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COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS AND COMPATIBLE MAPPINGS OF TYPE (A)

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Abstract. The purpose of this paper is to prove some theorems which generalize a theorem of M. Telči, K. Tas and B. Fisher on compatible mappings and compatible mappings of type (A).

Let S, T be two self-mappings of a metric space (X, d). Sessa [7] defines S and T to be weakly commuting if $d(STx, TSx) \leqslant d(Tx, Sx)$ for all x in X.

Jungck [3] defines T and S to be compatible iff $\lim_n d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Sx_n = \lim_n Tx_n = x$ for some $x \in X$. Clearly, commuting mappings are weakly commuting and weakly commuting maps are compatible, neither implication is reversible, Ex. 1 [9] and Ex. 2.2 [3].

Recently, Jungck, Muthy and Cho [4] have defined S and T to be compatible of type (A) if $\lim_n d(TSx_n, S^2x_n) = 0$ and $\lim_n d(STx_n, T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Sx_n = \lim_n Tx_n = t$ for some $t \in X$. Clearly, weakly commuting mappings are compatible of type (A). By [4], Ex. 2.2, it follows that this implication is not reversible. By [4], Ex. 2.1 and Ex. 2.2, it follows that the notions of compatible mappings and compatible mappings of type (A) are independent.

Lemma [4] *Let $S, T\colon (X, d) \to (X, d)$ be two mappings. If A and T are compatible of type (A) and $S(t) = T(t)$ for some $t \in X$, then $ST(t) = TT(t) = TS(t) = SS(t)$.*

Let $\mathcal{F}$ be the set of all functions $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that:

(i) f is isotone, i.e. if $t_1 \leqslant t_2$, then $f(t_1) \leqslant f(t_2)$ for all $t_1, t_2 \in \mathbf{R}_+$;

(ii) f is upper semi-continuous;

(iii) $f(t) < t$ for each $t > 0$.

The following theorem which generalizes the results from [1], [2], [5], [6] and [8] is proved in [10].

Theorem 1. *Let S, T, I and J be self-mappings of a complete metric space (X, d) satisfying*

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1° *$T(X) \subset I(X)$ and $S(X) \subset J(X)$;*

2° *the inequality*

$$[1+p\cdot d(Ix,Jy)]d(Sx,Ty) \leqslant p\max\{d(Ix,Sx)\cdot d(Jy,Ty), d(Ix,Ty)\cdot d(Jy,Sx)\}+ \\ + f(\max\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), \tfrac{1}{2}(d(Ix,Ty)+d(Jy,Sx))\}) \quad (1)$$

holds for all x, y in X, where $p \geqslant 0$ and $f \in \mathcal{F}$;

3° *one of S, T, I and J is continuous;*

4° *S and T weakly commute with I and J, respectively.*

Then S, T, I and J have a common fixed point z. Further, z is the unique common fixed point of S and I and T and J.

The purpose of this paper is to prove some theorems which generalize Theorem 1 for compatible mappings of type (A) and for compatible mappings.

Theorem 2. *Let S, T, I and J be self-mappings of a complete metric space (X,d) satisfying the conditions 1°, 2°, 3° of Theorem 1. and*

4° *S and I are compatible of type (A) and T and J are compatible of type (A).*

Then S, T, I and J have common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

Proof. Suppose x_0 is an arbitrary point in X. Then since 1° holds, we can define a sequence

$$\{Sx_0, Tx_1, Sx_2, Tx_3, \dots, Sx_{2n}, Tx_{2n+1}, \dots\} \quad (2)$$

inductively so that $Sx_{2n} = Jx_{2n+1}$, $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0,1,2,\dots$.

By [10], Lemma 2.3, the sequence (2) is a Cauchy sequence. Since X is complete, the sequence (2) converges to a point z in X. Hence z is also the limit of the sequences $\{Sx_{2n} = Jx_{2n+1}\}$ and $\{Tx_{2n-1} = Ix_{2n}\}$ of (2).

Now suppose that I is continuous. Then the sequence $\{I^2x_{2n}\}$ converges to Iz and $d(SIx_{2n}, Iz) \leqslant d(SIx_{2n}, I^2x_{2n}) + d(I^2x_{2n}, Iz)$. Since I is continuous and S and I are compatible of type (A) (by Prop. 2.5 [4]), letting n tend to infinity, it follows taht the sequence $\{SIx_{2n}\}$ also converges to Iz. Using (1), we have

$$[1+pd(I^2x_{2n}, Jx_{2n+1})]d(SIx_{2n}, Tx_{2n+1}) \leqslant \\ \leqslant p\cdot\max\{d(I^2x_{2n}, SIx_{2n})\cdot d(Jx_{2n+1}, Tx_{2n+1}), d(I^2x_{2n}, Tx_{2n+1}\cdot d(Jx_{2n+1}, SIx_{2n}) \\ + f(\max\{d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), \\ \tfrac{1}{2}(d(I^2x_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, SIx_{2n}))\}).$$

Letting n tend to infinity, we have

$$[1+pd(Iz,z)]d(Iz,z) \leqslant pd^2(Iz,z) + f(\max\{d(Iz,z),0\}) = pd^2(Iz,z) + f(d(Iz,z)).$$

It follows that $d(Iz,z) \leqslant f(d(Iz,z))$ and so $Iz = z$.

Using inequality (1) again, we have

$$[1 + d(Iz, Jx_{2n+1})]d(Sz, Tx_{2n+1}) \leqslant p \cdot \max\{d(Iz, Sz) \cdot d(Jx_{2n+1}, Tx_{2n+1}), d(Iz, Tx_{2n+1}) \cdot d(Jx_{2n+1}, Sz)\} + f(\max\{d(Iz, Jx_{2n+1}), d(Iz, Sz), d(Jx_{2n+1}, Tx_{2n+1}), \tfrac{1}{2}(d(Iz, Tx_{2n+1}) + d(Jx_{2n+1}, Sz))\}).$$

Letting n tend to infinity, we have $d(Sz, z) \leqslant f(d(Sz, z))$ and it follows that $Sz = z$ by (iii).

Now, since $S(X) \subset J(X)$, there exists a point u in X such that $Ju = z$. Using inequality (1), we have

$$d(z, Tu) = d(Sz, Tu) \leqslant f(\max\{d(z, Tu), \tfrac{1}{2}d(z, Tu)\}) = f(d(z, Tu))$$

and it follows that $Tu = z$ by (iii). Since T and J are compatible of type (A) and $Ju = Tu = z$ by Lemma follows that $TJu = JTu$ and so $Tz = TJu = JTu = Jz$. Thus from (1) we have $Tz = Jz = z$. Therefore, z is a common fixed point of S, T, I and J if I is continuous.

The same result holds if we assume that J is continuous instead of I.

Now suppose that S is continuous. Then the sequence $\{S^2x_{2n}\}$ converges to Sz. We have $d(ISx_{2n}, Sz) \leqslant d(ISx_{2n}, S^2x_{2n}) + d(S^2x_{2n}, Sz)$. Since S is continuous and S and T are compatible of type (A), letting n tend to infinity, it follows that $\{ISx_{2n}\}$ converges to Sz.

From inequality (1) we have

$$[1 + pd(ISx_{2n}, Jx_{2n+1})]d(S^2x_{2n}, Tx_{2n+1}) \leqslant p \cdot \max\{d(ISx_{2n}, S^2x_{2n}) \cdot d(Jx_{2n+1}, Tx_{2n+1}), d(ISx_{2n}, Tx_{2n+1}) \cdot d(Jx_{2n+1}, S^2x_{2n})\} + f(\max\{d(ISx_{2n}, Jx_{2n+1}), d(ISx_{2n}, S^2x_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), \tfrac{1}{2}[d(ISx_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, S^2x_{2n})]\}).$$

Letting n tend to infinity and using (ii), we obtain $d(Sz, z) \leqslant f(d(Sz, z))$ and so $Sz = z$ by (iii).

Since $S(X) \subset J(X)$, there exists a point u in X such that $Ju = z$. Using inequality (1) we have

$$[1 + pd(ISx_{2n}, Ju)]d(S^2x_{2n}, Tu) \leqslant p \cdot \max\{d(ISx_{2n}, S^2x_{2n}) \cdot d(Ju, Tu), d(ISx_{2n}, Tu) \cdot d(Ju, S^2x_{2n})\} + f(\max\{d(ISx_{2n}, Ju), d(ISx_{2n}, S^2x_{2n}), d(Ju, Tu), \tfrac{1}{2}(d(ISx_{2n}, Tu) + d(Ju, S^2x_{2n}))\})$$

and letting n tend to infinity, we have $d(z, Tu) \leqslant f(d(z, Tu))$. Thus $Tu = z$ by (iii). Since T and J are compatible of type (A) and $Tu = Ju = z$, then $Tz = TJu = JTu = Jz$. From (1) we now have

$$[1 + pd(Ix_{2n}, Jz)]d(Sx_{2n}, Tz) \leqslant p \max\{d(Ix_{2n}, Sx_{2n}) \cdot d(Jz, Tz), d(Ix_{2n}, Tz) \cdot d(Jz, Sx_{2n})\} + f(\max\{d(Ix_{2n}, Jz), d(Ix_{2n}, Sx_{2n}), d(Jz, Tz), \tfrac{1}{2}(d(Ix_{2n}, Tz) + d(Jz, Sx_{2n}))\}).$$

Letting n tend to infinity, we have $d(z,Tz) \leqslant f(d(z,Tz))$ and so $Tz = z = Jz$. On the other hand, since $T(X) \subset I(X)$, there exists a point u' in X such that $Iu' = z$ and from (1) we have $d(Su',z) \leqslant f(d(z,Su'))$, implying that $Iu' = Su' = z$. Then by the Lemma it follows that $z = Sz = SIu' = ISu' = Iz$. Thus, z is a common fixed point of S, T, I and J if S is continuous. The same result holds if we suppose that T is continuous instead of S.

Now suppose that S and I have another fixed point w. Then from (1) we have

$$[1+pd(w,z)]d(w,z) = [1+pd(Iw,Jz)]d(Sw,Tz) \leqslant pd^2(w,z) + f(d(w,z)),$$

which implies $w = z$. Analogously, z is the unique common fixed point of T and J. This completes the proof of the Theorem. ∎

For $f\colon (X,d) \to (X,d)$ we denote $F_f = \{\, x \in X \,:\, x = f(x)\,\}$.

THEOREM 3. *Let I, J, S, T be mappings from a metric space (X,d) into itself. If the inequality (1) holds for all x, y in X, then $(F_I \cap F_J) \cap F_S = (F_I \cap F_J) \cap F_T$.*

Proof. Let $x \in (F_I \cap F_J) \cap F_S$. Then

$$\begin{aligned}[1+pd(Ix,Jx)]d(x,Tx) &= [1+pd(Ix,Jx)]d(Sx,Tx) \leqslant p\max\{d(Ix,Sx)\cdot d(Jx,Tx),\\ &d(Ix,Tx)\cdot d(Jx,Sx)\} + f(\max\{d(Ix,Jx), d(Ix,Sx), d(Jx,Tx),\\ &\tfrac{1}{2}(d(Ix,Tx)+d(Jx,Sx))\}).\end{aligned}$$

Then $d(x,Tx) \leqslant f(d(x,Tx))$ which implies $x = Tx$. Thus $(F_I \cap F_J) \cap F_S \subset (F_I \cap F_J) \cap F_T$. Similarly, we have $(F_I \cap F_J) \cap F_T \subset (F_I \cap F_J) \cap F_S$. ∎

THEOREM 4. *Let I, J and $\{T_i\}_{i\in\mathbf{N}}$ be mappings from a complete metric space (X,d) into itself such that*

1° *$T_2(X) \subset I(X)$ and $T_1(X) \subset J(X)$;*

2° *one of I, J, T_1 and T_2 is continuous;*

3° *the pairs (T_1,I) and (T_2,J) are compatible of type (A);*

4° *the inequality*

$$\begin{aligned}[1+pd(Ix,Jy)]d(T_ix,T_{i+1}y) &\leqslant p\cdot\max\{d(Ix,T_ix)\cdot d(Jy,T_{i+1}y),\\ &d(Ix,T_{i+1}y)\cdot d(Jy,T_ix)\} + f(\max\{d(Ix,Jy), d(Ix,T_ix), d(Jy,T_{i+1}y),\\ &\tfrac{1}{2}(d(Ix,T_{i+1}y)+d(Jy,T_ix))\}) \qquad (3)\end{aligned}$$

holds for each x, y in X and all $i \in \mathbf{N}$, where $p \geqslant 0$.

Then I, J and $\{T_i\}_{i\in\mathbf{N}}$ have a common fixed point.

Proof. By Theorem 2, I, J, T_1 and T_2 have a common fixed point which is unique common fixed point for I and T_2 and thus is unique common fixed point for I, J and T_1. By Theorem 3, $(F_I \cap F_J) \cap F_{T_1} = (F_I \cap F_J) \cap F_{T_2}$ and thus z is the unique fixed point for I, J and T_2. By (3) and Theorem 3 it follows that z is unique common fixed point for I, J and $\{T_i\}_{i\in\mathbf{N}}$. ∎

REMARK. Similarly, we have two theorems, Theorem 2' and Theorem 4' with "compatibility" instead "compatibility of type (A)".

We conclude this paper by showing that our results (Theorem 2 and Remark) are appreciably superior than the result of Telci, Tas and Fisher (Theorem 1). To see the validity and generality of our results, we observe that even if we allow all the four maps to be continuous, we are able to keep the maps compatible or compatible of type (A) but not weakly commuting. It is well known that every weakly commuting pair of maps is compatible but the converse is not necessarily true (see [3]). Moreover, every compatible pair of maps is compatible of type (A) if the maps are continuous (see [4]).

EXAMPLE. Let $X = [0,1]$ with the Euclidean metric d. Define S, T, I, $J\colon X \to X$ by $Sx = \frac{1}{8}x^{1/2}$, $Tx = \frac{1}{4}x^{1/2}$, $Ix = \frac{1}{2}x^{1/2}$, $Jx = x^{1/2}$ for all x in X. Then really $T(X) = [0,1/4] \subset [0,1/2] = I(X)$ and $S(X) = [0,1/8] \subset [0,1] = J(X)$. By routine check-up (take a sequence $\{x_n\} \subset X$ such that $x_n \to 0$ as $n \to \infty$) one can see that the pairs of maps $\{S,I\}$ and $\{T,J\}$ are compatible as well as compatible of type (A).

Consider the function $f(t) = ht$ for $t > 0$, where $1/4 \leqslant h < 1/2$. Then f satisfies (i)–(iii) and so $f \in \mathcal{F}$. Furthermore, we obtain

$$\begin{aligned} d(Sx,Ty) &= \frac{1}{4}d(Ix,Jy) \\ &\leqslant f(\max\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), \tfrac{1}{2}(d(Ix,Ty)+d(Jy,Sx))\}) \qquad (*) \end{aligned}$$

and for any $p \geqslant 0$,

$$\begin{aligned} pd(Ix,Jy)\cdot d(Sx,Ty) &\leqslant pd(Ix,Ty)\cdot d(Jy,Sx) \\ &\leqslant p\max\{d(Ix,Sx)\cdot d(Jy,Ty), d(Ix,Ty)\cdot d(Jy,Sx)\}. \qquad (**) \end{aligned}$$

(Observe that for all $a,b \geqslant 0$, we have the inequality $\frac{1}{4}(a-b)^2 \leqslant |a = \frac{1}{4}b||\frac{1}{4}a - b|$). Adding $(*)$ and $(**)$, (1) holds for all x, y in X, where $p \geqslant 0$ and $f \in F$. Thus all the hypothesis of Theorem 2 are satisfied. Clearly, 0 is the unique common fixed point of S, T, I and J. But Theorem 1 is not applied since the maps are not weakly commuting. For, if $x \neq 0$ then $SIx \neq ISx$ and $d(SIx, ISx) \leqslant d(Sx, Ix)$ implies $1 \leqslant 3\sqrt{2}\,x^{1/4} \to 0$ as $x \to 0$ which is a contradiction. Similarly, we can show that T and J are not weakly commuting.

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