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ON WEIGHTED PARAMETRIC INFORMATION MEASURE

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Abstract. Shannon's measure of information plays a very important role for measuring diversity in plants and animals. But this measure does not deal with growth models other than exponential. Thus we need a parametric model which is suitable to all types of distributions. In this communication, one such measure depending upon two real parameters has been developed under a set of certain axioms and its particular cases have been studied.

1. Introduction

Let $P = \{(p_1, p_2, \dots, p_n), 0 \leqslant p_i \leqslant 1, \sum_{i=1}^n p_i = 1\}$ be a finite discrete probability distribution of a set of n events $E = (E_1, E_2, \dots, E_n)$ on the basis of an experiment whose predicted probability distribution is $Q = \{(q_1, q_2, \dots, q_n), 0 \leqslant q_i \leqslant 1, \sum_{i=1}^n q_i = 1\}$ and the utility distribution is $U = \{(u_1, u_2, \dots, u_n), u_i \geqslant 0\}$.

In Information theory, the following measures are well known:

$$I_n(P) = -\sum_{i=1}^n p_i \log p_i, \tag{1.1}$$

$$I_n(P;U) = -\sum_{i=1}^n u_i p_i \log p_i, \tag{1.2}$$

$$I_n(P/Q) = \sum_{i=1}^n p_i \log(p_i/q_i) \tag{1.3}$$

and

$$I_n(P;Q) = -\sum_{i=1}^n p_i \log q_i. \tag{1.4}$$

The information-theoretic measures (1.3) and (1.4), respectively, known as Kullback's [3] relative information and Kerridge's [2] inaccuracy are of great significance in statistical estimation and Physics. The measure (1.2) known as quantitative-qualitative measure of information introduced by Belis and Guiasu [1] has found deep applications in Theory of questionnaries.

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The Shannon's [8] measure (1.1) possesses very nice mathematical properties and is very useful in many fields. Specifically, in life sciences, we have diversity of animal and plant types which can be measured using Shannon's model. The greater the value of this measure, greater will be the diversity.

Where Shannon's model gives useful applications, it has a drawback. It always leads to exponential family of distributions. Since there are families of distributions other than exponential and there are laws of population growth other than exponential, we cannot confine ourselves to exponential families only and consequently, Shannon's measure may not be much applicable. Moreover, there are many factors like Physiography, Topography, Biotic interference, Anthropogenic, Climatic, Edaphic etc. which affect the diversity in plants. Let α, β etc. represent such factors upon which the information regarding such a cybernetic system $E_i/u_i/p_i/q_i$ depends. In this communication, we develop a generalized weighted information measure depending upon two real parameters α and β.

2. Weighted information measure of type (α, β)

Let the weighted measure of the type (α, β) be denoted by

$$I_n^{(\alpha,\beta)}[p_1, p_2, \dots, p_n; q_1, q_2, \dots, q_n; u_1, u_2, \dots, u_n].$$

To derive this measure, we assume that it satisfies the following axioms:

Axiom I. The weighted masure of information $I_n^{(\alpha,\beta)}[P;Q;U]$ is symmetric function of its arguments p_i's, q_i's and u_i's.

Axiom II. The measure $I_n^{(\alpha,\beta)}[P;Q;U]$ is proportional to its utility u_i, that is, for each non-negative λ, the following holds

$$I_n^{(\alpha,\beta)}[p_i; q_i; \lambda u_i] = \lambda I_n^{(\alpha,\beta)}[p_i; q_i; u_i].$$

Axiom III. The measure $I_n^{(\alpha,\beta)}[P;Q;U]$ satisfies the branching property of the type (α, β) given by

$$\begin{aligned} I_{n+1}^{(\alpha,\beta)}&[p_1, p_2, \dots, p_{i-1}, v_{i1}, v_{i2}, p_{i+1}, \dots, p_n;\\ &q_1, q_2, \dots, q_{i-1}, h_{i1}, h_{i2}, q_{i+1}, \dots, q_n; u_1, u_2, \dots, u_{i-1}, r_{i1}, r_{i2}, u_{i+1}, \dots, u_n]\\ &= I_n^{(\alpha,\beta)}[P;Q;U] + p_i^\alpha q_i^\beta I_2^{(\alpha,\beta)}\left[\frac{v_{i1}}{p_i}, \frac{v_{i2}}{p_i}; \frac{h_{i1}}{q_i}, \frac{h_{i2}}{q_i}; r_{i1}, r_{i2}\right] \end{aligned}$$

for every $v_{i1} + v_{i2} = p_i > 0$, $h_{i1} + h_{i2} = q_i > 0$ and $\dfrac{r_{i1}v_{i1} + r_{i2}v_{i2}}{v_{i1} + v_{i2}} = u_i > 0$; $i = 1$, 2, … , n; where $\alpha \neq 1$, $\beta \neq 0$ are arbitrary parameters.

The following theorem gives the generalized form of the weighted measure of type (α, β):

Theorem 1. *The quantitive-qualitative measure of type (α, β) satisfying axioms I to III together with the continuity of $I_n^{(\alpha,\beta)}[P;Q;U]$ determine the functions*

I_n as

$$I_n^{(\alpha,\beta)}[P;Q;U] = C(\alpha,\beta)\left[\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{\beta} - \sum_{i=1}^{n} u_i p_i\right], \qquad \alpha \neq 1, \beta \neq 0, \tag{2.1}$$

where $C(\alpha,\beta)$ ($\neq 0$) is a constant depending upon parameters α and β.

Proof. Parkash and Singh [4] proved the following result:

If

$$v_{ij} \geqslant 0, \quad j = 1,2,\ldots,m_i, \quad \sum_{j=1}^{m_i} v_{ij} = p_i > 0, \quad \sum_{i=1}^{n} p_i = 1,$$
$$h_{ij} \geqslant 0, \quad j = 1,2,\ldots,m_i, \quad \sum_{j=1}^{m_i} h_{ij} = q_i > 0, \quad \sum_{i=1}^{n} q_i = 1$$

and

$$r_{ij} \geqslant 0, \quad j = 1,2,\ldots,m_i, \quad \frac{\sum_{j=1}^{m_i} r_{ij} v_{ij}}{\sum_{j=1}^{m_i} v_{ij}} = u_i > 0,$$

for every $i = 1, 2, \ldots, n$, then

$$I_{nm_n}[V;H;R] = I_n[P;Q;U] + \sum_{i=1}^{n} p_i I_{m_i}\left[\frac{v_{i1}}{p_i},\ldots,\frac{v_{im_i}}{p_i};\frac{h_{i1}}{q_i},\ldots,\frac{h_{im_i}}{q_i};r_{i1},\ldots,r_{im_i}\right]. \tag{2.2}$$

With the help of Axiom III, the equation (2.2) takes the following form:

$$I_{nm_n}^{(\alpha,\beta)}[V;H;R] = I_n^{(\alpha,\beta)}[P;Q;U] + \sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta} I_{m_i}^{(\alpha,\beta)}\left[\frac{v_{i1}}{p_i},\ldots,\frac{v_{im_i}}{p_i};\frac{h_{i1}}{q_i},\ldots,\frac{h_{im_i}}{q_i};r_{i1},\ldots,r_{im_i}\right]. \tag{2.3}$$

Next, replacing m_i by m in (2.3) and substituting $v_{ij} = 1/mn$, $h_{ij} = 1/rs$, $r_{ij} = 1$ and $p_i = 1/m$, $q_i = 1/r$, $u_i = 1$, for every $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$ where m, n, r, s are positive integers such that $1 \leqslant m \leqslant r$, $1 \leqslant n \leqslant s$, we get

$$F^{(\alpha,\beta)}[mn;rs;1] = F^{(\alpha,\beta)}[m;r;1] + \left(\frac{1}{m}\right)^{\alpha-1}\left(\frac{1}{r}\right)^{\beta} F^{(\alpha,\beta)}[n;s;1] \tag{2.4}$$

where

$$F^{(\alpha,\beta)}[n;s;1] = I^{(\alpha,\beta)}\left[\frac{1}{n},\ldots,\frac{1}{n};\frac{1}{s},\ldots,\frac{1}{s};1,\ldots,1\right]. \tag{2.5}$$

Because of the symmetry of $I_n^{(\alpha,\beta)}[P;Q;1]$, (2.4) can be written as

$$F^{(\alpha,\beta)}[mn;rs;1] = F^{(\alpha,\beta)}[n;s;1] + \left(\frac{1}{n}\right)^{\alpha-1}\left(\frac{1}{s}\right)^{\beta} F^{(\alpha,\beta)}[m;r;1]. \tag{2.6}$$

Equations (2.4) and (2.6) give

$$\frac{F^{(\alpha,\beta)}[m;r;1]}{\left[1-\left(\frac{1}{m}\right)^{\alpha-1}\left(\frac{1}{r}\right)^{\beta}\right]}=\frac{F^{(\alpha,\beta)}[n;s;1]}{\left[1-\left(\frac{1}{n}\right)^{\alpha-1}\left(\frac{1}{s}\right)^{\beta}\right]}=-C(\alpha,\beta)\quad\text{(say)}. \tag{2.7}$$

Thus

$$F^{(\alpha,\beta)}[m;r;1]=C(\alpha,\beta)\Big[\Big(\frac{1}{m}\Big)^{\alpha-1}\Big(\frac{1}{r}\Big)^{\beta}-1\Big]. \tag{2.8}$$

Now we prove the result for rationals and the continuiuty of $I_n^{(\alpha,\beta)}$ proves the result for reals.

If m, r, r_i and t_i are positive integers such that $\sum_{i=1}^n r_i=m$, $\sum_{i=1}^n t_i=r$ and if we put $v_{ij}=1/m$, $h_{ij}=1/r$, $r_{ij}=1$ and $p_i=r_i/m$, $q_i=t_i/r$, $u_i=1$, for every $i=1,\,2,\,\ldots\,n$, then equation (2.3) becomes

$$I_n^{(\alpha,\beta)}[P;Q;1]=F^{(\alpha,\beta)}[m;r;1]-\sum_{i=1}^n p_i^\alpha q_i^\beta F^{(\alpha,\beta)}[r_i;t_i;1]. \tag{2.9}$$

Using (2.8), (2.9) gives

$$\begin{aligned}I_n^{(\alpha,\beta)}[P;Q;1]&=C(\alpha,\beta)\Big[\Big\{\Big(\frac{1}{m}\Big)^{\alpha-1}\Big(\frac{1}{r}\Big)^{\beta}-1\Big\}-\sum_{i=1}^n p_i^\alpha q_i^\beta\Big\{\Big(\frac{1}{r_i}\Big)^{\alpha-1}\Big(\frac{1}{t_i}\Big)^{\beta}-1\Big\}\Big]\\&=C(\alpha,\beta)\Big[\sum_{i=1}^n p_i^\alpha q_i^\beta-1\Big].\end{aligned}$$

Thus

$$I_n^{(\alpha,\beta)}[P;Q;1]=C(\alpha,\beta)\Big[\sum_{i=1}^n p_i^\alpha q_i^\beta-\sum_{i=1}^n p_i\Big]. \tag{2.10}$$

Next, setting $u_i=1$ and $\lambda=u_i$ in Axiom II, we get

$$I_n^{(\alpha,\beta)}[P;Q;U]=C(\alpha,\beta)\Big[\sum_{i=1}^n u_i p_i^\alpha q_i^\beta-\sum_{i=1}^n u_i p_i\Big],$$

which is (2.1). ■

Particular cases.

Case I. Equation (2.1) under the boundary conditions

$$I_2[P;P;U]=0 \tag{2.11}$$

and

$$I_2\Big[1,0;\frac{1}{2},\frac{1}{2};1,1\Big]=1 \tag{2.12}$$

gives

$$I_n^\alpha[P;Q;U]=(2^{\alpha-1}-1)^{-1}\Big[\sum_{i=1}^n u_i p_i^\alpha q_i^{1-\alpha}-\sum_{i=1}^n u_i p_i\Big],\quad \alpha\neq 1. \tag{2.13}$$

Taking $\alpha \to 1$, (2.13) gives

$$I_n[P;Q;U] = \sum_{i=1}^{n} u_i p_i \log \frac{p_i}{q_i},$$

a result studied by Taneja and Ruteja [6]. Also on ignoring utility, that is, taking $u_i = 1$, (2.13) gives

$$I_n^{\alpha}[P;Q] = (2^{\alpha-1}-1)^{-1}\left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - 1\right], \quad \alpha \neq 1,$$

which is a result studied by Rathie and Kannappan [5] and it reduces to Kullback's [3] relative information in the limiting case as $a \to 1$.

Case II. Equation (2.1) under the boundary conditions

$$I_3[p_1,p_2,p_3;q_1,q_2,q_2;u_1,u_2,u_3] = I_2\left[p_1, p_2+p_3; q_1, q_2; u_1, \frac{u_2p_2+u_3p_3}{p_2+p_3}\right] \tag{2.14}$$

and

$$I_2\left[\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};1,1\right] = 1 \tag{2.15}$$

gives

$$I_n^{\beta}[P;Q;U] = (2^{-\beta}-1)^{-1}\left[\sum_{i=1}^{n} u_i p_i q_i^{\beta} - \sum_{i=1}^{n} u_i p_i\right], \quad \beta \neq 0. \tag{2.16}$$

Taking $\beta \to 0$, $I_n^{\beta}[P;Q;U]$ reduces to $-\sum_{i=1}^{n} u_i p_i \log q_i$, a result studied by Taneja and Tuteja [7].

Further on ignoring utility, (2.16) gives

$$I_n^{\beta}[P;Q] = (2^{-\beta}-1)^{-1}\left[\sum_{i=1}^{n} p_i q_i^{\beta} - 1\right], \quad \beta \neq 0,$$

which reduces to Kerridge's [2] inaccuracy in the limiting case as $\beta \to 0$.

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