МАТЕМАТИЧКИ ВЕСНИК
48 (1996), 77–82

UDK 517.518.3
оригинални научни рад
research paper

A BASIS PROPERTY OF FAMILIES OF THE MITTAG-LEFFLER FUNCTIONS

Milutin Dostanić

Abstract. In this paper we study basis properties of the Mittag-Leffler functions $\{y(x,\lambda_k)\}$ where

$$y(x,\lambda)=|x|^{\omega/2}E_1(ix\lambda;1+\omega/2),\quad \omega\in(-1,1),$$

$E_1(z;\alpha)=\sum\limits_{k=0}^{\infty}\dfrac{z^k}{\Gamma(\alpha+k)}$ and $\{\lambda_k\}$ is a sequence of complex numbers. In the case $\omega=0$ this system is reduced to the exponential system.

1. Introduction

In [3] necessary and sufficient conditions for the system $\{x^{\alpha-1}E_1(i\lambda_n x;\alpha)\}$ to form an unconditional basis in $L^2(0,1)$ are obtained. The same conditions are obtained in [9] as a consequence of a more general result. These conditions are expressed in terms of the Muckenhoupt condition which should be satisfied by an entire function with zeros $\{\lambda_k\}$. Conditions that the system $\{e^{i\lambda_n x}\}$ is a Riesz basis of $L^2(-\pi,\pi)$ can also be expressed in terms of the Muckenhoupt condition [8].

These conditions are not always easy to check. In [5] some sufficient conditions that the exponential system is a Riesz basis in $L^2(-\pi,\pi)$ are given. In this paper we give some sufficient conditions that the system of the Mittag-Leffler functions is a Riesz basis in $L^2(-\pi,\pi)$.

2. Preliminaries

Given $f,g\colon X\to\mathbf{R}_+$ we write $f(x)\asymp g(x)$ $(x\in X)$ if there exist constants C_1, $C_2>0$ such that $C_1\leqslant f(x)/g(x)\leqslant C_2$ $(x\in X)$.

DEFINITION. [1] An entire function F of exponential type is said to be of sine type if

1) the zeros of F lie in $\{\, z\in\mathbf{C}\mid |\Im z|\leqslant h\,\}$ for some $h>0$.
2) there is $y_0\in\mathbf{R}$ such that $|F(x+iy_0)|\asymp 1$ $(x\in\mathbf{R})$ holds.

AMS Subject Classification: 30B60
Keywords and phrases: Basis property; Riesz basis; Mittag-Leffler functions
Supported by Ministry of Science and technology RS, grant number 04M01

If f is an entire function of exponential type, then by $h_f(\theta)$ we denote the indicator function

$$h_f(\theta) = \overline{\lim_{r\to\infty}} \frac{\ln |f(re^{i\theta})|}{r}.$$

By $W_\pi(\omega)$ we denote the class of entire functions of exponential type $\sigma \leqslant \pi$ such that $\int_{-\infty}^{\infty} |x|^\omega |f(x)|^2\, dx < \infty$ for some $\omega \in (-1,1)$. A characterization of this class is given by the following theorem:

THEOREM 1. [4]

$$W_\pi(\omega) = \left\{ f \;\middle|\; f(\lambda) = \int_{-\pi}^{\pi} E_1\left(i\lambda t; 1+\frac{\omega}{2}\right) |t|^{\omega/2}\varphi(t)\, dt \text{ and } \varphi \in L^2(-\pi,\pi) \right\}.$$

The function φ is almost everywhere determined by the formula

$$\frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty} \frac{e^{-itv}-1}{-iv} f(v)\left(|v|e^{i\frac{\pi}{2}\operatorname{sgn} v}\right)^{\frac{\omega}{2}} dv = \begin{cases} \varphi(t); & t \in (-\pi,\pi), \\ 0; & t \notin (-\pi,\pi). \end{cases}$$

The next theorem gives some sufficient conditions for the exponential system to be a Riesz basis in $L^2(-\pi,\pi)$.

THEOREM 2. [7], [8] *Let S be an entire function of sine type with zeros $\{\lambda_n\}_{n\in\mathbf{Z}}$ such that $h_s(\pm\pi/2) = \pi$ and $\inf_{m\neq n} |\lambda_n - \lambda_m| > 0$. Then the system of functions $\{e^{i\lambda_n x}\}_{n=-\infty}^{\infty}$ is a Riesz basis in $L^2(-\pi,\pi)$.*

3. Main result

Let the sequence $\{\lambda_n\}_{-\infty}^{\infty}$ satisfies the conditions of Theorem 2 and $\varphi(x) = |x|^{\omega/2}E_1(ix; 1+\omega/2)$.

THEOREM 3. *If $\omega \in (-1,1)$, then the system of functions $\{\varphi(x\lambda_n)\}_{-\infty}^{\infty}$ is a Riesz basis in $L^2(-\pi,\pi)$.*

First we prove the following lemma.

LEMMA. *If $f \in W_\pi(\omega)$ ($\omega \in (-1,1)$), $\{\lambda_n\}_{-\infty}^{\infty}$ is the sequence of zeros of a sine type function S which satisfies the conditions of Theorem 2, then*

$$\sum_{n=-\infty}^{\infty} |\lambda_n|^\omega |f(\lambda_n)|^2 \asymp \int_{\mathbf{R}} |x|^\omega |f(x)|^2\, dx. \tag{1}$$

Proof. Let $L_2^0(\mathbf{R})$ be the set of functions in $L^2(\mathbf{R})$ such that their Fourier transformations vanish a.e. on $\mathbf{R}\setminus[-\pi,\pi]$. Since $\frac{S(\lambda)}{S'(\lambda_n)(\lambda-\lambda_n)}$ is an entire function of exponential type and S is a sine type function, then

$$\frac{S(\lambda)}{S'(\lambda_n)(\lambda-\lambda_n)} \in W_\pi(\omega), \quad \omega \in (-1,1).$$

From the Theorem of B. Y. Levin and V. D. Golovin [5] it follows that the system of functions $\{e^{i\lambda_n x}\}_{-\infty}^{\infty}$ is a Riesz basis in $L^2(-\pi,\pi)$.

Since $\frac{S(\lambda)}{S'(\lambda_n)(\lambda-\lambda_n)} \in L^2(\mathbf{R})$ and it is an entire function of exponential type, from the Paley-Wiener theorem it follows that there exists a function $\psi_n \in L^2(-\pi,\pi)$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda x}\overline{\psi_n(x)}\,dx = \frac{S(\lambda)}{S'(\lambda_n)(\lambda-\lambda_n)}. \tag{2}$$

From (2) it follows that the system of functions $\{(2\pi)^{-1}\psi_n\}_{-\infty}^{\infty}$ is biorthogonal to the system $\{e^{i\lambda_n x}\}_{n=-\infty}^{\infty}$. Since the system $\{e^{i\lambda_n x}\}_{-\infty}^{\infty}$ is a Riesz basis in $L^2(-\pi,\pi)$, we conclude that the system $\{\psi_n\}_{-\infty}^{\infty}$ is also a Riesz basis in $L^2(-\pi,\pi)$. Hence, the system of functions $\{\psi_n(-x)\}_{-\infty}^{\infty}$ is also a Riesz basis in $L^2(-\pi,\pi)$.

Now we shall prove that the system $\left\{\frac{S(\lambda)}{S'(\lambda_n)(\lambda-\lambda_n)}\right\}_{-\infty}^{\infty}$ is a Riesz basis in $L_2^0(\mathbf{R})$.

It is enough to show ([2]) that this system is complete in $L_2^0(\mathbf{R})$ and that for arbitrary constants C_ν there holds

$$\left\|\sum_\nu C_\nu \frac{S(\lambda)}{S'(\lambda_n)(\lambda-\lambda_n)}\right\| \asymp \sum_\nu |C_\nu|^2. \tag{3}$$

But this follows directly from the Parseval relation, (2) and the fact that the system $\{\psi_n(x)\}$ is a Riesz basis in $L^2(-\pi,\pi)$.

Let $f \in W_\pi(\omega)$. From Theorem 1 it follows that $f(v)\left(|v|e^{i\frac{\pi}{2}\operatorname{sgn} v}\right)^{\omega/2} \in L_2^0(\mathbf{R})$,

$$f(x)\left(|x|e^{i\frac{\pi}{2}\operatorname{sgn} x}\right)^{\frac{\omega}{2}} = \sum_\nu d_\nu \frac{S(x)}{S'(\lambda_\nu)(x-\lambda_\nu)} \tag{4}$$

and

$$\sum_\nu |d_\nu|^2 \asymp \int_{-\infty}^{\infty} |x|^\omega |f(x)|^2\,dx \tag{5}$$

(because the system $\left\{\frac{S(x)}{S'(\lambda_n)(x-\lambda_n)}\right\}_{-\infty}^{\infty}$ is a Riesz basis in $L_2^0(\mathbf{R})$).

Since S is a sine type function, then (see [7]) $|S'(\lambda_n)| \geqslant \varepsilon > 0$ for each $n \in \mathbf{Z}$. Let $\delta = \inf_{m\neq n}|\lambda_n-\lambda_m|$ and $D = \bigcup_{\nu=-\infty}^{\infty}\{\,\lambda \mid |\lambda-\lambda_\nu| < \delta/3\,\}$. The series $\sum_{-\infty}^{\infty}|\lambda-\lambda_n|^{-2}$ converges uniformly on compact subsets of $\mathbf{C}\setminus D$ (because $S(\lambda_\nu)=0$ and the function S is of exponential type).

From the maximum modulus principle we get that the series $\sum_{-\infty}^{\infty}\frac{S^2(x)}{S'^2(\lambda_\nu)(x-\lambda_\nu)^2}$ converges uniformly on compact subsets of $\mathbf{C}$. Then from (5) it follows that the series $\sum_\nu d_\nu \frac{S(x)}{S'(\lambda_\nu)(x-\lambda_\nu)}$ converges uniformly on compact subsets of $\mathbf{C}$ and represents an entire function.

The convergence in (4) is convergence in the subspace $L_2^0(\mathbf{R})$. Since the function $f(x)\left(|x|e^{i\frac{\pi}{2}\operatorname{sgn} x}\right)^{\omega/2}$ is continuous for $x>0$ and the series $\sum_\nu d_\nu \frac{S(x)}{S'(\lambda_\nu)(x-\lambda_\nu)}$ converges uniformly on compact subsets, then from (4) it follows

$$f(x)\left(|x|e^{i\frac{\pi}{2}\operatorname{sgn} x}\right)^{\frac{\omega}{2}} = \sum_\nu d_\nu \frac{S(x)}{S'(\lambda_\nu)(x-\lambda_\nu)}, \quad x>0. \tag{6}$$

From (6), by the uniqueness theorem, we get

$$e^{\frac{i\pi\omega}{4}} f(\lambda)\lambda^{\frac{\omega}{2}} = \sum_\nu d_\nu \frac{S(\lambda)}{S'(\lambda_\nu)(\lambda-\lambda_\nu)},$$

(here $\lambda^{\frac{\omega}{2}} = e^{\frac{\omega}{2}\ln\lambda}$, $\ln\lambda = \ln|\lambda| + i\arg\lambda$, $-\pi \leqslant \arg\lambda < \pi$). The previous relation gives

$$|d_\nu| = |f(\lambda_\nu)|\,|\lambda_\nu|^{\omega/2}. \tag{7}$$

From (5) and (7) it follows (1). ∎

Proof of Theorem 3. It is well known [2] that a minimal system $\{e_k\}_{-\infty}^{\infty}$ is a Riesz basis in a Hilbert space H if and only if $\sum_k |(f,e_k)|^2 \asymp \|f\|^2$ for each $f\in H$ (with $(\cdot,\cdot)$ we denote the scalar product in H). Now we shall prove that the system of functions $\{\varphi(x\lambda_n)\}$ is minimal in $L^2(-\pi,\pi)$.

Really, the functions $\frac{S(\lambda)}{|\lambda_n|^{\omega/2}S'(\lambda_n)(\lambda-\lambda_n)}$ are in $W_\pi(\omega)$ and by Theorem 1 there exist functions $\varphi_n \in L^2(-\pi,\pi)$ such that

$$\int_{-\pi}^{\pi} E_1\left(i\lambda t; 1+\frac{\omega}{2}\right)|t|^{\frac{\omega}{2}}\varphi_n(t)\,dt = \frac{S(\lambda)}{|\lambda_n|^{\omega/2}S'(\lambda_n)(\lambda-\lambda_n)}. \tag{8}$$

From (8) it follows $(\varphi(x\lambda_m),\varphi_n)_{L^2(-\pi,\pi)} = \delta_{nm}$ which proves the minimality of $\{\varphi(x\lambda_n)\}$.

Let $e_n(x) = |\lambda_n x|^{\omega/2}E_1(i\lambda_n x; 1+\omega/2)$ and $h\in L^2(-\pi,\pi)$. To prove Theorem 3 it is enough to show that

$$\sum_n |(h,e_n)|^2 \asymp \|h\|^2. \tag{9}$$

Since $\sum_n |(h,e_n)|^2 = \sum_n |(e_n,h)|^2 = \sum_n |\lambda_n|^\omega |f(\lambda_n)|^2$ where

$$f(\lambda) = \int_{-\pi}^{\pi} E_1\left(i\lambda t; 1+\frac{\omega}{2}\right)|t|^{\frac{\omega}{2}}\overline{h(t)}\,dt \quad (\in W_\pi(\omega)),$$

the relation (9) follows from the Lemma. ∎

Notice that the basis property of $\{\varphi(x\lambda_n)\}$ (in $L^2(-\pi,\pi)$) does not follow from the basis property of $\{E_1(i\lambda_n x; 1+\omega/2)\}$ in $L^2(0,\pi)$ because $E_1(z_1+z_2;\alpha) \neq E_1(z_1;\alpha)\cdot E_1(z_2;\alpha)$.

THEOREM 4. *If* $\{\lambda_n\}_{-\infty}^{\infty}$ *is a complex sequence such that* $\sup_{n\in\mathbf{Z}}|\lambda_n - n| = q < 1/8$ *and* $-1 < \omega < 1-8q$, *then the system* $\{\varphi(x\lambda_n)\}_{-\infty}^{\infty}$ *is a Riesz basis in* $L^2(-\pi,\pi)$.

Proof. Let $G(\lambda) = (\lambda-\lambda_0)\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_{-n}}\right)\left(1-\frac{\lambda}{\lambda_n}\right)$. If we prove

a) $\dfrac{G(\lambda)}{G'(\lambda_n)(\lambda-\lambda_n)} \in W_\pi(\omega)$,

b) $|G'(\lambda_n)| \geqslant C(1+|\lambda_n|)^{-\varepsilon}$ ($C>0$ and does not depend on n and $\varepsilon < 1/2$),

the assertion of the Theorem will follow directly by the method used in the proof of Theorem 3.

Since $q<1/8$ and $-1<\omega<1-8q$ we have $|G(x)| \leqslant C_1(1+|x|)^{4q}$ (C_1 does not depend on x, see [6]) and hence the function $\frac{G(\lambda)}{G'(\lambda_n)(\lambda-\lambda_n)}$ satisfies the condition a). The function G satisfies the condition b) and in that case $\varepsilon = 4q$ ($<1/2$). In the proof we used the fact (see [6]) that if $\arg\lambda = \theta$, $\Re\lambda \geqslant 0$, $|\lambda| \geqslant 1/2$ and N is a natural number defined by $N-\frac{1}{2} \leqslant |\lambda|\sec\theta < N+\frac{1}{2}$, then

$$|G(\lambda)| \geqslant C_2\frac{|\lambda_N-\lambda|e^{\pi|\Im\lambda|}}{(1+|\lambda-N|)(1+|\lambda|)^{4q}} \tag{10}$$

where the constant C_2 does not depend on λ and N. Let $D_n = \{\,\lambda \mid |\lambda-\lambda_n| < \delta < 1-8q\,\}$. Since for $\lambda\in\partial D_n$ the inequality $N-\frac{1}{2} \leqslant |\lambda|\sec\theta < N+\frac{1}{2}$ holds for $N=n$, applying (10) we obtain

$$\left|\frac{G(\lambda)}{\lambda-\lambda_n}\right| \geqslant C_2'\frac{e^{\pi|\Im\lambda|}}{(1+|\lambda|)^{4q}} \tag{11}$$

where C_2' does not depend on $\lambda\in\partial D_n$ and n.

From (11) by the minimum modulus principle we get $|G'(\lambda_n)| \geqslant C(1+|\lambda_n|)^{-4q}$ where the constant C does not depend on n. Similarly we prove that the last inequality holds for $n=-1,-2,\ldots$ ■

REFERENCES

[**1**] S. A. Avdonin, I. Joo, *Riesz bases of exponentials and sine type functions*, Acta Math. Hung. **51**, 1–2 (1988), 3–14

[**2**] I. C. Gohberg, M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Moscow 1965.

[**3**] G. M. Gubreev, *A basis property of families of the Mittag-Leffler functions, Dzhrbashyan's transform and Muckenhoupt condition*, Funktsional. Anal. Prilozhen., **20**, 4 (1987), 71–72

[**4**] M. M. Dzhrbashyan, *Integral Transformation and Representation of Functions in Complex Domain*, Moscow 1966.

[**5**] V. E. Kaznelson, *Bases of exponential functions*, Funktsional. Anal. Prilozhen., **5**, 1 (1971), 37–47

[**6**] N. Levinson, *Gap and density theorems*, Colloquium publications **26**, 1940

[**7**] B. Ya. Levin, *Entire Functions*, Lectures, Moscow 1971.

[**8**] S. V. Hruščev, *Perturbation theorem for exponential system and Muckenhoupt condition*, Soviet Math. Dokl. **274**, 1 (1979), 44–48

[**9**] S. V. Hruščev, *Unconditional bases in* $L^2(0,a)$, Proc. Amer. Math. Soc. **99**, 4 (1987)

(received 04.12.1995.)

Faculty of Mathematics, Studentski trg 16, 11000 Beograd, Yugoslavia