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A WORD ON THE JOINT EMBEDDING PROPERTY

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Abstract. A possible generalization has been considered of the notion of the joint embedding property. Some preliminary results are obtained.

1. Preliminaries

For the reader's covinience we reapeat some definitions and introduce a notation which will be used in the sequel.

Throughout the article L is a first order language. The basic logical symbols will be $\neg$ (negation), $\wedge$ (conjunction) and $\exists$ (existential quantifier); the others are defined by the basic ones in the standard way. The choice of the logical symbols is of no importance for this story (finally, we can take all the familiar ones for the basic); however, since we will use some results of model-theoretic forcing, in which this combination of logical symbols is considered, we will keep it (of course, it does not mean that for any other choice we would not be able to reach the same results).

By a theory, T, of the first-order language L we assume a deductively closed set of sentences of L (thus, for a sentence φ, $T \vdash \varphi$ means $\varphi \in T$). The calss of models of a theory T will be denoted by $\mu(T)$. The models (of the language L) will be denoted by $\mathbf{A}$, $\mathbf{B}$, ... , while their domains will be A, B, ... If $\mathbf{A}$ is a model of the language L, then $L(A)$ is a simple expansion of the language L, obtained by adding to L the set of new constants which is in one to one correspondence with domain A. As usual, we will often make no difference in notation between the element a (from A) and the constant c_a joint to it; hence, whenever we are in "syntax", a is the constant (c_a), when we are in "semnatics", a is a. For a model $\mathbf{A}$, $Diag_n(\mathbf{A})$ is the set of Σ_n-, Π_n-sentences of the language $L(A)$ which hold in $\mathbf{A}$ (by Σ_m-formula any formula is understood equivalent to a formula in prenex normal form whose prenex consists of m blocks of quantifiers, the first one is the block of existential quantifiers—we allow the possibility of having "vacuous" quantifiers; Π_m-formulas sre defined similarly). In particular, for $n = 0$, in the books on model theory, $Diag_0(\mathbf{A})$ ($= Diag(\mathbf{A})$) has been considered as the set

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of the basic sentences (atomic and negation of atomic sentences) which hold in $\mathbf{A}$ rather than the set of all quantifier free sentences satisfiable in $\mathbf{A}$. For a formula φ, $fv(\varphi)$ is the set of its free variables. When we write $\varphi(v_{i_1},\dots,v_{i_m})$, it just means: $fv(\varphi)\subseteq\{v_{i_1},\dots,v_{i_m}\}$. We will sometimes write $\varphi(\tilde{v})$ instead of $\varphi(v_{i_1},\dots,v_{i_m})$. In general, $\tilde{v}$ will determine neither the variables nor the number of them. For a set of formulas Γ, $fv(\Gamma)=\bigcup_{\varphi\in\Gamma} fv(\varphi)$. If $fv(\Gamma)$ is finite, we say that Γ is a type. A type is n-type, (Σ_n-type, Π-type), $n\geq 0$, iff each of its formula is either Σ_n- or Π_n-formula; for $n=0$ we also say that a type is open. A type Γ is a type of a theory T iff some model $\mathbf{A}$ of T realizes it (which means that for some elements a_1, $\dots$, a_m from A, where $\{v_1,\dots,v_m\}=fv(\Gamma)$, $\mathbf{A}\models\varphi[a_1,\dots,a_m]$ for all $\varphi\in\Gamma$).

Definition 1.1. Let $\mathbf{A}$ be a submodel of a model $\mathbf{B}$. If $(\mathbf{B},a)_{a\in A}\models Diag_{n+1}(\mathbf{A})$, then $\mathbf{B}$ is a Σ_n-extension of a model $\mathbf{A}$ or, in other words, $\mathbf{A}$ is n-elementary submodel of $\mathbf{B}$; we will write: $\mathbf{A}\prec_n\mathbf{B}$ (therefore, $\mathbf{B}$ is an elementary extension of its submodel $\mathbf{A}$ iff it is an Σ_n-extension of $\mathbf{A}$ for each $n\in\omega$).

A model $\mathbf{A}$ is *n-embeded* into a model $\mathbf{B}$ iff for some embedding f of $\mathbf{A}$ into $\mathbf{B}$, the model $f(\mathbf{A})$ is n-elementary submodel of $\mathbf{B}$. The mapping f is then *n-embedding* of $\mathbf{A}$ into $\mathbf{B}$ (thus, f is an elementary embedding of $\mathbf{A}$ into $\mathbf{B}$ iff, for each $n\in\omega$, f is n-embedding).

Two models $\mathbf{A}$ and $\mathbf{B}$ of a same language L are *n-elementary equivalent*, in notation $\mathbf{A}\equiv_n\mathbf{B}$, iff they satisfy the same Π_n, Σ_n sentences of the language L.

A class of models $\mathcal{K}$ has the *n-joint embedding property* iff any two models from $\mathcal{K}$ can be n-embedded into some model from $\mathcal{K}$. For $n=0$ we have the standard joint embedding property, shortly denoted by JEP; in accordance with it, the n-joint embedding property is denoted by n-JEP.

A theory T has the *n-joint embedding property* iff any two of its models can be n-embeded into a third one (of course, if such embeddings exist we can always assume that one of them is just the inclusion), in other words, iff the class $\mu(T)$ has the n-joint embedding property.

A theory T has an *n-model-consistent completion* iff there is a complete theory T^* which has the common Π_{n+1}-segment with T, i.e. $T\cap\Pi_{n+1}=T^*\cap\Pi_{n+1}$, where $T\cap\Pi_m$ is the deductive closure of the set $\{\,\varphi\mid T\vdash\varphi$ and φ is a Π_m-sentence$\,\}$.

A theory T is *n-model complete* iff for any two models $\mathbf{A}$, $\mathbf{B}\in\mu(T)$, from $\mathbf{A}\prec_n\mathbf{B}$ it follows $\mathbf{A}\prec\mathbf{B}$.

For a theory T, T^{f_n} is its *n-finite forcing companion* (see [4], [5]), and T^{F_n} is its *n-infinite forcing companion* ([6]).

Note. Two theories have the same Π_{n+1}-segment iff each model of any of these theories can be n-embedded into a model of the other theory; in that sense, in accordance with the standard terminology, we will say that such two theories with the same Π_{n+1}-segment are *n-mutually model-consistent.* Thus, a theory T has the n-joint embedding property iff the class $\mu(T\cap\Pi_{n+1})$ has the n-joint embedding property (that is part of the assertion 3.2).

Let us also note that the Π_m-segment of a theory T is often denoted by $T_{\forall_m}$ or $T\cap\forall_m$.

2. Equivalents to the n-joint embedding property

The following theorem gives us (in case of first-order theories) some equivalents to the n-joint embedding property, either of semantic or syntactical nature.

THEOREM 2.1. *For the first order theory T of the language L (of arbitrary cardinality) the following conditions are equivalent:*

(1) *Theory T has the n-joint embedding property;*

(2) *If $\mathbf{A}$ and $\mathbf{B}$ are models of T and $a_1, \dots, a_m$ are some elements from A, then there is a model $\mathbf{C}$ of T, which is an n-extension of B and such that for some of its elements $c_1, \dots, c_m$ it holds: $(\mathbf{A}, a_1, \dots, a_m) \equiv_n (\mathbf{C}, c_1, \dots, c_m)$ (when $n = 0$ this just means that the submodels of $\mathbf{A}$ and $\mathbf{C}$ generated by the constants of the language L and those denoted are isomorphic);*

(3) *If ϕ and ψ are Π_{n+1} sentences of the language L such that $T \vdash \phi \vee \psi$ then either $T \vdash \phi$ or $T \vdash \psi$;*

(4) *If ϕ and ψ are Σ_{n+1} sentences such that both $T \cup \{\phi\}$ and $T \cup \{\psi\}$ are consistent sets of sentences then $T \cup \{\phi, \psi\}$ is consistent as well;*

(5) *T^{f_n} is a complete theory;*

(6) *If $\mathbf{A}$ and $\mathbf{B}$ are models of T then $T \cup Diag_n(\mathbf{A}) \cup Diag_n(\mathbf{B})$ is a consistent set of sentences;*

(7) *T^{F_n} is a complete theory;*

(8) *If Γ_1 and Γ_2 are n-types of the theory T such that $fv(\Gamma_1) \cap fv(\Gamma_2) = \emptyset$, then $\Gamma_1 \cup \Gamma_2$ is also an n-type of T;*

(9) *The same as in the previous condition but with Γ_1, Γ_2 being Σ_{n+1}-types;*

(10) *If $\mathbf{A}_\alpha$, $\alpha < \lambda$ are models of T then all of them can be n-embedded in some model of T;*

(11) *If $\Sigma = \{\, \phi_\alpha \mid \alpha < \lambda \,\}$ is the set of Σ_{n+1}-sentences, each of which is consistent with T, then $T \cup \Sigma$ is consistent;*

(12) *There is a model $\mathbf{A}$ of T such that any model of T can be n-embedded into some ultrapower of $\mathbf{A}$;*

(13) *Theory T has the n-model-consistent completion.*

Proof. The proof is based on the classical ones (see, for instance, [3], [9], [13]); however, we will follow the shortest line: (1) $\Longrightarrow$ (2) $\Longrightarrow \cdots \Longrightarrow$ (12) $\Longrightarrow$ (13) $\Longrightarrow$ (1). In proving all these implications we will assume in the begining that the assertion of the antecedent holds. Some informalities in the explanations and notations are pressumed.

(1) $\Longrightarrow$ (2). Let $\mathbf{A}$ and $\mathbf{B}$ be two models of the theory T and $a_1, \dots, a_m$ some elements from A. By (1), $\mathbf{A}$ and $\mathbf{B}$ are n-embedded into some model $\mathbf{C}$ of the theory T; let f and g be, respectively, these embedings of $\mathbf{A}$ and $\mathbf{B}$ into $\mathbf{C}$. But then $f(\mathbf{A}, a_1, \dots, a_m) \cong (f(\mathbf{A}), f(a_1), \dots, f(a_m)) \prec_n (\mathbf{C}, f(a_1), \dots, f(a_m))$, thus $(\mathbf{A}, a_1, \dots, a_m) \equiv_n (\mathbf{C}, f(a_1), \dots, f(a_m))$.

(2) $\implies$ (3). Let ϕ and ψ be Π_{n+1} sentences such that $T \vdash \phi \vee \psi$. If we supposed that neither $t \vdash \phi$ not $T \vdash \psi$, then for some models $\mathbf{A}$ and $\mathbf{B}$ of T we would have: $\mathbf{A} \models \neg\phi$, $\mathbf{B} \models \neg\psi$. Let $\neg\phi \iff \exists x_1 \dots \exists x_m \varphi$, where $\varphi(x_1, \dots, x_m)$ is an Π_n formula, and $\mathbf{A} \models \varphi[a_1, \dots, a_m]$. But then, if a model $\mathbf{C}$ of T is an n-elementary extension of $\mathbf{B}$ such that, for some elements $c_1, \dots, c_m$ of it, $(\mathbf{A}, a_1, \dots, a_m) \equiv_n (\mathbf{C}, c_1, \dots, c_m)$, it would hold: $\mathbf{C} \models \neg(\phi \vee \psi)$, a contradiction.

(3) $\implies$ (4). Let ϕ and ψ be Σ_{n+1} sentences such that both $T \cup \{\phi\}$ and $T \cup \{\psi\}$ are consistent. If $T \cup \{\phi \wedge \psi\}$ were inconsistent, it would follow $T \vdash \neg\phi \vee \neg\psi$, and consequently, $T \vdash \neg\phi$ or $T \vdash \neg\psi$, a contradiction.

(4) $\implies$ (5). Let ϕ be a sentence such that neither $T^{f_n} \vdash \phi$ nor $T^{f_n} \vdash \neg\phi$. Then, for some conditions $p(\tilde{a})$ and $q(\tilde{b})$, which are the finite sets of Σ_n-, Π_n-sentences of some simple expansion $L(A)$ of the language L (where A is a denumerable set of new constants disjoint with L) consistent with T, $p \Vdash_n \phi$, $q \Vdash_n \neg\phi$. We point out in p and q only the "new" constants. Of course, we can assume that p and q have only the constants from L in common. Since $T \cup \{\exists\tilde{v} \bigwedge p(\tilde{v})\}$ and $T \cup \{\exists\tilde{u} \bigwedge q(\tilde{u})\}$ are consistent (we choose variables such that there is no common variable in $\tilde{v}$ and $\tilde{u}$), $T \cup \{\exists\tilde{v}\exists\tilde{u}(\bigwedge p(\tilde{v}) \wedge \bigwedge q(\tilde{u}))\}$ is consistent as well. In particular $T \cup p(\tilde{a}) \cup q(\tilde{b})$ is consistent. But then $p(\tilde{a}) \cup q(\tilde{b})$ is a condition which n-forces both ϕ and $\neg\phi$, a contradiction.

(5) $\implies$ (6). Let $\mathbf{A}$ and $\mathbf{B}$ be models of T. We can assume that $A \cap B = \emptyset$. Let $p(\tilde{a})$ and $q(\tilde{b})$ be finite subsets of, respectively, $Diag_n(\mathbf{A})$, $Diag_n(\mathbf{B})$. Again, we point out only the constant not contained in L. Considering $p(\tilde{a})$ and $q(\tilde{b})$ as conditions of the n-finite forcing (connected with the theory T), we have $p(\tilde{a}) \Vdash \neg\neg \bigwedge p(\tilde{a})$, $q(\tilde{b}) \Vdash \neg\neg \bigwedge q(\tilde{b})$ (see Theorems 1.2 and 1.3 in [4]). Whence, being T^{f_n} complete, we obtain $\emptyset \Vdash \neg\neg\exists\tilde{v} \bigwedge p(\tilde{v}) \wedge \neg\neg\exists\tilde{u} \bigwedge q(\tilde{u})$, that is $T^{f_n} \vdash \exists\tilde{v} \bigwedge p(\tilde{v}) \wedge \exists\tilde{u} \bigwedge q(\tilde{u})$. Since T and T^{f_n} have the common Π_{n+1} segment, it follows that $T \cup \{\exists\tilde{v} \bigwedge p(\tilde{v}) \wedge \exists\tilde{u} \bigwedge q(\tilde{u})\}$ is consistent.

(6) $\implies$ (7). It is known that $T \cap \Pi_{n+1} = T^{F_n} \cap \Pi_{n+1}$ $(= T^{f_n} \cap \Pi_{n+1})$ as well as that if $\mathbf{A}$ and $\mathbf{B}$ are n-infinitely generic models and $\mathbf{A} \prec_n \mathbf{B}$, then $\mathbf{A} \prec \mathbf{B}$. If we suppose that T^{F_n} is not complete, then for some sentence φ and some n-infinitely generic models $\mathbf{A}$ and $\mathbf{B}$ it holds: $\mathbf{A} \models \varphi$, $\mathbf{B} \models \neg\varphi$ (we recall: $T^{F_n} = Th(\mathcal{L}_t^n)$— the theory of the class of all n-infinitely generic models of the theory T). Since T^{F_n} and T are n-mutually model-consistent, there exist models $\mathbf{A}_1$ and $\mathbf{B}_1$ of T into which the models $\mathbf{A}$ and $\mathbf{B}$ are, respectively, n-embedded. If $\mathbf{C}$ is a model of $T \cup Diag_n(\mathbf{A}_1) \cup Diag(\mathbf{B}_1)$ and $\mathbf{D}$ an n-infinitely generic model into which $\mathbf{C}$ is n-embedded, then $\mathbf{A}$ and $\mathbf{B}$ are n-embedded into $\mathbf{D}$, whence $D \models \varphi \wedge \neg\varphi$, a contradiction.

(7) $\implies$ (8). Let $\Gamma_1(\tilde{v})$ and $\Gamma_2(\tilde{u})$ be two n-types of the theory T, such that $fv(\Gamma_1) \cap fv(\Gamma_2) = \emptyset$ and let $\mathbf{A}$ and $\mathbf{B}$ be two models of T such that for some elements $\tilde{a} \in A$ and $\tilde{b} \in B$, $\mathbf{A} \models \Gamma_1[\tilde{a}]$ and $\mathbf{B} \models \Gamma_2[\tilde{b}]$. We can immediately assume that $\mathbf{A}$ and $\mathbf{B}$ are n-infinitely generic models (any model of T is n-embedded into such one), and because of the completeness of T^{F_n}, $\mathbf{A}$ and $\mathbf{B}$ are n-embedded into some model of T^{F_n}, which, on the other side, is n-embedded into some model of T.

Thus $T \cup \Gamma_1(\tilde{a}) \cup \Gamma_2(\tilde{b})$ is consistent.

(8) $\Longrightarrow$ (9). Let $\Gamma_1(\tilde{v})$ and $\Gamma_2(\tilde{u})$ be two Σ_{n+1}-types of T with no common free variable and let Δ_1 and Δ_2 be their finite subsets. But then $\bigwedge \Delta_1$ and $\bigwedge \Delta_2$ are equivalent to some Σ_{n+1} formulas in prenex normal forms, let us say, $\exists \tilde{w} \phi(\tilde{v}, \tilde{w})$ and $\exists \tilde{z} \psi(\tilde{u}, \tilde{z})$. We will assume immediately that $\tilde{w}$ and $\tilde{z}$ do not have common variable. Then $\{\phi(\tilde{v}, \tilde{w})\}$ and $\{\psi(\tilde{u}, \tilde{z})\}$ are n-types of T, hence $\{\phi(\tilde{v}, \tilde{w}), \psi(\tilde{u}, \tilde{z})\}$ is an n-type of T.

(9) $\Longrightarrow$ (10). Certainly, as before, we are free to assume that any $\alpha < \beta$ $(< \lambda)$, $A_\alpha \cap A_\beta = \emptyset$. By the compactness theorem, which we anyway tacitly use all the time, it is enough to show that $T \cup \{\phi_1(\tilde{a}_1), \dots, \phi_m(\tilde{a}_m)\}$, where $\phi_i(\tilde{a}_i)$, $i = 1$, $\dots$, m, is a conjunction of a finite set of sentences from $Diag_n(\mathbf{A}_{\alpha_i})$, is consistent for any choice of indeces and subsets of the corresponding diagrams. Since the sentences $\phi_i(\tilde{a}_i)$ are Σ_{n+1}, we have m Σ_{n+1}-types of the theory T $(\{\exists \tilde{v}_i \phi(\tilde{v}_i)\}$, $i = 1, \dots, m)$, but obviously if (9) holds for two Σ_{n+1}-types of T, it holds for any finite number, whence $T \cup \{\, \exists \tilde{v}_i \phi(\tilde{v}_i) \mid i = 1, \dots, m \,\}$ is consistent.

(10) $\Longrightarrow$ (11). Clear (repeating the same arguments would be a little bit tedious).

(11) $\Longrightarrow$ (12). Let the set $\mathcal{F}$ of all sets of Σ_{n+1} sentences of the language L consistent with T be ordered by inclusion. Since $\mathcal{F}$ is closed under unions of chains, it has a maximal element, but due to (11), there is only one such. Let it be Γ and let $\mathbf{A}$ be a model of $T \cup \Gamma$ and $\mathbf{B}$ any model of T with the domain $B = \{b_\alpha \mid \alpha < \lambda = |B|\}$; we fix one well-ordering of B. Now, if $\phi(b_{\alpha_1}, \dots, b_{\alpha_m}) \in Diag_n(\mathbf{B})$, where only the constant from $L(B)$ which are not in L are exposed, we have, after possible reindexation of variables, $\mathbf{B} \models \exists v_1 \dots \exists v_m \phi(v_1, \dots, v_m)$. Thus, $\exists v_1 \dots v_m \phi(v_1, \dots, v_m) \in \Gamma$ and $\mathbf{A} \models \exists v_1 \dots v_m \phi(v_1, \dots, v_m)$. Let us choose an λ-sequence of the elements from A, denoted by $\bar{a}^\phi = \langle a^\phi_\alpha \mid \alpha < \lambda \rangle$, such that $(\mathbf{A}, \bar{a}^\phi) \models \phi(b_{\alpha_1}, \dots, b_{\alpha_m})$. Further, since we chose for any sentence ψ from $Diag_n(\mathbf{B})$ the corresponding sequence $\bar{a}^\psi$, let $\mathcal{J}_\phi = \{\psi \in Diag_n(\mathbf{B}) \mid (\mathbf{A}, \bar{a}^\psi) \models \phi\}$ and $\mathcal{J} = \{\mathcal{J}_\phi \mid \phi \in Diag_n(\mathbf{B})\}$. Since $\mathcal{J}$ has the finite intersection property (obviously, $\mathcal{J}_{\phi_1} \cap \dots \cap \mathcal{J}_{\phi_k} = \mathcal{J}_{\phi_1 \wedge \dots \wedge \phi_k}$ $(\neq \emptyset)$), there exists some ultrafilter $\mathcal{I}$ containing $\mathcal{J}$. Let $f\colon B \to A^{Diag_n(\mathbf{B})}/\mathcal{I}$ be mapping defined by: $f(b_\alpha) = [f_\alpha]_\mathcal{I}$, where, for $\phi \in Diag_n(\mathbf{B})$, $f_\alpha(\phi) = a^\phi_\alpha$ (and, of course, $[f_\alpha]_\mathcal{I}$ is the equivalence class containing f_α). We claim: f is an n-embedding of $\mathbf{B}$ into $\mathbf{A}^{Diag_n(\mathbf{B})}/\mathcal{I}$. Really, if $\psi(b_{\alpha_1}, \dots, b_{\alpha_m}) \in Diag_n(\mathbf{B})$, the following relations are equivalent:

$$\mathbf{A}^{Diag_n(\mathbf{B})}/\mathcal{I} \models \psi[[f_{a_1}, \dots, [f_{\alpha_m}]],$$
$$\{\, \phi \in Diag_n(\mathbf{B}) \mid \mathbf{A} \models \psi[f_{\alpha_1}(\phi), \dots, f_{\alpha_m}(\phi)] \,\} \in \mathcal{I},$$
$$\{\, \phi \in Diag_n(\mathbf{B}) \mid \mathbf{A} \models \psi[a^\phi_{\alpha_1}, \dots, a^\phi_{\alpha_m}] \,\} \in \mathcal{I},$$
$$\{\, \phi \in Diag_n(\mathbf{B}) \mid (\mathbf{A}, \bar{a}^\phi) \models \psi(b_{\alpha_1}, \dots, b_{\alpha_m}) \,\} = \mathcal{J}_\psi \in \mathcal{I}.$$

(12) $\Longrightarrow$ (13). Let $\mathbf{A}$ be a model of T such that all other models of T are n-embedded into some ultrapower of it and let $Th(\mathbf{A}) \overset{\text{def}}{=} \{\, \phi \mid \phi$ is a sentence of the language L satisfiable in $\mathbf{A} \,\}$. Since $\mathbf{A}$ is elementary equivalent to any of its

ultrapowers, it follows $T \cap \Pi_{n+1} = Th(\mathbf{A}) \cap \Pi_{n+1}$; inclusion $\subseteq$ is obvious, while if a Π_{n+1} sentence ϕ is not a consequence of T and $\mathbf{B} \models T \cup \{\neg\phi\}$, then the ultrapower of $\mathbf{A}$ in which $\mathbf{B}$ is n-embeddable satisfies $\neg\phi$ as well, and consequently $\mathbf{A} \models \neg\phi$. Hence, $Th(\mathbf{A})$ is n-model consistent completion of T.

(13) $\implies$ (1). Let $\mathbf{A}$ and $\mathbf{B}$ be models of T, T^* an n-model consistent completion of T and $\mathbf{A}^*$ and $\mathbf{B}^*$ models of T^* in which the models $\mathbf{A}$ and $\mathbf{B}$ are, respectively, n-embeddable. As a complete theory, T^* has the n-joint embedding property (for any n), thus $\mathbf{A}^*$ and $\mathbf{B}^*$ are n-embeddable in some model $\mathbf{C}^*$ (of T^*), which is, on the other side, n-embeddable in some model $\mathbf{D}$ of T. Since the composition of n-embeddings is again an n-embedding, we conclude that T has the n-joint embedding property. ∎

Obviously, a theorem similar to the previous one can be formulated for the classes of models (thus, in general, not necessarily generalized elementary).

COROLLARY 2.2 *Let T be a theory of a countable language L. Then T has the n-joint property iff any two of its countable (finite or denumerable) models can be n-embedded into a third one.*

In general, if T is a theory of a language L of cardinality λ ($> \aleph_0$) with only infinite models, then T has the n-joint embedding property iff any two of its models of cardinality λ can be embedded into a third one.

Proof. Clearly; let just note that, for a given model $\mathbf{A}$, a finite subset of $Diag_n(\mathbf{A})$ is also a finite part of $Diag_n(\mathbf{B})$ for some countable elementary submodel $\mathbf{B}$ of $\mathbf{A}$. The rest is due to, for instance, the item (6) and, of course, the compactness argument.

In the general case, we use, if necessary, either the upward or downward Löwenheim-Skolem-Tarski theorem. ∎

3. A few remarks

The next lemmas are (more than) obvious (the proof of the second was in fact demonstrated in the proof of the previous theorem).

LEMMA 3.1. *A theory T is complete iff it has the n-joint embedding property for every $n \in \omega$ iff any two of its models can be elementary embedded into a third one.*

In model theory the following fact is often used: the completeness of a theory is a sufficient condition for the joint embedding property, but certainly it is not a necessary condition.

LEMMA 3.2. *If S and T are n-muttually model-consistent theories (of the same language L), then S has the n-joint embedding property iff T has the n-joint embedding property.*

The following observation shows that a lot of familiar theories which have the joint enbedding property, do not have the n-joint embedding property for any $n \geq 1$.

LEMMA 3.3. *No theory with equality and with finite models of different cardinalities has the* 1*-joint embedding property.*

Proof. Trivial; by a Σ_2-sentence it is said that a given finite normal model $\mathbf{A}$ (i.e. a model in which the interpretation of equality is just identity) has exactly $|A|$ elements. ■

If we assume that in the case of theories with equality only the normal models are to be considered (as we do in the "standard" mathematics), then it holds

COROLLARY 3.4. *A theory with equality and finite models which has the* 1*-joint embedding property is complete; it is the complete theory of its unique (up to isomorphism) model.*

The situation does not seem much better even if a theory has only infinite models. For we have

LEMMA 3.5. *If a theory T (of the language L) has the n-joint embedding property then for any Π_n-sentnece φ it holds: either $T \vdash \varphi$ or $T \vdash \neg\varphi$.*

If T has a model which is $(n+1)$-embedded into any other of its models, then T has the n-joint embedding property.

Proof. The first part of the lemma is already proved; see, for instance, the relation (1) $\iff$ (5) in 2.1 and recall that $T \cap \Pi_{n+1} = T^{f_n} \cap \Pi_{n+1}$.

The other part is equally trivial; if $\mathbf{A}$ is a model of T which is $(n+1)$-embedded into all others, then for any two models $\mathbf{B}$, $\mathbf{C}$ of T, $T \cup Diag_n(\mathbf{B}) \cup Diag_n(\mathbf{C})$ is consistent, moreover $T \cup Diag_n(\mathbf{B}) \cup \Gamma(\mathbf{C})$ is consistent ($\Gamma(\mathbf{C})$ is *the elementary diagram* of $\mathbf{C}$); for any Σ_{n+1}-existential sentence of the language $L(A)$ which holds in $\mathbf{B}$, holds in $\mathbf{A}$, hence in $\mathbf{C}$, too. ■

COROLLARY 3.6. *No sound axiomatic theory (i.e. a sound theory wuth effectively recursively enumerable set of axioms) has the* 1*-joint embedding property.*

Proof. By a sound theory any theory of the standard language of number theory is assumed which is contained in the so-called *complete number theory*—the complete theory of the standard models $\mathbf{N}$—$Th(\mathbf{N})$. By the Gödel's theorem, for any sound theory T with an effectively given recursively enumerable set of axioms, there exists an universal sentence φ such that $\varphi \in Th(\mathbf{N}) \setminus T$, and, because of soundness, also $\neg\varphi \notin T$. ■

COROLLARY 3.7. *No axiomatic theory T of the language of number theory that includes the axioms for addition and multiplication for natural numbers and the axioms of the form:*

$\overline{m} \neq \bar{n}$, $m \neq n$, $m, n \in \omega$,

$\forall v\, (v \leq \overline{m} \implies \bigvee_{i=1}^{m} v = \bar{i})$, $m \in \omega$,

$\forall v\, (\overline{m} \leq v \vee v \leq \bar{m})$, $m \in \omega$,

has 2*-joint embedding property.*

Proof. By the first incompleteness theorem there exists Σ_2 sentence such that neither $\varphi \in T$ nor $\neg\varphi \in T$. ■

In a lot of cases we have more precise results; for instance:

No number theory (that is a theory containig the Π_2-segment of Peano arithmetic) T with recursively enumerable Π_1-segment has the joint embedding property. In particular, Peano arithmetic does not have the joint embedding property [8].

No fragment of Peano arithmetic extending IE_1^- (bounded existential parameter-free induction) has the joint embedding property. Open induction and the usually studied stronger fragments of Peano arithmetic fail to have the joint embedding property [10], [11].

In order to weaken the conditions of the n-joint embedding property, we introduce

DEFINITION 3.8. A class of models $\mathcal{K}$ has the *almost n-joint embedding property* iff for any two models from $\mathcal{K}$, $\mathbf{A}$ and $\mathbf{B}$, there exists a model $\mathbf{C}$, into which the models $\mathbf{A}$ and $\mathbf{B}$ can be embedded so that at least one of these embeddings is an n-embedding.

A theory T has the almost n-joint embedding property iff the class $\mu(T)$ has the almost n-joint embedding property.

Of course, the almost 0-joint embedding property is just the joint embedding property. We also have

LEMMA 3.9. *A theory T has the almost 1-joint embedding property iff it has the almost n-joint embedding property for any $n > 1$.*

Proof. It is easy to see that for any $k \geq 1$, the almost k-joint embedding property is equivalent to the condition that any two models of T can be embedded into a third model of T so that one of these embeddings is elementary. ■

As an example of the class with 1-joint embedding property, trivially, one can take any class of models with the property that for any two of its models one of them can be embedded into the other, for instance the class of well ordered sets (clearly, this class does not have the 1-joint embedding property).

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