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ON TWO TENSOR FIELDS WHICH ARE ANALOGOUS TO THE CURVATURE AND TORSION TENSOR FIELDS

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1. Introducing of two tensor fields

Let M_n be a differentiable manifold, and let $\xi = (\mathcal{E}, \pi, M_n)$ be a vector bundle of rank k, endowed with a linear connection $\tilde{\Gamma}$. Apart from the linear connection we suppose that M_n is endowed with a linear connection Γ on the tangent bundle. So there exist two covariant derivations ∇ and $\tilde{\nabla}$, two curvature tensors

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$
$$\tilde{R}(X,Y) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]}$$

and one torsion tensor

$$T(X,Y) = \nabla_Y X - \nabla_X Y + [X,Y].$$

We introduce the following notation

$$\tilde{\tilde{\nabla}}_{(X,Y)} = \tilde{R}(X,Y) + \tilde{\nabla}_{T(X,Y)}$$

for X, $Y \in \chi(M_n)$. Now we define a mapping $R^*(X,Y,P,Q)\colon \tilde{\Gamma}\xi \to \tilde{\Gamma}\xi$ for given X, Y, P, $Q \in \chi(M_n)$ as follows

$$\begin{aligned} R^*(X,Y,P,Q) &= \tilde{\tilde{\nabla}}_{(X,Y)} \circ \tilde{\tilde{\nabla}}_{(P,Q)} - \tilde{\tilde{\nabla}}_{(P,Q)} \circ \tilde{\tilde{\nabla}}_{(X,Y)} - \tilde{\nabla}_{[T(X,Y),T(P,Q)]} \\ &+ \tilde{R}(\nabla_{T(P,Q)}X, Y) + \tilde{R}(X, \nabla_{T(P,Q)}Y) - \tilde{R}(\nabla_{T(X,Y)}P, Q) - \tilde{R}(P, \nabla_{T(X,Y)}Q) \\ &+ \tilde{R}(R(P,Q)X, Y) + \tilde{R}(X, R(P,Q)Y) - \tilde{R}(R(X,Y)P, Q) - \tilde{R}(P, R(X,Y)Q). \end{aligned} \tag{1.1}$$

Hence it follows that $R^*(X,Y,P,Q)\colon \tilde{\Gamma}\xi \to \tilde{\Gamma}\xi$ is a linear mapping such that $R^*(f_1X, f_2Y, f_3P, f_4Q) = f_1f_2f_3f_4R^*(X,Y,P,Q)$ for arbitrary functions $f_i \in$

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$\mathcal{F}(M_n)$ $(1 \leqslant i \leqslant 4)$ and arbitrary vector fields X, Y, P, $Q \in \chi(M_n)$, and thus R^* is a tensor field. Further we will find the components of the tensor field R^* in a local coordinate system.

Our convenience will be that the Latin indices take their values from the set $\{1,\dots,n\}$, and the Greek indices take their values from the set $\{1,\dots,k\}$. Let $X = \partial/\partial x^i$, $Y = \partial/\partial x^j$, $P = \partial/\partial x^r$, $Q = \partial/\partial x^s$. From (1.1) it follows that

$$(R^*(X,Y,P,Q)s)^\alpha = R^{*\alpha}_{\beta ijrs} s^\beta,$$

for arbitrary section s in the vector bundle, where

$$\begin{aligned} R^{*\alpha}_{\beta ijrs} = \tilde{R}^\alpha_{\gamma ij}\tilde{R}^\gamma_{\beta rs} - \tilde{R}^\alpha_{\gamma rs}\tilde{R}^\gamma_{\beta ij} + \tilde{R}^\alpha_{\beta pj}R^p_{irs} + \tilde{R}^\alpha_{\beta ip}R^p_{jrs} \\ - \tilde{R}^\alpha_{\beta ps}R^p_{rij} + \tilde{R}^\alpha_{\beta rp}R^p_{sij} + \tilde{R}^\alpha_{\beta rs;p}T^p_{ij} - \tilde{R}^\alpha_{\beta ij;p}T^p_{rs} + \tilde{R}^\alpha_{\beta pq}T^p_{ij}T^q_{rs}. \end{aligned} \tag{1.2}$$

From the definition (1.1) we obtain the following properties

$$\begin{aligned} R^*(X,Y,P,Q) &= -R^*(Y,X,P,Q), \\ R^*(X,Y,P,Q) &= -R^*(X,Y,Q,P), \\ R^*(X,Y,P,Q) &= -R^*(P,Q,X,Y). \end{aligned}$$

Analogously to (1.1) we can also define the following tensor

$$\tilde{R}^*(X,Y,P,Q) = \tilde{R}(X,Y) \circ \tilde{R}(P,Q) - \tilde{R}(P,Q) \circ \tilde{R}(X,Y) \tag{1.3}$$

which will be of special interest in the next sections. In local coordinate system, the components of this tensor are

$$\tilde{R}^{*\alpha}_{\beta ijrs} = \tilde{R}^\alpha_{\gamma ij}\tilde{R}^\gamma_{\beta rs} - \tilde{R}^\alpha_{\gamma rs}\tilde{R}^\gamma_{\beta ij}. \tag{1.4}$$

While the tensor R^* depends on the linear connection of the vector bundle ξ, the tensor $\tilde{R}^*$ depends on the connection of the vector bundle ξ only, and it can be obtained from the tensor R^* as a special case by putting $T = 0$ and $R = 0$.

Now we will introduce the second tensor field, which is analogous to the torsion tensor field. Let us define a mapping $T^* \colon \chi(M_n) \times \chi(M_n) \times \chi(M_n) \times \chi(M_n) \to \chi(M_n)$, by

$$\begin{aligned} &T^*(X,Y,P,Q) = \\ &\quad = T(\nabla_{T(P,Q)}X,Y) + T(X,\nabla_{T(P,Q)}Y) - T(\nabla_{T(X,Y)}P,Q) - T(P,\nabla_{T(X,Y)}Q) \\ &\qquad + T(R(P,Q)X,Y) + T(X,R(P,Q)Y) - T(R(X,Y)P,Q) - T(P,R(X,Y)Q). \end{aligned} \tag{1.5}$$

T^* is a linear mapping such that

$$T^*(f_1X, f_2Y, f_3P, f_4Q) = f_1f_2f_3f_4T^*(X,Y,P,Q)$$

for arbitrary functions $f_i \in \mathcal{F}(M_n)$ $(1 \leqslant i \leqslant 4)$ and arbitrary vector fields X, Y, P, $Q \in \chi(M_n)$, and hence T^* is a tensor field. This tensor depends on the linear connection Γ only. Similarly to the tensor field R^*, the components of the tensor field T^* in local coordinates are given by

$$T^{*m}_{ijrs} = T^m_{rs;p}T^p_{ij} - T^m_{ij;p}T^p_{rs} + T^m_{pq}T^p_{ij}T^q_{rs} + T^m_{pj}R^p_{irs} + T^m_{ip}R^p_{jrs} - T^m_{ps}R^p_{rij} - T^m_{rp}R^p_{sij}, \quad (1.6)$$

and it is obvious that

$$T^*(X,Y,P,Q) = -T^*(Y,X,P,Q),$$
$$T^*(X,Y,P,Q) = -T^*(X,Y,Q,P),$$
$$T^*(X,Y,P,Q) = -T^*(P,Q,X,Y).$$

To the end of this section we will consider the Ricci identities, and we will see that the tensors R^* and T^* naturally appear. Let $A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ be components of a ξ-tensor field A in the vector bundle ξ. We introduce the following notations for the antisymmetric covariant differentiation

$$A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q;(r,s)} = A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q;r;s} - A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q;s;r}, \quad (1.7)$$
$$A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q;((ij)(rs))} = A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q;(ij);(rs)} - A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q;(rs);(ij)}. \quad (1.8)$$

Indeed, $;(ij)$ is a differentiation and using the tensors R^* and T^*, the Ricci identity is given by

$$A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q;((ij)(rs))} = -\sum_{u=1}^{p} A^{\alpha_1\ldots\lambda\ldots\alpha_p}_{\beta_1\ldots\beta_q} R^{*\alpha_u}_{\lambda ijrs} + \sum_{u=1}^{q} A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\lambda\ldots\beta_q} R^{*\lambda}_{\beta_u ijrs} - A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q;u} T^{*u}_{ijrs}. \quad (1.9)$$

Although it is easy to obtain the formulas (1.2) and (1.6) starting from (1.9), it is not so easy to obtain the formulas (1.1) and (1.5). In the next sections we will consider the geometric interpretation and we will see that these two tensors satisfy analogous formulas as the ordinary curvature and torsion tensors.

2. Geometrical interpretation

In this section a geometrical interpretation of the tensor T^*, antisymmetric differentiation (1.8) and the tensor R^* will be given.

Let X, Y, P, $Q \in T_O(M_n)$ and let $\tau_1^{(1)}(t)$ $(0 \leqslant t \leqslant \Delta u)$, $\tau_2^{(1)}(t)$ $(0 \leqslant t \leqslant \Delta v)$, $\tau_3^{(1)}(t)$ $(0 \leqslant t \leqslant \Delta u)$ and $\tau_4^{(1)}(t)$ $(0 \leqslant t \leqslant \Delta v)$ be geodesics which connect the points O and B, B and C, C and D, D and E respectively (fig. 1), such that

$$(d\tau_1^{(1)}(t)/dt)_{t=0} = X, \qquad (d\tau_2^{(1)}(t)/dt)_{t=0} = Y_B,$$
$$(d\tau_3^{(1)}(t)/dt)_{t=0} = -X_C, \qquad (d\tau_4^{(1)}(t)/dt)_{t=0} = -Y_D,$$

Fig. 1

where Y_B is vector at B obtained by a parallel displacement along the curve $\tau_1^{(1)}$ using the linear connection Γ, and analogously X_C and Y_D are defined.

If $A \in C^2$ is a tensor field in a neighborhood of O and if $X = \partial/\partial x^p$ and $Y = \partial/\partial x^s$, then

$$A^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q ;(rs)} = - \lim_{\Delta U, \Delta v \to 0} \frac{1}{\Delta u \Delta v} \left[\varphi_E^O (A^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q}) - (A^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q})_O \right], \tag{2.1}$$

and it gives the geometrical interpretation of the diferentiation (1.7).

Now we will give similar geometrical interpretation of the differentiation (1.8). We transport the vectors P and Q parallelly to the point E through the points B, C, D, and using the vectors P_E and Q_E similarly we construct geodesics $\tau_1^{(2)}(t)$ $(0 \leqslant t \leqslant \Delta u')$, $\tau_2^{(2)}(t)$ $(0 \leqslant t \leqslant \Delta v')$, $\tau_3^{(2)}(t)$ $(0 \leqslant t \leqslant \Delta u')$, $\tau_4^{(2)}(t)$ $(0 \leqslant t \leqslant \Delta v')$, and obtain a point F. Similarly, using the parallelly transported vectors $-X_F$ and $-Y_F$ we can construct geodesics $\tau_1^{(3)}$, $\tau_2^{(3)}$, $\tau_3^{(3)}$ and $\tau_4^{(3)}$ and obtain a point G, and finally, using the vectors $-P_G$ and $-Q_G$ we find a point H. Further we are going to find the difference $(x^t)_H - (x^t)_O$ of the coordinates of the points H and O at fourth order approximation, neglecting all the terms of order higher than four everywhere.

We notice that

$$(x^t)_E - (x^t)_O = T^t_{rs} X^r Y^s \Delta u \Delta v + \vartheta_3(X,Y) + \vartheta_4(X,Y), \tag{2.2}$$

$$(x^t)_F - (x^t)_E = (T^t_{rs})_E (P^r)_E (Q^s)_E \Delta u' \Delta v' + \vartheta_3(P_E, Q_E) + \vartheta_4(P_E, Q_E), \tag{2.3}$$

where ϑ_i $(i = 3, 4)$ are the terms of i-th order of approximation, and $(T^t_{rs})_E$ are the components of the torsion tensor at the point E. At the second order of approximation it holds

$$(T^t_{rs})_E = T^t_{rs} + (\partial T^t_{rs}/\partial x^j) T^j_{pq} X^p Y^q \Delta u \Delta v, \tag{2.4}$$

and from (2.1) at the same order of approximation it holds

$$(P^r)_E = P^r - P^i(R^r_{ipq} + \Gamma^r_{ij}T^j_{pq})X^pY^q\Delta u\Delta v, \tag{2.5}$$

$$(Q^s)_E = Q^s - Q^i(R^s_{ipq} + \Gamma^s_{ij}T^j_{pq})X^pY^q\Delta u\Delta v. \tag{2.6}$$

By adding the equalities (2.2) and (2.3), and using the equalities (2.4), (2.5) and (2.6), we obtain

$$\begin{aligned}&(x^t)_F - (x^t)_O = T^t_{rs}X^rY^s\Delta u\Delta v + T^t_{rs}P^rQ^s\Delta u'\Delta v'\\ &+ U^t_{ijrs}X^iY^jP^rQ^s\Delta u\Delta v\Delta u'\Delta v' + \vartheta_3(X,Y) + \vartheta_4(X,Y) + \vartheta_3(P,Q) + \vartheta_4(P,Q),\end{aligned}$$

where

$$U^t_{ijrs} = \partial T^t_{rs}/\partial x^p \cdot T^p_{ij} - T^t_{rp}(R^p_{sij} + \Gamma^p_{sq}T^q_{ij}) - T^t_{ps}(R^p_{rij} + \Gamma^p_{rq}T^q_{ij}).$$

In the same way as the point F was obtained using vectors X, Y, P and Q, another point F' can be obtained using vectors P, Q, X and Y. Moreover, at the fourth order of approximation it holds

$$(x^t)_H - (x^t)_O = [(x^t)_F - (x^t)_O] - [(x^t)_{F'} - (x^t)_O].$$

Hence and using the following equality

$$\begin{aligned}&(x^t)_{F'} - (x^t)_O = T^t_{rs}P^rQ^s\Delta u'\Delta v' + T^t_{rs}X^rY^s\Delta u\Delta v\\ &+ U^t_{ijrs}P^iQ^jX^rY^s\Delta u\Delta v\Delta u'\Delta v' + \vartheta_3(P,Q) + \vartheta_4(P,Q) + \vartheta_3(X,Y) + \vartheta_4(X,Y),\end{aligned}$$

one can prove that

$$(x^t)_H - (x^t)_O = T^{*t}_{ijrs}X^iY^jP^rQ^s\Delta u\Delta v\Delta u'\Delta v', \tag{2.7}$$

which gives a geometrical interpretation of the tensor T^*. Especially, if $X = \partial/\partial x^i$, $Y = \partial/\partial x^j$, $P = \partial/\partial x^r$ and $Q = \partial/\partial x^s$, we obtain

$$(x^t)_H - (x^t)_O = T^{*t}_{ijrs}\Delta u\Delta v\Delta u'\Delta v'. \tag{2.8}$$

Further we will obtain the geometrical interpretation of the differentiation (1.8). In a neighborhood of the point O we choose a ξ-vector field A^α of class C^4. We parallelly transport the vector A^α from the point O to the point E along the previously mentioned geodesic lines, using the connection $\tilde\Gamma$. Then the fourth order of approximation leads to the following transported vector

$$(A^\alpha)_E = A^\alpha - A^\beta(R^\alpha_{\beta pq} + \Gamma^\alpha_{\beta t}T^t_{pq})X^pY^q\Delta u\Delta v + \vartheta_3(X,Y,A) + \vartheta_4(X,Y,A). \tag{2.9}$$

Similarly to (2.9) the parallelly transported vector $(A^\alpha)_E$ from E to F is

$$\begin{aligned}(A^\alpha)_F = (A^\alpha)_E - (A^\beta)_E[(R^\alpha_{\beta rs})_E + (\Gamma^\alpha_{\beta j})_E(T^j_{rs})_E](P^r)_E(Q^s)_E\Delta u'\Delta v'&\\ + \vartheta_3(P,Q,A) + \vartheta_4(P,Q,A),&\end{aligned}$$

where

$$(R^\alpha_{\beta rs})_E = R^\alpha_{\beta rs} + \partial R^\alpha_{\beta rs}/\partial x^a \cdot T^a_{pq} X^p Y^q \Delta u \Delta v,$$
$$(\Gamma^\alpha_{\beta j})_E = \Gamma^\alpha_{\beta j} + \partial \Gamma^\alpha_{\beta j}/\partial x^a \cdot T^a_{pq} X^p Y^q \Delta u \Delta v,$$
$$(T^j_{rs})_E = T^j_{rs} + \partial T^j_{rs}/\partial x^a \cdot T^a_{pq} X^p Y^q \Delta u \Delta v,$$

$(P^r)_E$, $(Q^s)_E$ and $(A^\alpha)_E$ are given by (2.5), (2.6) and (2.9). Hence, at the fourth order of approximation we obtain

$$\begin{aligned}(A^\alpha)_F = A^\alpha &- A^\beta(R^\alpha_{\beta pq} + \Gamma^\alpha_{\beta j} T^j_{pq}) X^p Y^q \Delta u \Delta v \\ &- A^\beta(R^\alpha_{\beta pq} + \Gamma^\alpha_{\beta j} T^j_{pq}) P^p Q^q \Delta u' \Delta v' + A^\beta V^\alpha_{\beta ijrs} X^i Y^j P^r Q^s \Delta u \Delta v \Delta u' \Delta v' \\ &+ \vartheta_3(X,Y,A) + \vartheta_4(X,Y,A) + \vartheta_3(P,Q,A) + \vartheta_4(P,Q,A),\end{aligned}$$

where

$$\begin{aligned}V^\alpha_{\beta ijrs} &= R^\alpha_{\gamma rs} R^\gamma_{\beta ij} + R^\alpha_{\gamma rs}\Gamma^\gamma_{\beta p} T^p_{ij} + R^\gamma_{\beta ij}\Gamma^\alpha_{\gamma p} T^p_{rs} + \Gamma^\gamma_{\beta p}\Gamma^\alpha_{\gamma q} T^p_{ij} T^q_{rs} + R^\alpha_{\beta ps} R^p_{rij} \\ &+ \Gamma^\alpha_{\beta p} T^p_{qs} R^q_{rij} + R^\alpha_{\beta ps}\Gamma^p_{rq} T^q_{ij} + \Gamma^\alpha_{\beta p} T^p_{qs}\Gamma^q_{rt} T^t_{ij} + R^\alpha_{\beta rp} R^p_{sij} + \Gamma^\alpha_{\beta p} T^p_{rq} R^q_{sij} \\ &+ R^\alpha_{\beta rp}\Gamma^p_{sq} T^q_{ij} + \Gamma^\alpha_{\beta p} T^p_{rq}\Gamma^q_{st} T^t_{ij} - \partial R^\alpha_{\beta rs}/\partial x^p \cdot T^p_{ij} - \Gamma^\alpha_{\beta p}\partial T^p_{rs}/\partial x^q \cdot T^q_{ij} - \partial\Gamma^\alpha_{\beta p}/\partial x^q \cdot T^p_{rs} T^q_{ij}.\end{aligned}$$

Symmetrically, it holds

$$\begin{aligned}(A^\alpha)_{F'} = A^\alpha &- A^\beta(R^\alpha_{\beta pq} + \Gamma^\alpha_{\beta j} T^j_{pq}) P^p Q^q \Delta u' \Delta v' \\ &- A^\beta(R^\alpha_{\beta pq} + \Gamma^\alpha_{\beta j} T^j_{pq}) X^p Y^q \Delta u \Delta v + A^\beta V^\alpha_{\beta ijrs} P^i Q^j X^r Y^s \Delta u \Delta v \Delta u' \Delta v' \\ &+ \vartheta_3(P,Q,A) + \vartheta_4(P,Q,A) + \vartheta_3(X,Y,A) + \vartheta_4(X,Y,A).\end{aligned}$$

Thus, at the fourth order of approximation we obtain

$$(A^\alpha)_H = (A^\alpha)_O + [(A^\alpha)_F - (A^\alpha)_{F'}] = A^\beta(V^\alpha_{\beta ijrs} - V^\alpha_{\beta rsij}) X^i Y^j P^r Q^s \Delta u \Delta v \Delta u' \Delta v'.$$

According to (2.8), the ξ-vector field A^α at the point H has coordinates

$$A^\alpha + \partial A^\alpha/\partial x^p \cdot T^{*p}_{ijrs} X^i Y^j P^r Q^s \Delta u \Delta v \Delta u' \Delta v'.$$

Subtracting the parallelly transported vector $(A^\alpha)_H$ from this vector, we obtain the following vector

$$\begin{aligned}&(A^\alpha_{;p} T^{*p}_{ijrs} + A^\beta R^{*\alpha}_{\beta ijrs}) X^i Y^j P^r Q^s \Delta u \Delta v \Delta u' \Delta v' = \\ &\qquad = -A^\alpha_{;((ij)(rs))} X^i Y^j P^r Q^s \Delta u \Delta v \Delta u' \Delta v'.\end{aligned}$$

Especially, if $X = \partial/\partial x^i$, $Y = \partial/\partial x^j$, $P = \partial/\partial x^r$ and $Q = \partial/\partial x^s$, we obtain

$$A^\alpha_{;((ij)(rs))} = -\lim_{\Delta u, \Delta v, \Delta u', \Delta v' \to 0} \frac{1}{\Delta u \Delta v \Delta u' \Delta v'}[(A^\alpha)_H - \varphi^O_H(A^\alpha)], \tag{2.10}$$

where φ^O_H denotes the parallel displacement from O to H with respect to the connection $\tilde{\Gamma}$, and $(A^\alpha)_H$ denotes the vector of the field A^α at the point H. Now (2.10) represents the geometrical interpretation of the differentiation (1.8). Note that (2.10) also holds if the vector field A is substituted by an arbitrary ξ-tensor field.

Further we will give geometrical interpretation of the tensor $\tilde{R}^*$. Let X, Y, P, $Q \in T_O(M_n)$, and let τ' and τ'' be two infinitesimal parallelograms with adjacent edges $\Delta u \cdot X$ and $\Delta v \cdot Y$ for τ', and $\Delta u' \cdot P$ and $\Delta v' \cdot Q$ for τ''. Let A be an arbitrary ξ-tensor field in a neighborhood of O. We will denote by $\varphi(A)$ the ξ-tensor at the point O which is obtained by parallel displacement of the ξ-tensor A_O along the curve $\tau' \cdot \tau'' \cdot (\tau')^{-1} \cdot (\tau'')^{-1}$ using the connection $\tilde{\Gamma}$. It is easy to verify that at the fourth order of approximation, it holds

$$(\varphi(A))^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} - A^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = \Big[- \sum_{w=1}^{p} A^{\alpha_1 \ldots \gamma \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \tilde{R}^{*\alpha_w}_{\gamma ijrs} + \sum_{w=1}^{q} A^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \gamma \ldots \beta_q} \tilde{R}^{*\gamma}_{\beta_w ijrs} \Big] X^i Y^j P^r Q^s \Delta u \Delta v \Delta u' \Delta v', \quad (2.11)$$

which gives the required geometrical interpretation.

3. Structure equation and Bianchi identity

Further we will not use the linear connection Γ and the tensors R^* and T^*, and so our convenience will be to use the notations Γ, ∇, R and R^* instead of $\tilde{\Gamma}$, $\tilde{\nabla}$, $\tilde{R}$ and $\tilde{R}^*$.

First we will obtain the structure equation and the Bianchi identity for the curvature tensor R^* in the case of the vector bundle $(\mathcal{E}, \pi, M_n)$, and further we will consider the general case when a connection is given in a principal bundle $(\mathcal{E}, \pi, \mathcal{E}/G)$.

Like in the non-commutative differential geometry [1], the need of introducing of differential forms of the following kind naturally appears here. Differential r-form is a correspondence of an element of $\Lambda^r(T'_x(M_n))$ to each point $x \in M_n$, where $T'_x(M_n)$ is the vector space of bivectors on $T_x(M_n)$, i.e.

$$T'_x(M_n) = \left\{ a^{ij} \left[\left(\frac{\partial}{\partial x^i} \right)_x \wedge \left(\frac{\partial}{\partial x^j} \right)_x \right] \ : \ a^{ij} \in \mathbf{R} \right\}.$$

For example, an arbitrary differential 3-form in local coordinates has the following form

$$w = f_{i_1 j_1 i_2 j_2 i_3 j_3} (dx^{i_1} \wedge dx^{j_1}) \wedge (dx^{i_2} \wedge dx^{j_2}) \wedge (dx^{i_3} \wedge dx^{j_3}).$$

For the sake of simplicity our convenience is to write the differential forms as in the following form $w = f_{i_1 j_1 i_2 j_2 i_3 j_3} dx^{i_1 j_1} \wedge dx^{i_2 j_2} \wedge dx^{i_3 j_3}$. Note that

$$dx^{ij} \left(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \right) = \delta^i_p \delta^j_q - \delta^i_p \delta^j_q .$$

The need of consideration of differential forms with values in the vector bundle $T^p_q \xi$ also appears. Using the tensor $R^\alpha_{\beta ij}$ we obtain the following differential 1-form

$$\Omega^\alpha_\beta = \frac{1}{2} R^\alpha_{\beta ij} \, dx^{ij} \quad (3.1)$$

with values in $\mathrm{End}\, \xi = \xi^* \otimes \xi$.

Now we define an operator D^* as follows. If ϕ is a differential r-form with values in $T^p_q\xi$, then $D^*\phi$ is a differential $(r+1)$-form with values in the tensor bundle $T^p_q\xi$ and

$$(D^*\phi)^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} = \sum_{s=1}^{p} \Omega^{\alpha_s}_{\lambda} \wedge \phi^{\alpha_1\dots\lambda\dots\alpha_p}_{\beta_1\dots\beta_q} - \sum_{s=1}^{q} \Omega^{\lambda}_{\beta_s} \wedge \phi^{\alpha_1\dots\alpha_p}_{\beta_1\dots\lambda\dots\beta_q}. \tag{3.2}$$

The connection between the operator D^* and the differential operator D is given as follows. If

$$\psi^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} = A^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q s_1\dots s_r}\, dx^{s_1} \wedge \cdots \wedge dx^{s_r}$$

is an ordinary r-form (i.e. an element of $\Lambda^r(T(M_n))$) with values in $T^p_q\xi$, then it is well known that

$$(D\psi)^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} = \Big[\partial A^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q s_1\dots s_r}/\partial x^u + \sum_{s=1}^{p} \Gamma^{\alpha_s}_{\lambda u} A^{\alpha_1\dots\lambda\dots\alpha_p}_{\beta_1\dots\beta_q s_1\dots s_r} - \sum_{s=1}^{q} \Gamma^{\lambda}_{\beta_s u} \phi^{\alpha_1\dots\alpha_p}_{\beta_1\dots\lambda\dots\beta_q s_1\dots s_r}\Big]\, dx^u \wedge dx^{s_1} \wedge \cdots \wedge dx^{s_r}. \tag{3.3}$$

It can be verified that

$$(DD\psi)^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} = \sum_{s=1}^{p} \Omega^{\alpha_s}_{\lambda} \wedge \psi^{\alpha_1\dots\lambda\dots\alpha_p}_{\beta_1\dots\beta_q} - \sum_{s=1}^{q} \Omega^{\lambda}_{\beta_s} \wedge \psi^{\alpha_1\dots\alpha_p}_{\beta_1\dots\lambda\dots\beta_q}, \tag{3.4}$$

where Ω^α_β is the differential 2-form of curvature. According to (3.2) and (3.4) it seems to be that $D^2 = D^*$, but it is not because D^2 acts on the elements of $\Lambda^r(T(M_n))$ but D^* acts on the elements of $\Lambda^r(T'(M_n))$.

The definition (3.2) of D^* gives the following properties:

(i) If $p = q = 0$, then $D^*\phi = 0$.

(ii) The operator D^* commutes with the contraction.

(iii) If ϕ_1 is a differential r-form with values in $T^p_q\xi$ and ϕ_2 is a differential s-form with values in $T^u_v\xi$, then

$$D^*(\phi_1 \wedge \phi_2) = (D^*\phi_1) \wedge \phi_2 + (-1)^r \phi_1 \wedge (D^*\phi_2).$$

The operator D^2 satisfies the properties (i) and (ii) also, and instead of (iii) it satisfies

$$D^2(\phi_1 \wedge \phi_2) = (D^2\phi_1) \wedge \phi_2 + \phi_1 \wedge (D^2\phi_2).$$

So D^* is more convenient than D^2. Using the tensor R^* we obtain a differential 2-form with values in $\operatorname{End}\xi = \xi^* \otimes \xi$, as follows

$$\Omega^{*\alpha}_{\beta} = \frac{1}{2} R^{*\alpha}_{\beta ijrs} dx^{ij} \wedge dx^{rs}. \tag{3.5}$$

From (3.1), (3.5) and (1.4) it is easy to obtain

$$\Omega^{*\alpha}_{\beta} = \Omega^{\alpha}_{\gamma} \wedge \Omega^{\gamma}_{\beta} \tag{3.6}$$

and

$$\Omega_{\beta}^{*\alpha} = D^*(\Omega_{\beta}^{\alpha}). \tag{3.7}$$

The equation (3.7) is the structural equation for the form of curvature Ω^*. Note that if Ω_{β}^{α} was considered as a differential 2- form (an element of $\Lambda^2(T(M_n))$) we would obtain $DD(\Omega_{\beta}^{\alpha}) = 0$. That shows the need of consideration of new class of differential forms as elements of $\Lambda^r(T'(M_n))$.

From (3.2) and (3.6) it follows that

$$D^*\Omega_b^{*\alpha} = 0 \tag{3.8}$$

which is the Bianchi identity for the curvature form Ω^*.

If ϕ is a differential r-form, then (3.2) and (3.6) imply the following identity

$$(D^*D^*\psi)^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \sum_{s=1}^{p} \Omega_{\lambda}^{*\alpha_s} \wedge \psi^{\alpha_1\ldots\lambda\ldots\alpha_p}_{\beta_1\ldots\beta_q} - \sum_{s=1}^{q} \Omega_{\beta_s}^{*\lambda} \wedge \psi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\lambda\ldots\beta_q}, \tag{3.9}$$

which has the same form as the operator $D \cdot D$. So we can introduce a new class of differential forms which correspond an element of $\Lambda^r(T''_x(M_n))$ to each point $x \in M_n$, where $T''_x = T'_x(T'_x(M_n))$. Thus

$$\Omega_{\beta}^{*\alpha} = \frac{1}{2} R^{*\alpha}_{\beta ijrs} dx^{ijrs} \tag{3.10}$$

is a differential 1-form with values in $\operatorname{End}\xi$ and for arbitrary r-form ϕ with values in $T^p_q\xi$, we define the following differentiation

$$(D^{**}\phi)^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \frac{1}{2}\left[\sum_{s=1}^{p} \Omega_{\lambda}^{*\alpha_s} \wedge \phi^{\alpha_1\ldots\lambda\ldots\alpha_p}_{\beta_1\ldots\beta_q} - \sum_{s=1}^{q} \Omega_{\beta_s}^{*\lambda} \wedge \phi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\lambda\ldots\beta_q}\right]. \tag{3.11}$$

This procedure can be further continued, and the corresponding structural equations and the Bianchi identities can be obtained.

Now we will generalize the previous considerations to the case of a principal G-bundle $(\mathcal{E}, \pi, \mathcal{E}/G)$. Let ω be the form of connection and Ω be the form of curvature. Then it is known that the tensorial forms of type $\operatorname{Ad} G$ can be identified by $\xi[\mathrm{Ad}]$-valued forms on $M_n = \mathcal{E}/G$, where $\xi[\mathrm{Ad}]$ is the associated vector bundle ([3]). Thus we will use the tensorial forms of type $\operatorname{Ad} G$ on $\mathcal{E}$, instead of $\xi[\mathrm{Ad}]$-valued forms on M_n.

Let ψ be an ordinary tensorial r-form of type $\operatorname{Ad} G$, in sense that at each point $u \in \mathcal{E}$ we have an element of $\Lambda^r(T_u(\mathcal{E}))$. Then it is well known that

$$D\psi = d\psi + [\omega, \psi]. \tag{3.12}$$

From the definition of the operation $[\,]$, the following identities can be verified

$$d[\psi_1, \psi_2] = [d\psi_1, \psi_2] - [\psi_1, d\psi_2] \tag{3.13}$$

and

$$[\psi_1, [\psi_1, \psi_2]] = \frac{1}{2}[[\psi_1, \psi_1], \psi_2], \tag{3.14}$$

where ψ_1 is a differential 1-form and ψ_2 is a differential r-form. Using the formulas (3.12), (3.13) and (3.14), if ϕ is an arbitrary r-form of type $\mathrm{Ad}\,G$, we obtain

$$\begin{aligned}(D\cdot D)\psi &= D(d\psi+[\omega,\psi]) = d[\omega,\psi]+[\omega, d\psi+[\omega,\psi]]\\ &= [d\omega,\psi]-[\omega,d\psi]+[\omega,[\omega,\psi]]+[\omega,d\psi]\\ &= [d\omega,\psi]+\frac{1}{2}[[\omega,\omega],\psi]=[\Omega,\psi]. \end{aligned}\tag{3.15}$$

Further we are going to define tensorial forms of type $\mathrm{Ad}\,G$ in sense that for each point $u\in\mathcal{E}$ we have an element of $\Lambda^r(T_u'(\mathcal{E}))$. So we first define that the bivector S is horizontal, if it can be represented in the following form

$$S=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}X_i\wedge X_j,$$

where $X_1,\,\dots\,,\,X_n$ are horizontal vectors. Besides, it is naturally to define

$$(R_\alpha)_*(Y_1\wedge Y_2)=(R_\alpha)_*Y_1\wedge(R_\alpha)_*Y_2,$$

where $Y_1,\,Y_2\in T_u(\mathcal{E})$. Now we define a new class of tensorial r-forms of type $\mathrm{Ad}\,G$ as g-valued differential r-forms, which correspond to each point $u\in\mathcal{E}$ an element of $\Lambda^r(T_u'(\mathcal{E}))$ such that:

(i) $R_a^*\phi=(\mathrm{Ad}\,a^{-1})\phi$ for each $a\in G$, and

(ii) $\phi(S_1,\dots,S_r)=0$, if at least one of the bivectors $S_1,\,\dots\,,\,S_r$ is not horizontal.

Now the tensorial form of the curvature can be interpreted as a tensorial 1-form Ω' of type $\mathrm{Ad}\,G$ in the sense of the previous definition, if we suppose that $\Omega'(X\wedge Y)=\Omega(X,Y)$. Thus, if ϕ is a tensorial r-form of type $\mathrm{Ad}\,G$ in the sense of the previous definition, then the formula (3.15) leads us to introduce an operator D^* by

$$D^*\phi=[\Omega',\phi]\tag{3.16}$$

and now $D^*\phi$ is a tensorial $(r+1)$-form of type $\mathrm{Ad}\,G$. From the definition of the operation [] one can verify that

$$D^*(\phi_1\wedge\phi_2)=(D^*\phi_1)\wedge\phi_2+(-1)^r\phi_1\wedge(D^*\phi_2),$$

where ϕ_1 is a tensorial r-form and ϕ_2 is a tensorial s-form of type $\mathrm{Ad}\,G$.

Further we define tensorial 2-form of curvature Ω^* by $\Omega^*=[\Omega',\Omega']$, i.e.

$$\Omega^*=D^*(\Omega').\tag{3.17}$$

The formula (3.17) represents the structural equation. It is easy to verify that

$$D^*(\Omega^*)=0,\tag{3.18}$$

which is the Bianchi identity.

The equality (3.14) implies

$$(D^* \cdot D^*)\phi = \frac{1}{2}[\Omega^*, \phi]. \tag{3.19}$$

This formula leads us to introduce a new operator, if we consider Ω^* as a tensorial 1-form of type $\mathrm{Ad}\, G$. This procedure can be continued further, and the corresponding structural equations and the Bianchi identities can be obtained.

4. Holonomy group

Let $\Phi_0(u)$ be the restricted holonomy group for the considered principal bundle $(\mathcal{E}, \pi, \mathcal{E}/G)$ at point $u \in \mathcal{E}$, and we will denote by $f(u)$ the corresponding Lie algebra. We suppose that $\mathcal{E}/G$ is a connected manifold. We will prove the following proposition.

Proposition. *Suppose that there exists a point $v \in \mathcal{E}$ such that $f(v)$ is generated by the elements of the form $\Omega_v(A, B)$ where A, $B \in H_v$ are arbitrary horizontal vectors at the point v. If Ω^* iz zero form on $\mathcal{E}$, then $\Phi_0(u)$ is a commutative group for each $u \in \mathcal{E}$.*

Conversely, if $\Phi_0(u)$ is a commutative group, then Ω^* iz a zero form on $\mathcal{E}$.

Proof. Let $f(v)$ is generated by the elements $\Omega_v(A, B)$, A, $B \in H_v$ and let $\Omega^* = 0$. Hence

$$\begin{aligned}[]
[\Omega_v(A,B), \Omega_v(C,D)] &= [\Omega'_v(A \wedge B), \Omega'_v(C \wedge D)] = \frac{1}{2}[\Omega'_v, \Omega'_v](A \wedge B, C \wedge D) \\
&= \frac{1}{2}\Omega^*_v(A \wedge B, C \wedge D) = 0
\end{aligned}$$

for each A, B, C, $D \in H_v$, and so $[X, Y] = 0$ for each X, $Y \in f(v)$. Since $\Phi_0(v)$ is a connected Lie group whose Lie algebra is $f(v)$, it follows that $\Phi_0(v)$ is a commutative group, and hence $\Phi_0(u)$ is commutative for each $u \in \mathcal{E}$.

Let $\Phi_0(\omega)$ is a commutative group, and let $u \in \mathcal{E}$ be an arbitrary point. Then $\Phi_0(u)$ is also a commutative group. So, $[X, Y] = 0$ for each X, $Y \in f(u)$ and hence Ω^*_u iz a zero form. ∎

Moreover, if there exists $v \in \mathcal{E}$ which can be connected with u by a horizontal path and such that $f(u)$ is generated by the elements of the form $\Omega_v(A, B)$ where A, $B \in H_v$, then one can prove that the derived group $\Phi'_0(u)$ is a Lie group and its Lie algebra is generated by the elements of the form $\Omega^*_v(A \wedge B, C \wedge D)$ where A, B, C, $D \in H_v$.

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