $\lambda\notin\sigma_{ab}(A)$. Hence, $\sigma_{ea}(A)=\sigma_{ab}(A)$.

Now we have that

$$\sigma_{ab}(A)=\sigma_{ea}(A)\subseteq\liminf\sigma_{ea}(A_n)\subseteq\liminf\sigma_{ab}(A_n)\,.$$

Since σ_{ab} is always upper semi-continuous we have that $\lim_{n\to\infty}\sigma_{ab}(A_n)=\sigma_{ab}(A)$. ∎

THEOREM 2.3. *If A_n^*, A^* are quasihyponormal operators in $B(H)$ and $A_n\to A$, then* $\lim_{n\to\infty}\sigma_a(A_n)=\sigma_a(A)$.

Proof. Since σ_a is upper semi-continuous [2], we have to show that

$$\sigma_a(A)\subset\liminf\sigma_a(A_n)\,.$$

Let λ_0 be in $\sigma_a(A)$. If $\lambda_0\in\sigma_{ea}(A)$, then by Theorem 2.1. we have that

$$\lambda_0\in\sigma_{ea}(A)\subset\liminf\sigma_{ea}(A_n)\subset\liminf\sigma_a(A_n)\,.$$

Suppose that $\lambda_0 \in \sigma_a(A) \setminus \sigma_{ea}(A)$. Now we have that λ_0 is an isolated point of $\sigma(A)$ and by [8, Theorem IV 3.16] we have that $\lambda_0 \in \liminf \sigma(A_n)$. There exists a sequence $\lambda_n \in \sigma(A_n)$ such that $\lambda_n \to \lambda$. If $\lambda_n \in \sigma(A_n) \setminus \sigma_a(A_n)$, then we have that $\alpha(A_n - \lambda_n) = 0$. Since A_n^* are quasihyponormal and $\beta(A_n - \lambda_n)^* = \alpha(A_n - \lambda_n) < \infty$ by the proof of Theorem 2.1. we have that

$$\beta(A_n - \lambda_n) = \alpha(A_n - \lambda_n)^* \leq \beta(A_n - \lambda_n)^* = \alpha(A_n - \lambda_n) = 0\,.$$

Then by conditions $\alpha(A_n - \lambda_n) = \beta(A_n - \lambda_n) = 0$ and $R(A_n - \lambda_n)$ is closed follows that $\lambda_n \notin \sigma(A_n)$. This is a contradiction and we get that $\lambda_n \in \sigma_a(A_n)$ and $\lambda \in \liminf \sigma_a(A_n)$. ■

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