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OCTAHEDRAL NONCOMPACT HYPERBOLIC SPACE FORMS WITH FINITE VOLUME

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Abstract. Following Poincarè's geometric method, we construct two new nonorientable noncompact hyperbolic space forms by the regular octahedron in Fig. 1. The construction is motivated by Thurston's example [6], discussed also by Apansov [1] in details. Our new space forms will be denoted by

$$\tilde{D}_1 = H^3/G_1 \quad \text{and} \quad \tilde{D}_2 = H^3/G_2,$$

where $\tilde{D}_1$ and $\tilde{D}_2$ are obtained by pairing faces of D via isometries of groups G_1 and G_2, respectively, acting discontinuously and freely on the hyperbolic 3-space H^3 (Fig. 2, Fig. 3). These groups are defined by generators and relations in Sect. 3. The complete computer classification of possible space forms by our octahedron will be discussed in [4], where it turns out that our two space forms are isometric, i.e. G_1 and G_2 are conjugated by an isometry φ of H^3, i.e. $G_2 = \varphi^{-1}G_1\varphi$,

$$G_1 = (g_1, g_2, \bar{g}_1, \bar{g}_2 \text{ —— } g_1\bar{g}_1^{-1}g_2\bar{g}_2^{-1} = g_1g_1g_2g_2 = \bar{g}_1\bar{g}_1\bar{g}_2\bar{g}_2 = 1),$$
$$G_2 = (t_1, t_2, \bar{g}_1, \bar{g}_2 \text{ —— } t_1\bar{g}_1^{-1}t_2^{-1}\bar{g}_2 = t_1t_2t_1^{-1}t_2^{-1} = \bar{g}_1\bar{g}_1\bar{g}_2\bar{g}_2 = 1).$$

1. Introduction

A complete connected Riemannian n-manifold of constant sectional curvature, i.e. a space form, can be considered as an orbit space $\boldsymbol{M}^n/G$, where $\boldsymbol{M}^n$ is one of the spaces $\boldsymbol{S}^n$, $\boldsymbol{E}^n$, $\boldsymbol{H}^n$ (spherical, Euclidean and hyperbolic n-space, respectively) and G is an isometry group acting discontinuously and freely on $\boldsymbol{M}^n$. Having investigated hyperbolic space forms in earlier papers [3,5], by the method of polyhedron identification due to Poincarè, we shall construct two isometry groups G_i $(i = 1, 2)$ acting discontinuously and freely on the hyperbolic space $\boldsymbol{H}^3$. Each group is given by the same noncompact fundamental Dirichlet polyhedron $\boldsymbol{D}$ (Fig. 1) and by pairing its faces via isometries. The identifying isometries generate the corresponding group G_i. In this way we can obtain all the hyperbolic space forms $\tilde{\boldsymbol{D}} = \boldsymbol{H}^3/G$, whose fundamental polyhedron is the given octahedron.

This paper is related with [3,5], where more details of the method are described.

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Fig. 1

2. The construction of the octahedron $\boldsymbol{D}$

Take a triangle $\boldsymbol{A}_0\boldsymbol{A}_1\boldsymbol{A}_2$ on a hyperbolic plane $\boldsymbol{H}^2$ with a right angle at the vertex $\boldsymbol{A}_1$, with angle $\angle \boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_0 = \pi/4$ and with the ideal vertex $\boldsymbol{A}_0$ (Fig. 1). Thus the angle of parallelism $\Pi(\boldsymbol{A}_1\boldsymbol{A}_2)$ equals $\pi/4$. By reflections in the lines $\boldsymbol{A}_1\boldsymbol{A}_2$ and $\boldsymbol{A}_0\boldsymbol{A}_2$, we construct a regular hyperbolic quadrangle $\boldsymbol{P}$, with all ideal vertices. On the plane containing the line $\boldsymbol{A}_1\boldsymbol{A}_2$ and the line $\boldsymbol{o}$ perpendicular to the plane of the polygon $\boldsymbol{P}$ at the point $\boldsymbol{A}_2$ we construct the rays $\boldsymbol{A}_1\boldsymbol{A}_3$ and $\boldsymbol{A}_1\boldsymbol{A}_3^*$, so that both of them are parallel to the line $\boldsymbol{o}$, with the angle of parallelism $\Pi(\boldsymbol{A}_1\boldsymbol{A}_2) = \pi/4$.

We get the polyhedron $\boldsymbol{D}$, a regular tetragonal bipyramid with ideal vertices. All its faces form angles $\pi/4$ with the base, and all its intersecting faces form angles $\pi/2$. $\boldsymbol{D}$ will be our required noncompact fundamental Dirichlet polyhedron to the centre A_2.

Since all vertices of this polyhedron are ideal points, all its edges concurrent to such an ideal vertex belong to a parabolic bundle. A horosphere centred at this ideal vertex is orthogonal to this bundle and intersects the bipyramid $\boldsymbol{D}$ in a horospherical polygon. A horosphere is isometric to the Euclidean plane $\boldsymbol{E}^2$, so these polygons are Euclidean.

In this sense, the apices of this polyhedron $\boldsymbol{D}$ are related to Euclidean regular quadrangles with angles $\pi/2$.

3. Identifications on $\boldsymbol{D}$

Now, we are going to identify the faces of $\boldsymbol{D}$. The result is symbolized in the Schlegel diagrams of $\tilde{\boldsymbol{D}}_1 = \boldsymbol{H}^3/G_1$ (Fig. 2) and $\tilde{\boldsymbol{D}}_2 = \boldsymbol{H}^3/G_2$ (Fig. 3), where we omit f's from the face symbols.

Fig. 2

The pairs of faces $\boldsymbol{g}_j^{-1}$ and $\boldsymbol{g}_j$ $(j = 1, 2)$ are identified by horospherical glide reflections g_j or their inverses g_j^{-1} respectively. The pairs of faces $\bar{\boldsymbol{g}}_j$ and $\bar{\boldsymbol{g}}_j^{-1}$ $(j = 1, 2)$ are identified by horospherical glide reflections $\bar{g}_j$ and $\bar{g}_j^{-1}$ respectively. These will be fixed by the requirement that the centre A_2 will be mapped onto the centres of the neighbouring (along side faces) octahedra.

The identifications of the faces of $\tilde{\boldsymbol{D}}_1$ are induced by those of the sides of the regular quadrangle $\boldsymbol{P}^1$ for the group G^1 in the Euclidean plane $\boldsymbol{E}^2$ (Fig. 2), where g_j $(j = 1, 2)$ are glide reflections in $\boldsymbol{E}^2$. This observation motivated us to construct the group G_1.

Now we list the edge equivalence classes of the polyhedron $\tilde{\boldsymbol{D}}_1$, induced by the above identifications, writing down for each edge class the defining relation which expresses that every point on these edges has a trivial stabilizer. We establish incidentally the stabilizer subgroup for each end class (cusp).

The octahedron $\tilde{\boldsymbol{D}}_1$ has 12 side edges, divided into 3 classes with 4 edges in each class :

(1) $g_1\bar{g}_1^{-1}g_2\bar{g}_2^{-1} = 1$ at the edge class $\Longrightarrow\Longrightarrow$

(2) $g_1g_1g_2g_2 = 1$ at the edge class $\longrightarrow\longrightarrow$

(3) $\bar{g}_1\bar{g}_1\bar{g}_2\bar{g}_2 = 1$ at the edge class $\dashrightarrow\dashrightarrow$

each of the related dihedral angle is $\pi/2$, so $4 \cdot \pi/2 = 2\pi$, which guarantees the trivial stabilizer for points on these edges [5].

We conclude that $\tilde{\boldsymbol{D}}_1$ is a fundamental polyhedron for the group G_1 with presentation

$$G_1 = (g_1, g_2, g_3, g_4 \;—\; g_1 g_3^{-1} g_2 g_4^{-1} = g_1 g_1 g_2 g_2 = g_3 g_3 g_4 g_4 = 1). \tag{4}$$

Forming the orbit $\boldsymbol{O}_1^G$, where $\boldsymbol{O} := \boldsymbol{A}_2$ is the centre of the polyhedron $\tilde{\boldsymbol{D}}_1$ (Fig. 1), we obtain by a symmetry argument that $\tilde{\boldsymbol{D}}_1 = \tilde{\boldsymbol{D}}_{1O}$ is a Dirichlet polyhedron corresponding to $\boldsymbol{O}$ and its G_1-orbit.

The stabilizer for the end classes of $\boldsymbol{A}_3$ (and $\boldsymbol{A}_3^*$) will be the Euclidean plane crystallographic group $G^1 = pg$, coresponding to the fundamental quadrangle $\tilde{P}^1 = \boldsymbol{E}^2/G^1$ (Fig. 2), with presentation $g_1 g_1 g_2 g_2 = 1$. We also have one end class on the base of $\tilde{\boldsymbol{D}}_1$, corresponding to $\boldsymbol{A}_0$, with the plane translation group $\boldsymbol{p1}$ as stabilizer.

Fig. 3

The pair of faces $\boldsymbol{g}_j^{-1}$ and $\boldsymbol{g}_j$ $(j = 1, 2)$ are identified by horospherical glide reflections g_j or their inverses g_j^{-1}, respectively.

The pairs of faces $\boldsymbol{t}_j^{-1}$ and $\boldsymbol{t}_j$ $(j = 1, 2)$ are identified by horospherical translations t_j and t_j^{-1}, respectively.

The identifications of the faces of $\tilde{\boldsymbol{D}}_2$, joining the end A_3, are induced by those of the sides of the regular quadrangle $\boldsymbol{P}_2$ for the group $G^2 = p1$ in the Euclidean plane $\boldsymbol{E}^2$, where t_j $(j = 1, 2)$ are translations, with relation $t_1 t_2 t_1^{-1} t_2^{-1} = 1$ (Fig. 3). This observation motivated us to construct the group G_2.

Now we list the edge equivalence classes of the polyhedron $\tilde{\boldsymbol{D}}_2$, induced by the above identifications, writing down for each edge class the defining relation which expresses that every point on these edges has a trivial stabilizer. We establish incidentally the stabilizer subgroup for each end class (cusp).

The octahedron $\tilde{\boldsymbol{D}}_2$ has 12 side edges, divided into 3 classes with 4 edges in each class :

(5) $t_1 \cdot g_1^{-1} \cdot t_2^{-1} \cdot g_2 = 1$ at the edge class $\Longrightarrow\Longrightarrow$

(6) $t_1 \cdot t_2 \cdot t_1^{-1} \cdot t_2^{-1} = 1$ at the edge class $\longrightarrow\longrightarrow$

(7) $g_1 \cdot g_1 \cdot g_2 \cdot g_2 = 1$ at the edge class $\dashrightarrow\dashrightarrow$

We conclude that $\tilde{\boldsymbol{D}}_2$ is a fundamental polyhedron for the group G_2 with presentation

$$(8) \qquad G_2 = (g_1, g_2, t_1, t_2 \text{ --- } t_1 g_1^{-1} t_2^{-1} g_2 = t_1 t_2 t_1^{-1} t_2^{-1} = g_1 g_1 g_2 g_2 = 1).$$

Forming the orbit $\boldsymbol{O}_2^G$, by a symmetry argument we conclude that $\tilde{\boldsymbol{D}}_2 = \tilde{\boldsymbol{D}}_{2O}$ is a Dirichlet polyhedron corresponding to $\boldsymbol{O}$ and its G_2-orbit.

The stabilizers for the end classes of the $\boldsymbol{A}_3$ and $\boldsymbol{A}_3^*$ will be the Euclidean plane crystallographic groups $G_1 = pg$ and $G_2 = p1$, corresponding to the fundamental quadrangles $\tilde{\boldsymbol{P}}^1 = \boldsymbol{E}^2/G^1$ (Fig. 2) and $\tilde{\boldsymbol{P}}^2 = \boldsymbol{E}^2/G^2$ (Fig. 3), respectively. We also have one end class on the base of $\tilde{\boldsymbol{D}}_2$, corresponding to $\boldsymbol{A}_0$, with stabilizer pg.

We have seen that each pairing $\tilde{\boldsymbol{D}}_i$ $(i = 1, 2)$, by isometries on the faces of the bipyramid $\boldsymbol{D}$, induces equivalence classes of edges so that Poincarè's angle conditions hold for each edge class. This guarantees the free action of G_i, generated by the pairing, at the points of edges of $\tilde{\boldsymbol{D}}_i$.

The orthogonal projections of the centre $\boldsymbol{O} := \boldsymbol{A}_2$ on the faces and edges of $\boldsymbol{D}$ will be mapped by the pairing onto each other, respectively. Hence each element of G_i, preserving an end of $\tilde{\boldsymbol{D}}_i$, is parabolic. This means that only the fixed end is invariant under the non(trivial isometry from the stabilizer of the end, so each horosphere, centred at the end considered, is invariant under this stabilizer. Indeed, we have determined a fundamental end domain for each stabilizer and recognized the corresponding Euclidean plane group as one of $p1$ and pg. So we have checked the so-called cusp condition which guarantees that G_i is discrete on $\boldsymbol{H}^3$ and the hyperbolic metric of $\boldsymbol{H}^3/G_i$ is complete ([2], [3]).

By the Poincarè's theorem these facts are sufficient for $\tilde{\boldsymbol{D}}_i = \boldsymbol{H}^3/G_i$ to be a hyperbolic space form.

4. Metric construction of $\boldsymbol{D}$ in a vector model of $\boldsymbol{H}^3$

We are going to indicate the analitical treatment of our problem (see [2] and [3] for more details).

We consider the 4-dimensional real vector space $\boldsymbol{V}^4$, whose dual space, i.e. the space of its linear forms, is denoted by $\boldsymbol{V}_4^*$. In the usual way the projective 3-space $\boldsymbol{P}^3(\boldsymbol{V}^4, \boldsymbol{V}_4^*)$ can be introduced. The 1-dimensional subspaces of $\boldsymbol{V}^4$ (or the 3-spaces of $\boldsymbol{V}_4^*$) represent the points of $\boldsymbol{P}^3$ and the 1-subspaces of $\boldsymbol{V}_4^*$ (or the 3-subspaces of $\boldsymbol{V}^4$) represent the planes of $\boldsymbol{P}^3$. The point $\boldsymbol{X}(\boldsymbol{x})$ and the plane $\alpha(\boldsymbol{a})$ are incident iff $\boldsymbol{x} \cdot \boldsymbol{a} = 0$, i.e. the value of the linear form $\boldsymbol{a}$ on the vector $\boldsymbol{x}$ is equal to zero ($\boldsymbol{x} \in \boldsymbol{V}^4$, $\boldsymbol{x} \neq 0$ and $\boldsymbol{a} \in \boldsymbol{V}_4^*$, $\boldsymbol{a} \neq 0$). The straight lines of $\boldsymbol{P}^3$ are characterized by 2-subspaces of $\boldsymbol{V}^4$ or of $\boldsymbol{V}_4^*$, respectively. If $\{\boldsymbol{e}_i\}$ is a basis in $\boldsymbol{V}^4$ and $\{\boldsymbol{e}^j\}$ is its dual basis in $\boldsymbol{V}_4^*$, i.e. $\boldsymbol{e}_i \cdot \boldsymbol{e}^j = \delta_i^j$ (the Kronecker symbol), then the form $\boldsymbol{a} = \boldsymbol{e}^j a_j$ takes the value $\boldsymbol{x} \cdot \boldsymbol{a} = x^i \cdot a_i$ on the vector $\boldsymbol{x} = x^i \cdot \boldsymbol{e}_i$. We use the summation convention for the same upper and lower indices.

In order to embed $\boldsymbol{H}^3$ into the real projective space $\boldsymbol{P}^3(\boldsymbol{V}^4, \boldsymbol{V}_4^*)$ we introduce a hyperbolic projective metric by giving a bilinear form [2]

$$\langle\ ;\ \rangle : \boldsymbol{V}_4^* \times \boldsymbol{V}_4^* \to \mathbf{R}, \qquad \langle \boldsymbol{b}^i u_i; \boldsymbol{b}^j v_j \rangle = u_i b^{ij} v_j,$$

by means of the Schläfli matrix

$$(1) \qquad (\langle \boldsymbol{b}^i; \boldsymbol{b}^j \rangle) = (b^{ij}) = \begin{bmatrix} 1 & -\cos\pi/4 & 0 & 0 \\ -\cos\pi/4 & 1 & -\cos\pi/4 & 0 \\ 0 & -\cos\pi/4 & 1 & -\cos\pi/4 \\ 0 & 0 & -\cos\pi/4 & 1 \end{bmatrix}$$

where the basis $\{\boldsymbol{b}^i\}$ in the dual vector space $\boldsymbol{V}_4^*$ represents the planes $\boldsymbol{m}_i$ of $\boldsymbol{P}^3$ in connection with the octahedron $\boldsymbol{D}$ ($i = 0,\ 1,\ 2,\ 3$) in $\boldsymbol{H}^3 \subset \boldsymbol{P}^3$ (Fig. 1). The planes $\boldsymbol{m}_0 = \boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3$ and $\boldsymbol{m}_1 = \boldsymbol{A}_0\boldsymbol{A}_2\boldsymbol{A}_3$ will be the symmetry ones of $\boldsymbol{D}$, $\boldsymbol{m}_3 = \boldsymbol{A}_0\boldsymbol{A}_1\boldsymbol{A}_2$ will be a base plane and $\boldsymbol{m}_2 = \boldsymbol{A}_0\boldsymbol{A}_1\boldsymbol{A}_3$ will be a side plane of $\boldsymbol{D}$.

We compute

$$(2) \qquad B = \det(b^{ij}) = -0.25 < 0$$

and the inverse matrix (a_{ij}) of the matrix (b^{ij}). The equation $b^{ij} \cdot a_{jk} = \delta^i_k$ holds iff

$$(3) \qquad \begin{gathered} a_{00} = 0,\ a_{01} = a_{10} = -\sqrt{2} < 0,\ a_{02} = a_{20} = -2 < 0,\ a_{03} = a_{30} = -\sqrt{2} < 0, \\ a_{11} = -2 < 0,\ a_{12} = a_{21} = -2\sqrt{2} < 0,\ a_{13} = a_{31} = -2 < 0, \\ a_{22} = -2 < 0,\ a_{23} = a_{32} = -\sqrt{2} < 0, \\ a_{33} = 0. \end{gathered}$$

Now, let $\{\boldsymbol{a}_j\}$ be the basis in the vector space $\boldsymbol{V}^4$ dual to the given basis $\{\boldsymbol{b}^i\}$ in $\boldsymbol{V}_4^*$, defined by $\boldsymbol{a}_j \cdot \boldsymbol{b}^i = \delta^i_j$. Geometrically, the vectors $\boldsymbol{a}_j$ represent the vertices of the simplex $\boldsymbol{S} = A_0A_1A_2A_3$ in $\boldsymbol{H}^3 \subset \boldsymbol{P}^3$, whose side planes are described by the forms $\boldsymbol{b}^i$. The induced bilinear form

$$\langle\ ;\ \rangle : \boldsymbol{V}^4 \times \boldsymbol{V}^4 \to \mathbf{R}, \qquad \langle x^i \boldsymbol{a}_i; y^j \boldsymbol{a}_j \rangle = x^i a_{ij} y^j$$

is defined by the matrix

$$(4) \qquad (\langle \boldsymbol{a}_i; \boldsymbol{a}_j \rangle) = (a_{ij})$$

with entries in (3).

We easily see that the bilinear form $\langle\ ;\ \rangle$ has a signature $(+,+,+,-)$, so that the projective metric in $\boldsymbol{P}^3(\boldsymbol{V}^4, \boldsymbol{V}_4^*)$ is hyperbolic [2].

In general, the proper points $\boldsymbol{X}(\boldsymbol{x})$ of $\boldsymbol{H}^3 \subset \boldsymbol{P}^3$ are defined by "time-like" 1-subspaces of $\boldsymbol{V}^4$

$$(5) \qquad \{\, (\boldsymbol{x}) \subset \boldsymbol{V}^4 :\ \langle \boldsymbol{x}; \boldsymbol{x} \rangle < 0 \,\},$$

while the ideal points of $\boldsymbol{H}^3$ (the ends of absolute) are described by

(6) $$\{\,(\boldsymbol{y}) \subset \boldsymbol{V}^4 :\ \langle \boldsymbol{y};\boldsymbol{y}\rangle = 0\,\}.$$

The proper planes $\boldsymbol{U}(\boldsymbol{u})$ of $\boldsymbol{H}^3$ are

(7) $$\{\,(\boldsymbol{u}) \subset \boldsymbol{V}_4^* :\ \langle \boldsymbol{u};\boldsymbol{u}\rangle > 0\,\}.$$

All elements of the main diagonal of the matrix (1) are positive. This means that all planes $\boldsymbol{m}_i$ of the polyhedron $\boldsymbol{D}$, represented by the forms $\boldsymbol{b}^i$, are proper planes in the space $\boldsymbol{H}^3 \subset \boldsymbol{P}^3$. From the relations (3) of this section, we see that the vertices $\boldsymbol{A}_1(\boldsymbol{a}_1)$ and $\boldsymbol{A}_2(\boldsymbol{a}_2)$ of the simplex $\boldsymbol{S}$, described by the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, are proper points, while the vertices $\boldsymbol{A}_0(\boldsymbol{a}_0)$ and $\boldsymbol{A}_3(\boldsymbol{a}_3)$ of the simplex $\boldsymbol{S}$, described by the vectors $\boldsymbol{a}_0$ and $\boldsymbol{a}_3$, are ideal points (ends). This implies that $\boldsymbol{D}$ is a polyhedron with ideal vertices.

Applying formulas valid for $\boldsymbol{H}^3 \subset \boldsymbol{P}^3$ [2], other data of the simplex $\boldsymbol{S}$ can be computed from the matrices (a_{ij}) and (b_{ij}). Thus we can check that the Coxeter diagram (Fig. 1) correctly describes the dihedral angles of simplex $\boldsymbol{S}$, e.g., $\pi/4$ is the angle of planes $\boldsymbol{m}_0 = A_1A_2A_3$ and $\boldsymbol{m}_1 = A_0A_2A_3$, furthermore, $\boldsymbol{m}_0$ is perpendicular to $\boldsymbol{m}_2$ and $\boldsymbol{m}_3$.

The Coxeter group C, generated by reflections in mirrors $\boldsymbol{m}_j$ $(j = 0, 1, 2, 3)$ of the simplex $\boldsymbol{S}$, is a supergroup of index 16 of each group G_i, since $\boldsymbol{D}$ is the union of 16 congruent copies of simplex $\boldsymbol{S}$. Therefore, we could also express the generators of each group G_i by matrices with respect to the bases $\{\boldsymbol{b}^i\}$ or $\{\boldsymbol{a}_i\}$.

In general, isometries of $\boldsymbol{H}^3$ can be described by linear transformations of $\boldsymbol{V}^4$, or $\boldsymbol{V}_4^*$, which preserve the bilinear form $\langle\ ;\ \rangle$.

The most important radius ρ of the inscribed ball of $\boldsymbol{D}$ can be computed. This is the distance of $\boldsymbol{A}_2(\boldsymbol{a}_2)$ from the plane $\boldsymbol{m}_2 = \boldsymbol{A}_0\boldsymbol{A}_1\boldsymbol{A}_3$ represented by $\boldsymbol{b}^2$. We have

(8) $$\begin{gathered}\operatorname{sh}(\rho/k) = \boldsymbol{a}_2\cdot\boldsymbol{b}^2/\sqrt{-\langle \boldsymbol{a}_2;\boldsymbol{a}_2\rangle\cdot\langle \boldsymbol{b}^2;\boldsymbol{b}^2\rangle} = 1/\sqrt{-a_{22}} = \sqrt{2}/2,\\ \rho = k\cdot \text{area sh}\,\sqrt{2}/2.\end{gathered}$$

Here $k = \sqrt{-1/K}$ is the metric constant of $\boldsymbol{H}^3$, and $K < 0$ is the sectional curvature.

Other data of the simplex $\boldsymbol{S}$ or of $\boldsymbol{D}$ can be computed in a similar way [2].

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