МАТЕМАТИЧКИ ВЕСНИК
49 (1997), 85–88

UDK 517.535
оригинални научни рад
research paper

ON CERTAIN CLASS OF RATIONAL FUNCTIONS WHOSE DERIVATIVES HAVE POSITIVE REAL PARTS

S. B. Joshi and Milutin Obradović

Abstract. For a certain class of rational functions we give a sufficient condition such that their derivatives have positive real parts.

1. Introduction and preliminaries

Let A denote the class of functions f analytic in the unit disc $U = \{\, z : |z| < 1 \,\}$ with $f(0) = f'(0) - 1 = 0$. Let R denote the subclass of A for which $\operatorname{Re}\{f'(z)\} > 0$, $z \in U$. It is well known that the previous condition implies univalence of functions $f \in A$.

In their paper [3] Ozaki and Nunokawa proved the following criterion for univalence in U.

THEOREM A. *If $f \in A$ and*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \quad z \in U, \tag{1}$$

then f is univalent in U.

But the condition (1) doesn't imply $\operatorname{Re}\{f'(z)\} > 0$, $z \in U$. For example, for the function

$$f(z) = \frac{z}{(1 - z\sqrt{k})^2}, \qquad 0 < k \leqslant 1,$$

we have that

$$f'(z) = \frac{1 + z\sqrt{k}}{(1 - z\sqrt{k})^3}; \qquad \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| = |-kz^2| < k, \quad z \in U;$$

and $\operatorname{Re}\{f'(i)\} = \operatorname{Re}\{(1-6k+k^2)/(1+k)^3\} < 0$, for $0.17157\ldots = 3 - 2\sqrt{2} < k \leqslant 1$.

AMS Subject Classification: 30 C 45

The work of the second author was supported by Ministry of Science and technology RS, grant number 04M03 through Matematički institut

In that sense we have just showed that for $3-2\sqrt{2}<k\leqslant 1$ the condition

$$\left|\frac{z^2f'(z)}{f^2(z)}-1\right|<k, \quad z\in U \tag{2}$$

does not imply that $f\in R$.

In this paper we give some additional conditions to (2) such that they provide $f\in R$. We apply these results to a certain class of rational functions.

We also need the next lemma.

Lemma 1. [1] *Let Ω be a subset of the complex plane $\mathbf{C}$ and suppose that the function $\psi\colon \mathbf{C}^2\times U\to\mathbf{C}$ satisfies the conditions $\psi(ix,y;z)\notin\Omega$, for all real x, $y\leqslant -(1+x^2)/2$ and all $z\in U$. If the function p is analytic in U, $p(0)=1$ and $\psi[p(z),zp'(z);z]\in\Omega$, then $\operatorname{Re}\{p(z)\}>0$.*

2. Results and consequences

First, we prove

Theorem 1. *Let $f\in A$ satisfy the condition (2) and the condition*

$$\left|\arg\frac{z}{f(z)}\right|\leqslant\frac{1}{2}\arctan\frac{\sqrt{1-k^2}}{k}, \qquad z\in U, \tag{3}$$

for some $0<k\leqslant 1$. Then $f\in R$.

Proof. If we put $f'(z)=p(z)$ and $(z/f(z))^2=g(z)$, then by the conditions (2) and (3) of the theorem we obtain

$$|g(z)p(z)-1|<k \quad\text{and}\quad |\arg g(z)|=2\left|\arg\frac{z}{f(z)}\right|\leqslant\arctan\frac{\sqrt{1-k^2}}{k}, \tag{4}$$

for some $0<k\leqslant 1$. Also, if we put $g(z)=u+iv$, then by (4) we get $v^2\leqslant\frac{1-k^2}{k^2}u^2$ and so

$$|g(z)(ix)-1|^2-k^2=|(u+iv)(ix)-1|-k^2=(u^2+v^2)x^2+2vx+1-k^2\geqslant 0.$$

By Lemma 1 we conclude that $\operatorname{Re}\{p(z)\}=\operatorname{Re}\{f'(z)\}>0$, $z\in U$, i.e. $f\in R$. ∎

From Theorem 1 we easily obtain the following

Corollary 1. *Let $f\in A$ satisfy the condition (2) and let*

$$\left|\frac{z}{f(z)}-1\right|\leqslant\sqrt{\frac{1-k}{2}}, \qquad z\in U,$$

for some $0<k\leqslant 1$. Then $f\in R$.

In the next theorem we give an application to a certain class of rational functions.

THEOREM 2. *Let*

$$f(z) = z\left\{1 + \sum_{n=1}^{\infty} b_n z^n\right\}^{-1} \tag{5}$$

and let

$$\sum_{n=2}^{\infty} (n-1)|b_n| \leqslant k \quad \textit{and} \quad \sum_{n=1}^{\infty} |b_n| \leqslant \sqrt{\frac{1-k}{2}}, \tag{6}$$

for some $0 \leqslant k \leqslant 1$. *Then* $f \in R$.

Proof. First let us suppose that the function (5) satisfies (6) and let $0 < k < 1$ and $\sum_{n=2}^{\infty} |b_n| > 0$. Then we have

$$\begin{aligned}\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| &= \left|-z^2\left[\frac{1}{f(z)} - \frac{1}{z}\right]'\right| \leqslant \left|\left[\frac{1}{f(z)} - \frac{1}{z}\right]'\right| \\ &= \left|\sum_{n=2}^{\infty} (n-1) b_n z^{n-2}\right| < \sum_{n=2}^{\infty} (n-1)|b_n| \leqslant k, \quad z \in U.\end{aligned}$$

and

$$\left|\frac{z}{f(z)} - 1\right| = \left|\sum_{n=1}^{\infty} b_n z^{n-1}\right| \leqslant \sum_{n=1}^{\infty} |b_n| \leqslant \sqrt{\frac{1-k}{2}}, \quad z \in U.$$

Hence from Corollary 1 we conclude that $f \in R$.

The other cases are simple. Namely, for $k = 0$ or for $\sum_{n=2}^{\infty} |b_n| = 0$ we get $b_2 = b_3 = \cdots = 0$ with $|b_1| \leqslant \sqrt{2}/2$, and the function (5) becomes $f(z) = z/(1 + b_1 z)$. For this function we can obtain that $\mathrm{Re}\{f'(z)\} > 0$, $z \in U$, for $|b_1| \leqslant \sqrt{2}/2$, i.e. that $f \in R$. For $k = 1$ we obtain the function $f(z) = z$, which also belongs to R. ■

Finally we have

THEOREM 3. *Let the function* f *be given by (5) and let*

$$\sum_{n=2}^{\infty} (n-1)|b_n| + 2\left(\sum_{n=1}^{\infty} |b_n|\right)^2 \leqslant 1. \tag{7}$$

Then $f \in R$.

Proof. Let us put $\sum_{n=2}^{\infty} (n-1)|b_n| = k$. Then from (7) we have that $0 \leqslant k \leqslant 1$ and $\sum_{n=1}^{\infty} |b_n| \leqslant \sqrt{(1-k)/2}$, which by Theorem 2 means that $f \in R$. ■

EXAMPLE. For the function $f(z) = z\left\{1 + \sum_{n=1}^{\infty} \frac{z^n}{2^{n+1}}\right\}^{-1}$ we have that $b_n = 2^{-(n+1)}$ and $\sum_{n=2}^{\infty} (n-1)|b_n| + 2\left(\sum_{n=1}^{\infty} |b_n|\right)^2 = 1$. By the previuous theorem we conclude that this function belongs to R, i.e. it is univalent with $\mathrm{Re}\{f'(z)\} > 0$, $z \in U$.

We note that the method given here is the same as in [2].

REFERENCES

[**1**] S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 167–171.

[**2**] M. Obradović, *Starlikeness and certain class of rational functions*, to appear

[**3**] S. Ozaki, M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc. **33** (2) (1972), 392–394.

(received 14.09.1995, in revised form 04.06.1997.)

Department of Mathematics, Walchand College of Engineering, Sangli, 416415 India

Department of Mathematics, Faculty of Technology and Metallurgy, University of Belgrade, 4 Karnegieva St., 11000 Belgrade, Yugoslavia