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H-PROJECTING IN n-DIMENSIONAL EUCLIDEAN SPACE E^n

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Abstract. Several types of projecting in n-dimensional Euclidean spaces are known. In this article we define a new type of projecting of the n-dimensional Euclidean space onto its fixed plane. We shall prove some properties of this projecting. It will be shown that so defined projecting is a central projecting with an $(n-3)$-dimensional subspace as a center.

1. Central projecting

By E^n we denote the n-dimensional Euclidean space. An m-dimensional subspace of E^n will be denoted by E^m. It is known that E^n can be extended to the projective n-dimensional space P^n by adding a hyperplane E^{n-1}_∞. The subspaces of P^n will be denoted the same way as the subspaces of E^n. The lower index ∞ will denote that a subspace of P^n is in E^{n-1}_∞. If E^n_1 and E^n_2 are subspaces of P^n, then their intersection is also a subspace. If $n_1+n_2-n \geq 0$ and the subspaces E^n_1 and E^n_2 are in general position, their intersection is the subspace $E^{n_1} \cap E^{n_2} = E^{n_1+n_2-n}$.

Let M be a point of E^n and $\{S_{1\infty}, \dots, S_{(n-2)\infty}\}$ a simplex of a subspace E^{n-3}_∞. The points $S_{1\infty}, \dots, S_{(n-2)\infty}$, M determine a subspace E^{n-2}_M of E^n. Let E^2_0 be a fixed plane of E^n in general position with respect to the simplex $\{S_{1\infty}, \dots, S_{(n-2)\infty}\}$, i.e. such that they span E^n. We define that $E^2_0 \cap E^{n-2}_M = E^0 = M'$, is the projection of the point M by a subspace E^{n-3}_∞. The subspace E^{n-2}_M is called the projecting subspace. To determine the projection of any other point N onto the plane E^2_0, it is sufficient to intersect that plane by the subspace determined by the points $S_{1\infty}, \dots, S_{(n-2)\infty}$, N. The subspace E^{n-3}_∞ is called the center of projecting of E^n onto the plane E^2_0. The point M' is called the central projection of the point M by the center E^{n-3}_∞.

2. Projecting of E^n by E^{n-3}_∞

Let $Ox_1 \dots x_n$ be a coordinate system of E^n. Let $X_{i\infty} = x_i \cap E^{n-1}_\infty$ $(i = 1, \dots, n)$ and let $E^{n-1}_{1,\dots,n-1}$ be the coordinate hyperplane $Ox_1 \dots x_{n-1}$. If M

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is a point of E^n then its projection onto the hyperplane $E^{n-1}_{1,\dots,n-1}$ is $M_{1,\dots,n-1} = E^{n-1}_{1,\dots,n-1} \cap MX_{n\infty}$. Let H_n be a point of the line $X_{1\infty}X_{n\infty}$, $H_n \neq X_{1\infty}$, $X_{n\infty}$. The point $M^n_{1,\dots,n-1} = H_nM \cap X_{1\infty}M_{1,\dots,n-1}$ is the projection of M by H_n onto the hyperplane $E^{n-1}_{1,\dots,n-1}$. If H_{n-1} is a point of the line $X_{1\infty}X_{(n-1)\infty}$, then the procedure of projecting will be continued from the points $X_{(n-1)\infty}$ and H_{n-1}.

We shall obtain the point $M_{1,\dots,n-2} = (M_{1,\dots,n-1})_{1,\dots,n-2} = E^{n-2}_{1,\dots,n-2} \cap M_{1,\dots,n-1}X_{(n-1)\infty}$ and the point $M^n_{1,\dots,n-2} = \left(M^n_{1,\dots,n-1}\right)_{1,\dots,n-2} = E^{n-2}_{1,\dots,n-2} \cap M^n_{1,\dots,n-1}X_{(n-1)\infty}$ as the projections of the points $M_{1,\dots,n-1}$ and $M^n_{1,\dots,n-1}$ from $X_{(n-1)\infty}$ onto the coordinate subspace $E^{n-2}_{1,\dots,n-2} = Ox_1 \dots x_{n-2}$. Also $M^{n-1}_{1,\dots,n-2} = (M_{1,\dots,n-1})^{n-1}_{1,\dots,n-2} = E^{n-2}_{1,\dots,n-2} \cap H_{(n-1)}M_{1,\dots,n-1}$, will be the projection of the point $M_{1,\dots,n-1}$ from H_{n-1} onto the subspace $E^{n-2}_{1,\dots,n-2}$.

After $n-2$ such steps, we shall obtain the set of $n-1$ points:

$M_{1,2} = \left((\dots((M_{1,\dots,n-1})_{1,\dots,n-2})\dots)_{1,2,3}\right)_{1,2}$,

$M^3_{1,2} = \left((\dots((M_{1,\dots,n-1})_{1,\dots,n-2})\dots)_{1,2,3}\right)^3_{1,2}$,

$M^4_{1,2} = \left((\dots((M_{1,\dots,n-1})_{1,\dots,n-2})\dots)^4_{1,2,3}\right)_{1,2}$,

$\dots$

$M^{n-1}_{1,2} = \left((\dots((M_{1,\dots,n-1})^{n-1}_{1,\dots,n-2})\dots)_{1,2,3}\right)_{1,2}$,

$M^n_{1,2} = \left((\dots((M^n_{1,\dots,n-1})_{1,\dots,n-2})\dots)_{1,2,3}\right)_{1,2}$.

We call this projecting the H-projecting of E^n onto the plane $E^2_{1,2} = Ox_1, x_2$. We shall prove that the central projecting defined in section 1 is actually an H-projecting.

LEMMA 1. *Let E^{n-k-l} be a subspace of E^n and let $\{S_{1\infty}, \dots, S_{(k+l)\infty}\}$ be a simplex of E^{k+l-1}_∞ in general position with respect to E^{n-k-l}. Let E^{k-1}_∞ be the (afine) span of $\{S_{1\infty}, \dots, S_{k\infty}\}$, E^{l-1}_∞ the span of $\{S_{(k+1)\infty}, \dots, S_{(k+l)\infty}\}$, and E^{n-l} the span of E^{n-k-l} and E^{k-1}_∞. If M' is the central projection of M from E^{l-1}_∞ onto the subspace E^{n-l}, M'' the central projection of M' from E^{k-1}_∞ onto the subspace E^{n-k-l}, and M''' the central projection of M from E^{k+l-1}_∞ onto the subspace E^{n-k-l}, then $M''' = M''$.*

Proof. Let E^l be the span of E^{l-1}_∞ and M, E^{k+l} the span of E^{k+l-1}_∞ and M, and E^k the span of E^{k-1}_∞ and M'. Then $E^l \subset E^{k+l}$ implies $M' \in E^{k+l}$, which implies $E^k \subset E^{k+l}$, and the last relation in turn implies $M'' \in E^{k+l} \cap E^{n-k-l} = \{M'''\}$. ∎

THEOREM 1. *The point $M_{1,2}$ defined by the H-projecting is equal to the point M', defined by the central projecting for $S_{i\infty} = X_{i\infty}$, $i = 3, \dots, n$, and $E^2_0 = Ox_1x_2$.*

Proof. We shall prove the theorem by induction. The statement is true for $n = 3$, we assume it is true for $n = m - 1$, and let $n = m$. By Lemma 1 the point M' is the central projection of the point $M_{1,2,\dots,m-1}$ from the span of the points $X_{i\infty}$, $i = 3, \dots, m-1$, onto the plane E^2_0, which is (by induction) the point $M_{1,2}$. ∎

We shall also prove the following essential property of the H-projecting.

THEOREM 2. *The points $M_{12}, M^3_{1,2}, \dots, M^n_{1,2}$ are on a line parallel to the x_1 axis.*

Proof. The planes which contain the points H_i and $X_{i\infty}$ do contain the point $X_{1\infty} \in H_iX_{i\infty}$. Hence, 2-dimensional planes spanned by the points H_i, $X_{i\infty}$ and $M_{1,2,\dots,i}$ intersect the coordinate planes $E^{i-1}_{1,2,\dots,i-1}$ along the lines which are parallel to the x_1 axis. Since $M^i_{1,2,\dots,i-1} = E^{i-1}_{1,2,\dots,i-1} \cap M_{1,2,\dots,i}H_i$, and $M_{1,2,\dots,i-1} = E^{i-1}_{1,2,\dots,i-1} \cap M_{1,2,\dots,i}X_{i\infty}$, we conclude that the lines $M^i_{1,2,\dots,i-1}M_{1,2,\dots,i-1}$ are paralel to the x_1 axis.

By central and H projectings onto the plane Ox_1x_2 the lines parallel to the x_1 axis remain parallel to it. Hence, the lines $M_{12}, M^i_{1,2}$ are parallel to the x_1 axis, and therefore they coincide, as the central projections of the lines $M^i_{1,2,\dots,i-1}M_{1,2,\dots,i-1}$ onto the plane E^2_0. ∎

By an H-projecting the point M is maped onto an $(n-1)$-touple of colinear points $M_{12}, M^3_{1,2}, \dots, M^n_{1,2}$. We shall prove that this correspondence is bijective.

THEOREM 3. *The mappings $M \xrightarrow{H} (M_{1,\dots,i}, M^{i+1}_{1,\dots,i}, \dots, M^n_{1,\dots,i})$, $i = 2, 3, \dots, n-1$, are bijections of E^n onto the set of $(n-i+1)$-touples of points (of the coordinate planes $E^i_{1,2,\dots,i}$) which are on lines parallel to the x_1-axis.*

Proof. We need only to prove that given an $(n-i+1)$-tuple of points $(N_1,\dots,N_{n-i+1})$ of a line from $E^i_{1,\dots,i}$ parallel to the x_1 axis, there is a unique point M such that

$$(N_1,\dots,N_{n-i+1}) = (M_{1,\dots,i}, M^{i+1}_{1,\dots,i},\dots,M^n_{1,\dots,i}). \tag{1}$$

The proof will go by induction. Let $i=n-1$. The point M, if it exists, is on the lines $N_1X_{n\infty}$ and N_2H_n. Since $N_1=N_2$, or $N_1N_2 \ni X_{1\infty}$ and $X_{1\infty}\in X_{n\infty}H_n$, the lines $N_1X_{n\infty}$ and N_2H_n are coplanar nonparallel and intersect at a unique point M.

Assuming the statement is true for $i=k\le n-1$ we prove it is true for $i=k-1$. Let us suppose that (1) holds for some point M and $i=k-1$. As we have just proved, the points $N_1=M_{1,\dots,k-1}$ and $N_2=M^k_{1,\dots,k-1}$ give rise to a unique point $N\in E^k_{1,2,\dots,k}$ such that $N=M_{1,2,\dots,k}$. The point $N=M_{1,2,\dots,k}$ in turn determines the line through it parallel to the x_1 axis which, as we have shown in the proof of the previous theorem, should contain the points $M^j_{1,2,\dots,k}$, $j=k+1,\dots,n$. The points $M^j_{1,2,\dots,k}$, $j=k+1,\dots,n$, are therefore unique intersections of the line through $N=M_{1,2,\dots,k}$ which is parallel to the x_1 axis, and the lines through the points $N_{j-k+2}=M^j_{1,\dots,k-1}$, $j=k+1,\dots,n$, which are parallel to the x_k axis. Now, using the induction, we conclude that $n-k+1$-tuple $(M_{1,\dots,k}, M^{k+1}_{1,\dots,k},\dots,M^n_{1,\dots,k})$ uniquely determines M. ■

REMARK. If the coordinate system $Ox_1\dots x_n$ is orthogonal, and the directions H_i, $i=3,\dots,n$ disect the right angles defind by the directions $X_{1\infty}$ and $X_{i\infty}$, then $M_{1,2}M^i_{1,2}$ equals to the distance of the point M to the hyperplane $E^{n-1}_{1,\dots,i-1,i+1,\dots,n}$.

REFERENCES

[**1**] Glazunov, E. A., Četveruhin, N. F., *Aksonometriya*, Moskva, 1953.

[**2**] Filipov, P. V., *Nachertatelnaya geometriya mnogomernogo prostranstva i ee prilozheniya*, Izd-vo Leningrad. un-ta, 1979.

[**3**] Janić, M., *Projekcija euklidskog prostora E^n na neku njegovu ravan*, Zbornik radova, Tehnički Fakultet, Bor, 1979.

[**4**] Pryanishnikova, Z. I., *Obobshchenie proekcii E. S. Fedorova*, Metody nachertatelnoĭ geometrii i ee prilozheniya, Moskva, 1955.

[**5**] Šnajder, Z., *Određivanje tragova ravni u četvorodimenzionalnom prostoru i tragova $(n-2)$-dimenzionalnoga prostora u n-dimenzionalmom prostoru*, Bull. Soc. math. phys. R. P. de Serbie, **IX**, 1–2, (1957).

[**6**] Šnajder, Z., *Eine Eigenschaft der Spurenbestimmung eines $(n-2)$-dimensionalen Raumes in einem E_n Raum*, Bull. Soc. math. phys. R. P. de Serbie, **XI**, 1–4, (1959).

[**7**] Bell, P. O., *Generalized theorems of Desarques for n-dimensional space*, Proc. Amer. Math. Soc. **6** (1955), 675–681.

[**8**] Schoute, P. H., *Mehrdimensionale Geometrie, t. I, Die linearen Räume*, Leipzig, 1902.

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