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ON THE IMPOSSIBILITY OF ONE RULER-AND-COMPASS CONSTRUCTION

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Abstract. We study the problem to determine all integers $n \geqslant 4$ for which there exists the ruler-and-compass construction of the cyclic polygon whose sides are equal to n given segments. The impossibility of such construction for $n = 5$ is proved.

1. Introduction

From the elementary geometry we know that each side of a polygon is smaller than the sum of the remaining sides of that polygon. The converse also holds: if each of n given segments is less than the sum of the remaining $n-1$ segments, then there exists a polygon whose sides are equal to the given segments. For $n \geqslant 4$ that polygon is not uniquely determined.

Among isoperimetric problems there exists a problem to determine a polygon with maximal area, whose sides are equal to the given segments. Cramer's theorem states that the solution of that problem exists and it is the cyclic polygon.

Several different proofs of Cramer's theorem exist and they can be put into two groups. The characteristic representative of the first group of proofs is given in [1], §12. Such proofs consist of two parts. In the first part, using methods of Mathematical Analysis, the existence of the solution is proved. In the second part there is the proof that the solution of the problem must be a cyclic polygon. In such a way an indirect proof of the existence of a polygon, whose sides are equal to the given segments, is obtained. The characteristic representative of the second group of proofs of Cramer's theorem is given in [2], §11. These proofs consist also of two parts. The first part is the proof that there exists a cyclic polygon, whose sides are equal to the given segments (provided each segment is smaller than the sum of the remaining segments). That part is proved using methods of Mathematical Analysis. The second part is the proof that the cyclic polygon has greater area than any other polygon with equal sides.

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There is a tradition in geometry to prove existence theorems using ruler-and-compass construction and in general, whenever possible, to solve the problems using elementary methods. That is how many famous problems appeared: trisection of the angle, duplication of the cube, quadrature of the circle, construction of the regular polygon, the third Hilbert problem. In connection with Cramer's theorem the following questions arises: why in all proofs of the Cramer's theorem the existence part is proved using methods of Mathematical Analysis and not using constructive methods? In other words, there is the question: given n segments satisfying that each of them is smaller than the sum of the remaining $n-1$ segments, is it possible to construct an n-gon (using ruler and compass) whose sides are equal to the given n segments? For $n = 4$ the answer is yes (see [1], §12). We shall prove here that for $n = 5$ the answer is no.

2. Solution of the problem

THEOREM. *The cyclic pentagon whose sides are equal to the given five segments can not be constructed by ruler and compass.*

Proof. It is enough to prove that it is impossible to construct the inscribed pentagon whose four sides have the length 1 and one side has the length 2, using ruler and compass. Let r be the radius of the circle circumscribed about the pentagon. Let the angle subtended by the side of the length 1 at the center of the circle be equal 2θ. Then the angle subtended by the side of the length 2 at the center of the circle is equal $2\pi - 8\theta$. According to the law of sines we have

$$1 = 2r\sin\theta, \quad 2 = 2r\sin(\pi - 4\theta) = 2r\sin 4\theta.$$

Since $\sin 4\theta = 4\sin\theta\cos\theta(1 - 2\sin^2\theta)$, and so

$$\sin^2 4\theta = 16\sin^2\theta(1-\sin^2\theta)(1-2\sin^2\theta)^2,$$

it follows that

$$\frac{1}{r^2} = 16\,\frac{1}{4r^2}\left(1 - \frac{1}{4r^2}\right)\left(1 - \frac{1}{2r^2}\right)^2,$$

and therefore $12r^6 - 20r^4 + 8r^2 - 1 = 0$. The polynomial

$$P(x) = 12x^3 - 20x^2 + 8x - 1$$

is irreducible over the field of rational numbers. That follows from the fact that $P(x)$ has no rational roots, which is easy to prove. According to the Theorem 5.2 from [3], it is impossible to construct (using ruler and compass) a line segment whose length is a root of such a polynomial. It follows that it is impossible to construct the radius r and therefore it is impossible to construct our polygon. The proof is completed. ■

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