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GEOMETRY OF k-LAGRANGE SPACES OF SECOND ORDER

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Introduction

Let M be a smooth manifold coordinated by $(U, x^1, \dots, x^n)$ and $x^i = x^i(t^1, \dots, t^k)$, $\operatorname{rank}\left(\dfrac{\partial x^i}{\partial t^\alpha}\right) = k$, $\alpha = 1, \dots, k$ its k-dimensional submanifold, $k = 1, \dots, n-1$. The Latin indices will range from 1 to n and the Greek indices will run from 1 to k. The Einstein convention on summation will work for both kinds of indices.

A real valued smooth function $L\left(x^i, \dfrac{\partial x^i}{\partial t^\alpha}, \dfrac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta}\right)$ will be called a k-Lagrangian of second order.

For an open set $\mathcal{O}$ in the range of parameters $(t^1, \dots, t^k)$ with the property that its closure $\overline{\mathcal{O}}$ is compact, one considers the multiple integral $\displaystyle\int_{\overline{\mathcal{O}}} L\left(x^i(t), \frac{\partial x^i}{\partial t^\alpha}(t), \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta}(t)\right) dt^{(\alpha)}$, where $dt^{(\alpha)} = dt^1\, dt^2 \dots dt^k$.

One may ask to find among the k-submanifolds with the same frontier those which afford extremal values for the above multiple integral.

Our purpose is to provide a geometrization of the k-Lagrangians of second order as a framework of the variational problem sketched above.

First, we introduce in §1 a manifold $J_k^2 M$ fibered over M on which such Lagrangians are living.

In §2 we consider a nonlinear connection on $J_k^2 M$ and exhibit the basis adapted to it. Various geometrical structures on $J_k^2 M$ are pointed out, too. In §3 a special class of linear connections on $J_k^2 M$ is considered.

The geometry of the manifold $J_k^2 M$ is interesting for itself since for $k = 1$ it reduces to the manifold $\operatorname{Osc}^2 M$, see R. Miron [3], and for $k = n$ it is the prolongation of second order of the frame manifold.

More facts from this geometry will appear elsewhere.

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1. Manifold $J_k^2 M$

Let M be a smooth manifold of dimension n and $J_{o,p}(\mathbf{R}^k, M)$ the set of germs of smooth mappings $f\colon \mathbf{R}^k \to M$ with $f(0) = p \in M$. We say that $f, g \in J_{o,p}(\mathbf{R}^k, M)$ are equivalent up to order q if there exists a chart (U, φ) around p such that

$$d_0^h(\varphi \circ f) = d_0^h(\varphi \circ g), \quad 1 \le h \le q, \tag{1.1}$$

where d denotes Frechet differentiation. It can be seen that if (1.1) holds for a chart (U, φ), it holds for any other chart (V, ψ) around p.

We denote by $j^q_{0,p}f$ the equivalence class of f (the coset of f) and set $J^q_{0,p} = \{\, j^q_{o,p}f,\ f \in J_{0,p}(\mathbf{R}^k, M)\,\}$. Then we put $J^q_k M = \bigcup_{p\in M} J^q_{0,p}$ and define $\pi\colon J^q_k M \to M$ by $\pi(J^q_{0,p}) = p$.

One can see that $J^q_k M$ has a structure of smooth manifold.

We notice that for $k = 1$, this manifold is just the manifold $\mathrm{Osc}^q M$ studied by R. Miron [3], which reduces to the tangent manifold for $q = 1$. For $k = n$ and $q = 1$, we get the manifold of frames over M and for $k \in \{2, 3, \ldots, n-1\}$ and $q = 1$ it can be identified with $TM \otimes \cdots \otimes TM$ (k times) which is the manifold supporting the k-Lagrange geometry, see R. Miron, M. Kirkovits, Mihai Anastasiei [5].

For these reasons we confine ourselves to the cases $k = 2, 3, \ldots, n-1$ and for the sake of simplicity we take $q = 2$. The case q greater than 2 can be similarly treated.

We also notice that $J^2_k M$ is the manifold of 2-jets of the sections of the fibre bundle $\mathbf{R}^k \times M \to \mathbf{R}^k$ but the theory of jets from the book by D.J. Saunders [6] cannot be applied since the typical fibre M of this bundle is too general.

Instead of that theory we follow the ideas and techniques from the k-Lagrange geometry and from the geometry of $\mathrm{Osc}^q M$ spaces as well, see [1], [2], [4].

Let us return to (1.1) for $q = 2$. Letting $\varphi \circ f, \varphi \circ g : \mathbf{R}^k \to \mathbf{R}^n$ as $f^i = f^i(t^1, \ldots, t^k)$, $g^i = g^i(t^1, \ldots, t^k)$ this condition becomes

$$f^i(0) = g^i(0) = \varphi(p),\ \frac{\partial f^i}{\partial t^\alpha}(0) = \frac{\partial g^i}{\partial t^\alpha}(0),\ \frac{\partial^2 f^i}{\partial t^\alpha \partial t^\beta}(0) = \frac{\partial^2 g^i}{\partial t^\alpha \partial t^\beta}(0), \tag{1.1$'$}$$

for $\alpha, \beta = 1, 2, \ldots, k$. Let us set $\partial_i : \partial/\partial x^i$, $\partial_\alpha := \partial/\partial t^\alpha$.

Now, for another local chart (V, ψ) around p such that $\psi \circ \varphi^{-1}\colon x^{i'} = x^{i'}(x^1, \ldots, x^n)$, rank $\left(\dfrac{\partial x'}{\partial x^k}\right) = n$, taking $\psi \circ f$ and $\psi \circ g$ as $f^{i'} = f^{i'}(t^1, \ldots, t^p)$ and $g^{i'} = g^{i'}(t^1, \ldots, t^p)$, respectively, we get $f^{i'} = x^{i'}(f^j(t^1, \ldots, t^p))$, $g^{i'} = x^{i'}(g^j(t^1, \ldots, t^p))$ as well as

$$\begin{aligned} \frac{\partial f^{i'}}{\partial t^\alpha}(0) &= \frac{\partial x^{i'}}{\partial x^j}(\varphi(p)) \frac{\partial f^j}{\partial t^\alpha}(0), \\ \frac{\partial^2 f^{i'}}{\partial t^\alpha \partial t^\beta} &= \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}(\varphi(p)) \frac{\partial f^j}{\partial t^\alpha}(0) \frac{\partial f^k}{\partial t^\beta}(0) + \frac{\partial x^{i'}}{\partial x^j} \frac{\partial^2 f^j}{\partial t^\alpha \partial t^\beta} \end{aligned} \tag{1.2}$$

By (1.2) the independence of (1.1) on the chosen local chart follows.

For $f\colon \mathbf{R}^k \to M$ with $f(0) = \varphi(p)(x^1, \dots, x^n)$ we set $y^i_\alpha = \dfrac{\partial f^i}{\partial t^\alpha}(0)$, $z^i_{\alpha\beta} = \dfrac{1}{2}\dfrac{\partial^2 f^i}{\partial t^\alpha \partial t^\beta}(0)$ and define a mapping $\phi\colon \pi^{-1}(U) \to \varphi(U) \times \mathbf{R}^{kn} \times \mathbf{R}^{\frac{k(k+1)}{2}n}$ by $\phi([f]_p) = (x^i, y^i_\alpha, z^i_{\alpha\beta})$.

The mapping ϕ is invertible, its inverse associating to $(x^i, y^i_\alpha, z^i_{\alpha\beta})$ the coset of the mapping $\varphi^{-1} \circ T$, where T is the Taylor polynomial $x^i + y^i_\alpha t^\alpha + z^i_{\alpha\beta} t^\alpha t^\beta$. If we similarly define ψ in connection with the local chart (V, ψ), $\psi \circ \phi^{-1}$ has the following form

$$\left.\begin{aligned} & x^{i'} = x^{i'}(x^i, \dots, x^n),\ \operatorname{rank}\left(\frac{\partial x^{i'}}{\partial x^i}\right) = n \\ & y^{i'}_\alpha = \frac{\partial x^{i'}}{\partial x^i} y^i_\alpha, \qquad \alpha = 1, \dots, k \\ & z^{i'}_{\alpha\beta} = \frac{1}{2}\frac{\partial^2 x^{i'}}{\partial x^j \partial x^k} y^j_\alpha y^k_\beta + \frac{\partial x^{i'}}{\partial x^j} z^j_{\alpha\beta}. \end{aligned}\right\} \tag{1.3}$$

It results that $\psi \circ \phi^{-1}$ is smooth. Thus $\{(\pi^{-1}(U), \phi)\}$ associated to the atlas $\{(U, \varphi)\}$ on M provides a smooth manifold structues for $J^2_k M$. At the same time (1.3) gives the allowable coordinate transformations with respect to the fibration $\pi : J^2_k M \to M$. This fibration is a locally trivial bundle with typical fibre $\mathbf{R}^{kn} \times R^{\frac{k(k+1)}{2}n}$. The bundle chart associated to (U, φ) is $(\pi^{-1}(U), \bar\phi)$ where $\bar\phi\colon \pi^{-1}(U) \to U \times \mathbf{R}^{nk} \times R^{\frac{k(k+1)}{2}n}$, $\bar\phi([f]_p) = (p, y^i_\alpha, z^i_{\alpha\beta})$.

We notice that $\pi\colon J^2_k M \to M$ is not a vector bundle although its typical fibre is so, because the mappings $\bar\psi_p \circ \bar\phi_p^{-1}$ are not linear. Here $\bar\phi_p$ is the restriction of $\bar\phi$ to the fibre $\pi^{-1}(p)$ and $\bar\psi, \bar\psi_p$ are similarly constructed in connection with (V, ψ).

The mapping $j^2_{0,p} f \mapsto j^1_{0,p} f$ induces a mapping $\pi_{2,1} : J^2_k M \to J^1_k M$ which in the local charts previously introduced has the form $(x^i, y^i_\alpha, z^i_{\alpha\beta}) \mapsto (x^i, y^i_\alpha)$. Thus, it is a surjective submersion. We set $\pi_1\colon J^1_k M \to M$ and so $\pi = \pi_1 \circ \pi_{2,1}$. For a local chart (U, φ) around $p \in M$, let $\phi_1\colon \pi_1^{-1}(U) \to U \times \mathbf{R}^{kn}$, $\phi_1(j^1_{0,p} f) = (p, y^i_\alpha)$ and $\psi_1\colon \pi_1^{-1}(V) \to V \times \mathbf{R}^{kn}$ similarly associated to (V, ψ). Then $\psi_{1,p} \circ \phi^{-1}_{1,x} : y^{i'}_\alpha = \dfrac{\partial x^{i'}}{\partial x^i} y^i_\alpha$ is a linear mapping from $R^{kn} \to \mathbf{R}^{kn}$. Hence $(J^1_k M, \pi_1, M)$ is a vector bundle of rank kn. Now let $\phi_2(J^2_{0,p} f) = (p, j^1_{0,p} f, z^i_{\alpha\beta})$ and ψ_2 similarly defined in connection with (V, ψ). Then $\psi_{2,(p,y)} \circ \phi^{-1}_{2,(p,y)} : z^{i'}_{\alpha\beta} = \dfrac{1}{2}\dfrac{\partial^2 x^{i'}}{\partial x^i \partial x^j} y^i_\alpha y^j_\beta + \frac{\partial x^{i'}}{\partial x^i} z^i_{\alpha\beta}$ with $y := j^1_{0,p} f$ is an affine morphism of the space $\mathbf{R}^{\frac{k(k+1)}{2}n}$ endowed with the standard affine structure. Hence $(J^2_k M, \pi_{2,1}, J^1_k M)$ is an affine bundle.

2. Nonlinear connections on $J^2_k M$. Adapted basis

Let us put $E := J^2_k M$ and let $\pi\colon E \to M$ be the canonical projection. Since π is a submersion, the linear spaces $V_u E := \ker \pi_{*,u}$, $u \in E$ define a distribution $V : u \to V_u E$ on E (vertical distribution).

DEFINITION. A nonlinear connection on $J_k^2M =: E$ is a distribution $H\colon u \to H_uE$ on E which is supplementary to the vertical distribution i.e.

$$T_uE = H_uE \oplus V_uE \quad \text{(direct sum)}, \tag{2.1}$$

holds for every $u \in E$.

Let us introduce the following notation:

$$\partial_i := \frac{\partial}{\partial x^i},\ \partial_i^\alpha := \frac{\partial}{\partial y_\alpha^i},\ \partial_i^{\alpha\beta} = \frac{\partial}{\partial z_{\alpha\beta}^i} = \partial_i^{\beta\alpha}. \tag{2.2}$$

The natural basis $\bar{B}$ of $T_u(J_k^2M)$ is

$$\bar{B} = (\partial_i, \partial_i^\alpha, \partial_i^{\alpha\beta}). \tag{2.3}$$

If the change of coordinates (1.3) is performed, the elements of $\bar{B}$ are transformed as follows:

$$\begin{aligned} &\partial_i = (\partial_i x^{i'})\partial_{i'} + (\partial_i\partial_j x^{i'})y_\alpha^j\partial_{i'}^\alpha + [2^{-1}(\partial_i\partial_j\partial_k x^{i'})y_\alpha^j y_\beta^k + (\partial_i\partial_j x^{i'})z_{\alpha\beta}^j]\partial_{i'}^{\alpha\beta},\\ &\partial_i^\alpha = (\partial_i x^{i'})\partial_{i'}^\alpha + (\partial_i\partial_j x^{i'})y_\beta^j\partial_{i'}^{\alpha\beta},\ \partial_i^{\alpha\beta} = (\partial_i x^{i'})\partial_{i'}^{\alpha\beta}. \end{aligned} \tag{2.4}$$

We note that the vertical distribution is locally spanned by $(\partial_i^\alpha, \partial_i^{\alpha\beta})$. For each $\alpha \in \{1, 2, \dots, k\}$ we define a linear operator $\overset{\alpha}{J}\colon T_uE \to T_uE$ on basis $\bar{B}$ as follows:

$$\overset{\alpha}{J}(\partial_i) = \partial_i^\alpha,\ \overset{\alpha}{J}(\partial_i^\beta) = \partial_i^{\alpha\beta},\ \overset{\alpha}{J}(\partial_i^{\beta\gamma}) = 0. \tag{2.5}$$

One checks using (2.4) that $\overset{\alpha}{J}$ is well-defined. Also, one easily verifies

$$\overset{\alpha}{J} \circ \overset{\beta}{J} = \overset{\beta}{J} \circ \overset{\alpha}{J},\ \overset{\alpha}{J} \circ \overset{\beta}{J} \circ \overset{\gamma}{J} = 0,\ \overset{\alpha}{J}{}^3 = 0,\ \text{for every } \alpha, \beta, \gamma \in \{1, 2, \dots, k\}. \tag{2.6}$$

Thus $\overset{\alpha}{J}$ is a 3-tangent structure on E and so J_k^2M is endowed with k natural 3-tangent structures which commute with each other.

The restriction of $\pi_{*,u}$ to T_uE is an isomorphism $H_u \to T_{\pi(u)}M$. Denoting by h its inverse and setting $\delta_i = h(\partial_i)$ one gets a local basis of the horizontal distribution. Since $\pi_*(\delta_i) = \partial_i$, the local vector fields δ_i will have the form

$$\delta_i = \partial_i - N_{i\alpha}^j(x, y, z)\partial_{j\alpha} - N_{i\alpha\beta}^j(x, y, z)\partial_j^{\alpha\beta}, \tag{2.7}$$

where the minus sign is taken for convenience and because of $\delta_i = (\partial_i x^{i'})\delta_i$, the functions $N_{i\alpha}^j, N_{i\alpha\beta}^j$ have to satisfy

$$\begin{aligned} N_{i'\alpha}^{j'}(\partial_i x^{i'}) &= (\partial_j x^{j'})N_{i\alpha}^j - \partial_i(y_\alpha^{j'})\\ N_{i'\alpha\beta}^{j'}(\partial_i x^{i'}) &= (\partial_j x^{j'})N_{i\alpha\beta}^j + N_{i\gamma}^j\partial_j^\gamma(z_{\alpha\beta}^{j'}) - \partial_i z_{\alpha\beta}^{j'}. \end{aligned} \tag{2.8}$$

Conversely, a set of functions $N = (N_{i\alpha}^j(x, y, z), N_{i\alpha\beta}^j(x, y, z))$ verifying (2.8) completely determine (δ_i) which in turn defines a nonlinear connection on E.

Now if we consider $\delta_i^\alpha := \overset{\alpha}{J}(\delta_i) = \partial_i^\alpha - N_{i\beta}^j \partial_j^{\alpha\beta}$ for all $\alpha \in \{1, \dots, k\}$, we get kn linearly independent local vector fields verifying,

$$\delta_i^\alpha = (\partial_i x^{i'}) \delta_{i'}^\alpha, \quad \alpha = 1, \dots, k. \tag{2.9}$$

Setting $H_u E := N_0(u)$, from the above it follows that (δ_i^α) span a subdistribution $N_1(u)$ of V and $\delta_i^{\alpha\beta} := (\overset{\alpha}{J} \circ \overset{\beta}{J})(\delta_i)$ span a second subdistribution $N_2(u)$ of V. Clearly, we have

$$T_u E = N_0(u) \oplus N_1(u) \oplus N_2(u), \quad u \in E. \tag{2.10}$$

Notice that each distribution N_1 and N_2 decomposes in k and, respectively, $k(k+1)/2$ n-dimensional distributions.

The adapted basis with respect to the decomposition (2.10) is

$$B = (\delta_i, \delta_i^\alpha, \delta_i^{\alpha\beta}); \quad \delta_i^{\alpha\beta} = \partial_i^{\alpha\beta} = \partial_i^{\beta\alpha}. \tag{2.11}$$

Notice that we have

$$\delta_i = (\partial_i x^{i'}) \delta_i, \ \delta_i^\alpha = (\partial_i x^{i'}) \delta_{i'}^\alpha, \ \delta_i^{\alpha\beta} = (\partial_i x^{i'}) \delta_{i'}^{\alpha\beta}. \tag{2.12}$$

These equations provide the main advantage of B when comparing with $\bar{B}$.

The dual basis of $\bar{B}$ is $\bar{B}^* = (dx^i, dy_\alpha^i, dz_{\alpha\beta}^i)$. By the change of coordinates (1.3) the elements of $\bar{B}^*$ are transformed as follows

$$\begin{aligned}
dx^{i'} &= (\partial_i x^{i'}) dx^i, \\
dy_\alpha^{i'} &= (\partial_i \partial_j x^{i'}) y_\alpha^j dx^i + (\partial_i x^{i'}) dy_\alpha^i, \\
dz_{\alpha\beta}^{i'} &= (\partial_i z_{\alpha\beta}^{i'}) dx^i + (\partial_j^\gamma z_{\alpha\beta}^{i'}) dy_\gamma^j + (\partial_i^{\gamma\delta} z_{\alpha\beta}^{i'}) dz_{\gamma\delta}^i \\
&= [2^{-1} (\partial_i \partial_j \partial_h x^{i'}) y_\alpha^j y_\beta^h + (\partial_i \partial_j x^{i'}) z_{\alpha\beta}^j] dx^i \\
&\quad + 2^{-1} (\partial_j \partial_h x^{i'}) (y_\beta^h dy_\alpha^j + y_\alpha^h dy_\beta^j) + (\partial_i x^{i'}) dz_{\alpha\beta}^i.
\end{aligned} \tag{2.13}$$

The dual basis of B is $B^* = (dx^j, \delta y_\alpha^j, \delta z_{\alpha\beta}^j)$, where

$$\begin{aligned}
\delta y_\alpha^j &= dy_\alpha^j + M_{i\alpha}^j dx^i \\
\delta z_{\alpha\beta}^j &= dz_{\alpha\beta}^j + M_{i\alpha\beta}^{j\gamma} dy_\gamma^i + M_{i\alpha\beta}^j dx^i.
\end{aligned} \tag{2.14}$$

The functions M are, for the time being, undetermined.

Proposition 2.1. *The necessary and sufficient conditions for the basis B and B^* to be dual to each other (when $\bar{B}$ and $\bar{B}^*$ are dual) are the following equations:*

$$\begin{aligned}
M_{i\alpha}^j &= N_{i\alpha}^j, \\
M_{i\alpha\beta}^{j\gamma} &= N_{i\alpha}^j \text{ for } \gamma = \beta \text{ and zero for } \beta \neq \gamma, \\
M_{i\alpha\beta}^j &= N_{i\alpha\beta}^j + N_{h\alpha}^j N_{i\beta}^h.
\end{aligned} \tag{2.15}$$

The proof follows by a straightforward calculation.

In the following we shall need the formulae which express the elements of $\bar{B}$ as functions of elements of B. These are getting in the form

$$\begin{aligned} \partial_i &= \delta_i + N^j_{i\alpha}\delta^\alpha_j + (N^h_{i\alpha}N^j_{h\beta} + N^j_{i\alpha\beta})\delta^{\alpha\beta}_h, \\ \partial^\alpha_i &= \delta^\alpha_i + N^j_{i\beta}\delta^{\alpha\beta}_j, \\ \partial^{\alpha\beta}_i &= \delta^{\alpha\beta}_i. \end{aligned} \tag{2.16}$$

We shall need also the brackets of vector fields from B

$$\begin{aligned} [\delta^{\alpha\beta}_i, \delta^{\varepsilon\gamma}_j] &= 0, \\ [\delta^\alpha_i, \delta^{\beta\gamma}_j] &= \partial^{\beta\gamma}_j(N^h_{i\varepsilon})\delta^{\alpha\varepsilon}_h, \\ [\delta_i, \delta^{\beta\gamma}_j] &= \partial^{\beta\kappa}_j(N^k_{i\alpha})\delta^\alpha_k + [\partial^{\beta\kappa}_j(N^k_{i\alpha})N^h_{k\varepsilon} + \partial^{\beta\kappa}_j(N^h_{i\alpha\varepsilon})]\delta^{\alpha\varepsilon}_h, \\ [\delta^\alpha_i, \delta^\beta_j] &= [\partial^\beta_j(N^k_{i\gamma}) - N^h_{i\varepsilon}\partial^{\beta\varepsilon}_h(N^k_{i\gamma})]\partial^{\alpha\gamma}_k - [\partial^\alpha_i(N^h_{j\varepsilon}) - N^h_{j\varepsilon}\partial^{\beta\varepsilon}_h(N^k_{i\gamma})]\partial^{\beta\gamma}_k, \\ [\delta_i, \delta_j] &= R^k_{ij\alpha}\partial^\alpha_k + \bar{R}^k_{ij\alpha\beta}\partial^{\alpha\beta}_k = R^k_{ij\alpha}\delta^\alpha_k + (\bar{R}^k_{ij\alpha\beta} - N^k_{h\beta}R^h_{ij\alpha}))\partial^{\alpha\beta}_k, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} R^k_{ij\alpha} &= \delta_j(N^k_{i\alpha}) - \delta_i(N^k_{j\alpha}), \\ \bar{R}^k_{ij\alpha\beta} &= \delta_j(N^k_{i\alpha\beta}) - \delta_i(N^k_{j\alpha\beta}). \end{aligned} \tag{2.18}$$

From (2.17) one reads

Proposition 2.2. *The horizontal distribution $u \mapsto N_0(u)$, $u \in E$ is integrable if and only if*

$$R^k_{ij\alpha} = 0, \; \bar{R}^k_{ij\alpha\beta} = 0. \tag{2.19}$$

By (1.3) and (2.4) it follows that $\overset{(\beta)}{\Gamma} = y^i_\alpha\partial^{\alpha\beta}_i$ are k-vector fields globally defined on J^2_kM. They are similar to the Liouville vector field on TM.

3. Distinguished connections on J^2_kM

Among the linear connections on $E = J^2_kM$ those which preserve by parallelism the decomposition (2.11) are remarkable ones. They are useful especially when a calculation in local coordinates is performed.

Let $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2$ be the sets of vector fields on E which take their values in the distributions N_0, N_1, N_2, respectively.

Definition 3.1. A linear connection D on E will be called a distinguished connection (d-connection, for brevity) if for any vector field $X \in \mathcal{X}(E)$ we have

$$(\forall Y \in \mathcal{N}_a)\, D_XY \in \mathcal{N}_a, \quad a = 0, 1, 2. \tag{3.1}$$

By (2.21), $P(X)=X \iff X\in\mathcal{N}_0$, $P(X)=-X \iff X\in\mathcal{N}_1\oplus\mathcal{N}_2$.

Using these formulae one finds

PROPOSITION 3.1. *A linear connection D on E is a d-connection if and only if $D_XP=0$.*

In the basis B, a d-connection D takes the form:

$$
\begin{aligned}
&D_{\delta_j}\delta_i=\overset{1}{F}{}^k_{ij}\delta_k,\ D_{\delta^\alpha_j}\delta_i=\overset{1}{V}{}^{k\alpha}_{ij}\delta_k,\ D_{\partial^{\alpha\beta}_j}\delta_i=\overset{1}{C}{}^{k\alpha\beta}_{ij}\delta_k,\\
&D_{\delta_j}\delta^\alpha_i=\overset{2}{F}{}^{k\alpha}_{ij\beta}\delta^\beta_k,\ D_{\delta^\beta_j}\delta^\alpha_i=\overset{2}{V}{}^{k\alpha\beta}_{ij\gamma}\delta^\gamma_k,\ D_{\partial^{\beta\gamma}_j}\delta^\alpha_i=\overset{2}{C}{}^{k\alpha\beta\gamma}_{ij\varepsilon}\delta^\varepsilon_k,\\
&D_{\delta_j}\partial^{\alpha\beta}_i=\overset{3}{F}{}^{k\alpha\beta}_{ij\mu\nu}\partial^{\mu\nu}_k,\ D_{\delta^\gamma_j}\partial^{\alpha\beta}_i=\overset{3}{V}{}^{k\alpha\beta\gamma}_{ij\mu\nu}\partial^{\mu\nu}_k,\ D_{\partial^{\gamma\varepsilon}_j}\partial^{\alpha\beta}_i=\overset{3}{C}{}^{k\alpha\beta\gamma\varepsilon}_{ij\mu\nu}\partial^{\mu\nu}_k.
\end{aligned}
\tag{3.2}
$$

DEFINITION 3.2. A d-connection D will be called strongly distinguished if $D\overset{\alpha}{J}=0$, $\alpha=1,\dots,k$.

A straightforward calculation gives

PROPOSITION 3.2. *A d-connection D is normal if and only if its local coefficients in* (3.1) *verify*

$$
\begin{aligned}
\overset{2}{F}{}^{k\alpha}_{ij\beta}&=\overset{1}{F}{}^k_{ij}\delta^\alpha_\beta,\ \overset{3}{F}{}^{k\alpha\beta}_{ij\mu\nu}=\overset{1}{F}{}^k_{ij}\delta^\alpha_\nu\delta^\beta_\mu,\\
\overset{2}{V}{}^{k\alpha\beta}_{ij\gamma}&=\overset{1}{V}{}^{k\beta}_{ij}\delta^\alpha_\gamma\ \ \overset{3}{V}{}^{k\alpha\gamma\beta}_{ij\mu\nu}=\overset{1}{V}{}^{k\beta}_{ij}\delta^\alpha_\nu\delta^\gamma_\mu,\\
\overset{2}{C}{}^{k\alpha\beta\gamma}_{ij\mu}&=\overset{1}{C}{}^{k\beta\gamma}_{ij}\delta^\alpha_\mu,\ \overset{3}{C}{}^{k\alpha\beta\gamma\varepsilon}_{ij\mu\nu}=\overset{1}{C}{}^{k\alpha\beta}_{ij}\delta^\varepsilon_\mu\delta^\gamma_\nu.
\end{aligned}
\tag{3.3}
$$

Thus a normal d-connection is completely determined by the local coefficients $D\Gamma=(\overset{1}{F}{}^k_{ij},\overset{1}{V}{}^{k\alpha}_{ij},\overset{1}{C}{}^{k\alpha\beta}_{ij})$.

When the local coordinates are changed by (1.3), these local coefficients are transformed as follows:

$$
\begin{aligned}
\overset{1}{F}{}^{k'}_{i'j'}&=(\partial_{i'}x^i)(\partial_{j'}x^j)(\partial_kx^{k'})\overset{1}{F}{}^k_{ij}-(\partial_{i'}x^i)(\partial_{j'}x^j)\frac{\partial^2x^{k'}}{\partial x^{i'}\partial x^{j'}}\\
\overset{1}{V}{}^{k'\alpha}_{i'j'}&=(\partial_{i'}x^i)(\partial_{j'}x^j)(\partial_kx^{k'})\overset{1}{V}{}^{k\alpha}_{ij}\\
\overset{1}{C}{}^{k'\alpha\beta}_{i'j'}&=(\partial_{i'}x^i)(\partial_{j'}x^j)(\partial_kx^{k'})\overset{1}{C}{}^{k\alpha\beta}_{ij}.
\end{aligned}
\tag{3.4}
$$

Notice that $\overset{1}{V}$ and $\overset{1}{C}$ are tensor fields and $\overset{1}{F}$ changes like the coefficients of a linear connection.

From (2.8) one sees that if $(N^j_{i\alpha}(x,y,z))$ do not depend on z in a local chart, then this happens also in any other local chart. In other words, the property $\partial^{\beta\gamma}_kN^j_{i\alpha}=0$ is a geometrical one. In fact, the functions $T^{j\beta\kappa}_{ik\alpha}=\partial^{\beta\kappa}_kN^j_{i\alpha}$, define a tensor field of type (1.2).

Using again (2.8) it follows that if $T^{j\beta\gamma}_{ik\alpha} = 0$ then $\partial^{\alpha}_j N^k_{i\alpha}$ and $\partial^{\alpha\beta}_j N^k_{i\alpha\beta}$ change under (1.3) as $\overset{1}{F}{}^k_{ij}$. Thus we have

PROPOSITION 3.3. *Let* $(N^k_{i\alpha}, N^k_{i\alpha\beta})$ *be a nonlinear connection on* E*. If* $(N^k_{i\alpha})$ *do not depend on* z *then* $B\overset{1}{\Gamma} = (\partial^{\alpha}_j N^k_{i\alpha}, 0, 0)$ *and* $B\overset{2}{\Gamma} = (\partial^{\alpha\beta}_j N^k_{i\alpha\beta}, 0, 0)$ *are normal* d*-connections on* E*.*

The connections from Proposition 3.3 are similar with the Berwald connection from Finsler geometry.

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