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PEDALS, AUTOROULETTES AND STEINER'S THEOREM

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Abstract. The area of the autoroulette of a closed curve is double the area of the roulette of that curve on a straight base line taken with respect to the same generating point. This is an equivalent to Steiner's theorem for pedals. Also, an extension of Steiner's theorem is asserted.

1. Pedals and Steiner's theorem

DEFINITION 1. The pedal of a given curve with respect to a point P is the locus of the foot of the perpendicular from P to a variable tangent line to the curve.

Steiner's theorem states as follows.

THEOREM 1. [2], [3] *When a closed curve rolls on a straight line, the area between the line and the roulette generated in a complete revolution by any point on the rolling curve is double the area of the pedal of the rolling curve, this pedal being taken with respect to the generating point.*

The arc of the roulette on a straight base line and the arc of the corresponding pedal have equal lengths.

For the sake of simplicity, Steiner's theorem have be stated only when P lies on the curve. This is not an essential restriction. In Theorem 4 an extension is given to the non-closed curves.

2. Roulettes

DEFINITION 2. The roulette $\mathcal{R}(c,l,P)$ is the curve traced out by a point P in fixed position with respect to a rolling curve c which rolls without slipping along a fixed base curve l.

Let $c = \overset{\frown}{C_1C_2}$ and $l = \overset{\frown}{L_1L_2}$ be oriented piecewise regular curves with parametric equations $r(t) = (x(t), y(t))$ and $\rho(t) = (u(t), v(t))$, $a \le t \le b$, respectively. A parametric representation is regular if vector function r is of class C^1 and

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$\dot r(t) = (\dot x(t), \dot y(t)) \neq 0$ for all $a \le t \le b$. Tangent vectors $\dot r$ can be translated so that they have same origin. The total curvature κ_c^T is the difference $\theta_1 - \theta_0$ in the values of inclination θ of the tangent to the curve at the end points of the curve. For the closed curve we have $\kappa_c^T = 2z\pi$, where z is an integer. The natural parameter is the length of the arc $s = \int |\dot r(t)|\, dt$, in which case tangent vectors have length 1. If c is smooth in some neighborhood of a point X, then the curvature $\kappa(X)$ of c at X is

$$\kappa(X) = \lim \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}, \qquad \Delta s \to 0, \qquad Y \to X,$$

where $\Delta\alpha$ is the angle between tangent vectors at X and Y, Δs is the length of the arc $\overset{\frown}{XY}$. The classical formula for the curvature is

$$\kappa(X) = \frac{\dot x(t)\ddot y(t) - \ddot x(t)\dot y(t)}{(\dot x(t)^2 + \dot y(t)^2)^{3/2}}.$$

At a singular point $S \in c$ the point curvature is $\kappa(S) = \measuredangle(\dot r_-(S), \dot r_+(S))$, so that $-\pi \le \kappa(S) \le \pi$. We take $\kappa(S) = \pm\pi$ if $\lim \angle(\dot r_-(S), \overset{\frown}{SX}) = \pm\pi$, $X \to S$, $x \in \overset{\frown}{SC_2}$. If the whole arc $\overset{\frown}{SC_2}$ is a straight segment opposite to $\dot r_-(s)$, we can choose either π or $-\pi$. Mention that the curvature κ as a function of the length of the arc is the natural equation of the curve.

Definition 3. The curvature measure κ on a curve c is defined at the regular points by the curvature functional $\kappa(t)$, and at the singular points the point-measure is equal to the point-curvature. The centroid B of curvature of a curve c is the barycenter of c while c is by the curvature measure κ pondered.

Note that $\kappa_c^T = \int_c d\kappa$. Also, the difference $\kappa_c^T - \kappa_l^T$ is equal to the angle ρ of rotation of c in tracing the roulette.

Let $\mathcal{R}(c, l, P) = \overset{\frown}{R_1R_2}$, and let $l = \overset{\frown}{L_1L_2}$ where L_1 and L_2 are the first and the last contact points respectively. The area $\mathcal{R}(c, l, P)$ is the oriented area $L_1L_2R_2R_1$ bounded by the base curve $l = \overset{\frown}{L_1L_2}$, the roulette $\overset{\frown}{R_2R_1}$, and straight segments L_2R_2 and R_1L_1.

Theorem 2. [1] *Let B be the barycenter of a curve c, let c be pondered by the difference of curvature measures $\kappa_c - \kappa_l$, and let κ^T denote the total curvature. Then, the area bounded by the roulette and the base curve l is*

$$\begin{aligned} \operatorname{area}\mathcal{R}(c, l, P) &= \operatorname{area}\operatorname{cone} Pc + \frac{1}{2}\int_c PX^2\, d(\kappa_c - \kappa_l), \qquad X \in c \\ &= \operatorname{area}\mathcal{R}(c, l, B) + \frac{\kappa_c^T - \kappa_l^T}{2} BP^2 + \operatorname{area}\square PC_1BC_2. \end{aligned}$$

For the closed rolling curve $\operatorname{area}\operatorname{cone} Pc = \operatorname{area} c$ *and* $\operatorname{area}\square PC_1BC_2 = 0$. *Further,*

$$\operatorname{length}\mathcal{R}(c, l, P) = \int_c PX\, d(\kappa_c - \kappa_l), \qquad X \in C.$$

If c is a closed curve then points from the circle with center B trace roulettes with equal areas. Note that $(\kappa_c^T - \kappa_l^T)BP^2/2 = \text{area sector}\,(BP, \kappa_c^T - \kappa_l^T)$, where the sector is circular, with radius BP, and which angle is equal to the angle of rotation of c.

3. Autoroulettes

Recall that the area under a cycloid cusp is $3\pi r^2$, and the area bounded by a cardioid is $6\pi r^2$. This cardioid-cycloid proportion also holds in the more general case.

DEFINITION 3. The autoroulette $\mathcal{AR}(c, P)$ is the roulette $\mathcal{R}(c, l, P)$, where c and l are symmetric curves $\kappa_l(t) = -\kappa_c(t)$, $a \le t \le b$.

If c is a closed curve, then $\mathcal{AR}(c, P)$ is closed also, and, obviously,

$$\operatorname{area} \mathcal{AR}(c, P) = \operatorname{area} l + \operatorname{area} \mathcal{R}(c, l, P).$$

THEOREM 3. *Let c and l be closed symmetric curves $\kappa_l(t) = -\kappa_c(t)$, $a \le t \le b$, and let L be a straight line; then*

$$\begin{aligned} \operatorname{area} \mathcal{AR}(c, P) &= 2 \operatorname{area} \mathcal{R}(c, L, P), \\ \operatorname{length} \mathcal{AR}(c, P) &= 2 \operatorname{length} \mathcal{R}(c, l, P). \end{aligned}$$

Proof. By Theorem 2 we have

$$\begin{aligned} \operatorname{area} \mathcal{AR}(c, P) &= \operatorname{area} l + \operatorname{area} \mathcal{R}(c, l, P) \\ &= \operatorname{area} l + \operatorname{area} c + \frac{1}{2} \int_c PX^2 \, d(2\kappa_c) \\ &= 2 \left(\operatorname{area} c + \frac{1}{2} \int_c PX^2 \, d\kappa_c \right) \\ &= 2 \operatorname{area} \mathcal{R}(c, L, P). \quad \blacksquare \end{aligned}$$

Theorem 3 is an equivalent to Steiner's Theorem 1.

PROPOSITION. *The autoroulette of a closed curve is double the corresponding pedal and*

area pedal : area straight-roulette : area autoroulette = 1 : 2 : 4
length pedal : length straight-roulette : length autoroulette = 1 : 1 : 2.

4. Extensions

THEOREM 4. *(a) Extended straight roulette — autoroulette proportion. Let c be piecewise regular curve and L be a straight line, then*

$$\operatorname{area} \mathcal{AR}(c, P) - \operatorname{area\,cone} Pc = 2 \left[\operatorname{area} \mathcal{R}(c, L, P) - \operatorname{area\,cone} Pc\right].$$

(b) Extended Steiner's theorem. When a piecewise regular curve rolls on a straight line, the area between the line and the roulette, whereby from the end points of the roulette are taken perpendiculars to the line, is double the area of the cone which vertex is P and which base is the pedal.

Proof. (a) It follows immediately from Theorem 2.

(b) Let $c = \overset{\frown}{C_1C_2}$, the pedal denote by $p = \overset{\frown}{P_1P_2}$, and let $\mathcal{AR}(c, P) = \overset{\frown}{A_1A_2}$. Notice that the all areas are oriented.

$$\begin{aligned}
\operatorname{area}\mathcal{R}(c, L, P) &= \frac{1}{2}\left[\operatorname{area}\mathcal{AR}(c, P) + \operatorname{area}\operatorname{cone} Pc\right] \\
&= \frac{1}{2}\operatorname{area} PC_1\overset{\frown}{A_1A_2}C_2P \\
&= \frac{1}{2}\left[\operatorname{area}\operatorname{cone} P(\overset{\frown}{A_1A_2}) + 2\Delta PC_1P_1 - 2\Delta PC_2P_2\right] \\
&= \frac{1}{2}\left[4\operatorname{area}\operatorname{cone} Pp + 2\Delta PC_1P_1 - 2\Delta PC_2P_2\right] \\
&= 2\operatorname{area}\operatorname{cone} Pp + \Delta PC_1P_1 - \Delta PC_2P_2.
\end{aligned}$$

If $\operatorname{Area}\mathcal{R}(c, L, P)$ denote the area bounded by the roulette, the base straight line and two perpendiculars from the end points of the roulette to the line, then

$$\operatorname{area}\mathcal{R}(c, L, P) = \operatorname{area}\mathcal{R}(c, L, P) + \Delta PC_1P_1 - \Delta PC_2P_2,$$

so that

$$\operatorname{Area}\mathcal{R}(c, L, P) = 2\operatorname{area}\operatorname{cone} Pp,$$

what ends the proof. ■

These two definitions of the area of roulettes, by connecting the endoints of the roulette by the endpoints of the base straight line, and taking perpendiculars from the end points to the base straight line, for closed curves give equal values.

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