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CORRECTIONS TO "SOME GENERALIZATIONS OF T_D-SPACES" AND "A GENERALIZATION OF NORMAL SPACES"

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Abstract. Some corrections to the papers "Some Generalizations of T_D-Spaces (Mat. Vesnik **34** (1982), 221–230)" and "A Generalization of Normal Spaces (ibid. **35** (1983), 1–10)" are given.

1. Semi-T_D spaces [1]

The first part of the proofs of Theorems 1.6 and 2.4 of [1] are incorrect. The following two Theorems and their proofs provide the statements and proofs of the first parts of the Theorems 1.6 and 2.4 of [1]. The proofs of Theorems 1.6 and 2.4 [1] given in the paper prove the second parts of the statements of the Theorems.

THEOREM 1.1. *A semi-T_1 space is semi-T_D.*

Proof. Let X be a semi-T_1 space and $x \in X$. Then either $\{x\}$ is nowhere dense or $\{x\} \subseteq \operatorname{int}(\operatorname{cl}\{x\})$ [3] where $\operatorname{int} A$ and $\operatorname{cl} A$ respectively denote the interior and closure of a set A. If $\{x\}$ is nowhere dense, $\operatorname{cl}(X - \operatorname{cl}\{x\}) = X$ and therefore, $X - \operatorname{cl}\{x\} \subseteq X - d\{x\} \subseteq \operatorname{cl}(X - \operatorname{cl}\{x\})$. So, $X - d\{x\}$ is semi-open, $d\{x\}$ being the derived set of $\{x\}$. If $\{x\}$ is not nowhere dense, $\{x\} \subseteq \operatorname{int}(\operatorname{cl}\{x\}) = \operatorname{scl}\{x\}$, the semi-closure of $\{x\}$ [3]. Since X is semi-T_1, $\{x\} = \operatorname{int}(\operatorname{cl}\{x\})$. Hence $d\{x\} = \operatorname{cl}\{x\} - \{x\}$, the boundary of an open set and hence a nowhere dense set. Therfore $X - d\{x\}$ is semi-open. Thus $d\{x\}$ is semi-closed. ■

THEOREM 1.2. *A pairwise semi-T_1 space is pairwise semi-T_D.*

Proof. Let $(X, \mathbf{T}_1, \mathbf{T}_2)$ be pairwise semi-T_1. Then $(X, \mathbf{T}_1)$ and $(X, \mathbf{T}_2)$ are semi-T_1 and hence both are semi-T_D, in view of the above Theorem. For $x \in X$, $\mathbf{T}_1$-$d\{x\}$ is $\mathbf{T}_1$-semi-closed and $\mathbf{T}_2$-$d\{x\}$ is $\mathbf{T}_2$-semi-closed. Suppose $y \notin (\mathbf{T}_1\text{-}d\{x\} \cap \mathbf{T}_2\text{-}d\{x\})$. If $y \notin (\mathbf{T}_1\text{-}d\{x\} \cup \mathbf{T}_2\text{-}d\{x\})$, then there is a $\mathbf{T}_1$-semi-open set U and a $\mathbf{T}_2$-semi-open set V, each containing y, and each having empty intersection with $(\mathbf{T}_1\text{-}d\{x\} \cap \mathbf{T}_2\text{-}d\{x\})$. Now suppose $y \in (\mathbf{T}_1\text{-}d\{x\} \cup \mathbf{T}_2\text{-}d\{x\})$ and $y \notin (\mathbf{T}_1\text{-}d\{x\} \cap \mathbf{T}_2\text{-}d\{x\})$. Suppose $y \in \mathbf{T}_1\text{-}d\{x\}$ and $y \notin \mathbf{T}_2\text{-}d\{x\}$. Then there is a $\mathbf{T}_2$-semi-open set

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V_2 containing y such that $V_2 \cap (\mathbf{T}_1\text{-}d\{x\} \cap \mathbf{T}_2\text{-}d\{x\}) = \varnothing$. Since $y \in \mathbf{T}_1\text{-}d\{x\}$, $(X - \mathbf{T}_1\text{-}d\{x\}) \cup \{y\}$ is a $\mathbf{T}_1$-semi-open set containing y. For, $X - \mathbf{T}_1\text{-}d\{x\}$ is $\mathbf{T}_1$-semi-open and every $\mathbf{T}_1$-open set containing y contains x and $x \in (X - \mathbf{T}_1\text{-}d\{x\})$. Hence $y \in \mathbf{T}_1\text{-cl}(X - \mathbf{T}_1\text{-}d\{x\})$. Thus $(X - \mathbf{T}_1\text{-}d\{x\}) \cup \{y\}$ is $\mathbf{T}_1$-semi-open and $((X - \mathbf{T}_1\text{-}d\{x\}) \cup \{y\}) \cap (\mathbf{T}_1\text{-}d\{x\} \cap \mathbf{T}_2\text{-}d\{x\}) = \varnothing$, since $y \notin \mathbf{T}_2\text{-}d\{x\}$ and $(X - \mathbf{T}_1\text{-}d\{x\}) \cap \mathbf{T}_1\text{-}d\{x\} = \varnothing$. The case when $y \in \mathbf{T}_2\text{-}d\{x\}$ and $y \notin \mathbf{T}_1\text{-}d\{x\}$ can be similarly handled. Therefore $\mathbf{T}_1\text{-}d\{x\} \cap \mathbf{T}_2\text{-}d\{x\}$ is both $\mathbf{T}_1$-semi-closed as well as $\mathbf{T}_2$-semi-closed. Thus $(X, \mathbf{T}_1, \mathbf{T}_2)$ is semi-T_D. ■

The following example of Prof. Hindman of the Depatrment of Mathematics, Howard University, Washington, D. C. shows that arbitrary product of semi-T_D spaces need not be semi-T_D and hence Theorem 1.15 and its analogue in bitopological spaces, Theorem 2.8, are incorrect.

Let $X = \{0, 1\}$, and $\mathbf{T} = \{\varnothing, \{0\}, X\}$. For each $n \in \mathbf{N}$, let $X_n = X$ and let $Y = \prod X_n$. X is T_D and hence is a semi-T_D space. The derived set of $\{(0)\}$ is $Y - \{(0)\}$ which is not semi-closed in Y.

However, as we shall prove in the following theorem, the product of finitely many semi-T_D spaces is semi-T_D. We shall state the following Lemma without proof.

Lemma 1.3. *If A, B, C and D are any four sets, then $A \times B - C \times D = ((A - C) \times B) \cup (A \times (B - D))$.*

Theorem 1.4. *If X and Y are two semi-T_D spaces, then $X \times Y$ is semi-T_D.*

Proof. Let $(x, y) \in X \times Y$. Then $d\{(x, y)\} = \text{cl}\{(x, y)\} - \{(x, y)\} = \text{cl}(\{x\} \times \{y\}) - (\{x\} \times \{y\}) = ((\text{cl}\{x\} - \{x\}) \times \text{cl}\{y\}) \cup \text{cl}\{x\} \times (\text{cl}\{y\} - \{y\})$, in view of the above Lemma. Therefore, $d\{(x, y)\} = (d\{x\} \times \text{cl}\{y\}) \cup (\text{cl}\{x\} \times d\{y\})$. Now $X \times Y - d\{(x, y)\} = X \times Y - ((d\{x\} \times \text{cl}\{y\}) \cup (\text{cl}\{x\} \times d\{y\})) = [((X - d\{x\}) \times Y) \cup (X \times (Y - \text{cl}\{y\}))] \cap [((X - \text{cl}\{x\}) \times Y) \cup (X \times (Y - d\{y\}))] = ((X - \text{cl}\{x\}) \times Y) \cup ((X - d\{x\}) \times (Y - d\{y\})) \cup ((X - \text{cl}\{x\}) \times (Y - \text{cl}\{y\})) \cup (X \times (Y - \text{cl}\{y\}))$. This is a union of three open sets and one semi-open set and hence is semi-open. Thus $d\{(x, y)\}$ is semi-closed. ■

Corollary 1.5. *Product of a finite collection of semi-T_D spaces is semi-T_D.*

2. s-Normal spaces

There are counterexamples to show that Lemma 7 in [2] is incorrect. To state the correct proposition we need the following definition.

Definition 2.1 [2] A real valued function f on X is said to be quasi-lower semi-continuous, denoted as q-l.s.c., (respectively, quasi-upper semi-continuous, denoted q-u.s.c.) if the set $\{x : f(x) > b\}$ (respectively, $\{x : f(x) < b\}$) is a semi-open subset of X where b is a real number.

LEMMA 2.2 *Let D be any dense subset of the space of positive real numbers with relative usual topology. If to each $t \in D$ there corresponds a semi-open subset U_t of a space X such that $t < s$ in D implies that $\operatorname{scl} U_t \subseteq U_s$ and $\bigcup_{t\in D} U_t = X$, then the function f defined as $f(x) = \inf\{t : x \in U_t\}$ is quasi-upper semmi-continuous and quasi-lower semi-continuous.*

The proof of Lemma 7 in [2] proves that the function satisfying the hypothesis of the Lemma is in fact q-l.s.c. and q-u.s.c.

The purpose of Lemma 7 in [2] is to prove an analogue of Urysohn's Lemma for s-normal spaces. However, here we shall prove analogues of Urysohn's Lemma for s-normal spaces and for semi-normal spaces without using the above Lemma. Thus the change in Lemma 7 in [2] does not affect Theorem 8 or any result consequential to Theorem 8 in [2].

THEOREM 2.3. *A space X is s-normal if and only if for any two disjoint semi-closed subsets A and B of X, there exists a semi-continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ for every $x \in A$ and $f(x) = 1$ for every $x \in B$.*

Proof. Let A and B be two disjoint semi-closed subsets of X and let X be s-normal. Then there are disjoint semi-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Consider the characteristic function $\chi_{\operatorname{scl} V}$. $\chi^{-1}_{\operatorname{scl} V}(a,1] = \operatorname{scl} V$, a semi-open set, being the semi-closure of a semi-open set. $\chi^{-1}_{\operatorname{scl} V}[0,a) = X - \operatorname{scl} V$, a semi-open set and $\chi^{-1}_{\operatorname{scl} V}(a,b) = \varnothing$ where $0 < a < b < 1$. Also $\chi_{\operatorname{scl} V}(A) = 0$ and $\chi_{\operatorname{scl} V}(B) = 1$ and $\chi_{\operatorname{scl} V}$ is semi-continuous.

The converse is easy to prove. ■

THEOREM 2.4. *A space X is semi-normal if and only if for any two disjoint closed subsets A and B of X, there is a semi-continuous function $f\colon X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.*

Proof. Let X be semi-normal and let A and B be two disjoint closed subsets of X. Then there are disjoint semi-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Now, $\operatorname{scl} V$ and $\operatorname{scl} U$ are disjoint semi-open sets and if h is the characteristic function of $\operatorname{scl} V$, then h is semi-continuous, $h(A) = 0$ and $h(B) = 1$.

Converse is easy to prove. ■

The proof of Theorem 19 in [2] proves only the following and hence the Theorem 19 should be stated as follows:

THEOREM 2.5. *If X is an s-normal space and f and g are functions on X such that f is q-l.s.c. and g is q-u.s.c. and $g(x) \leq f(x)$ for every $x \in X$, then there is a q-u.s.c. and q-l.s.c. function h on X such that $g(x) \leqslant h(x) \leqslant f(x)$.*

REFERENCES

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