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ON PROBABILITY MEASURE GENERATED BY SPECTRAL DENSITY ESTIMATE

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Abstract. We study the periodogram based estimate of spectral density of a strictly stationary random sequence and consider this estimate as a random function on the whole domain of frequency. We renormalize the scale in this domain and investigate the probability measure generated by obtained process in the space of continuous functions.

1. Introduction

In this paper we shall consider a strictly stationary real random sequence $X(t)$, $t \in D = \{\dots,-1,0,1,\dots\}$, with the mean $EX(t)=0$ and the spectral density $f(\lambda)$, $\lambda \in [-\pi,\pi]$. Let $(X(1),\dots,X(N)$ be a sample of size N from this random sequence. The spectral density estimate of the Grenander-Rosenblatt type is given by

$$\widehat{f}_N(\lambda) = \int_{-\pi}^{\pi} \varphi_N(x-\lambda)\frac{1}{2\pi N}\left|\sum_{t=1}^{n} X(t)e^{-itx}\right|^2 dx,$$

where: $\varphi_N(x) = B_N^{-1}\varphi(xB_N^{-1})$, $x \in [-\pi,\pi]$; $\varphi : \mathbf{R} \to \mathbf{R}$ is a weight function for which we suppose that is symetric about 0, has a bounded first derivative and such that $\varphi(0)=1$, $\int_{-\pi}^{\pi}\varphi(x)\,dx = 1$ and $\varphi(x)=0$ for $|x| \ge \pi$. The sequence (B_N) of real numbers is such that $B_N \to 0$ and $NB_N \to \infty$ when $N \to \infty$. We assume that $B_N = N^{-\varepsilon}$, where $\dfrac{1}{3} < \varepsilon < \dfrac{1}{2}$, and that the functions f and φ_N are defined on the whole real line $\mathbf{R}$ and 2π-periodic.

Let us define the random process $\widetilde{\xi}_N(\lambda)$ and $\widetilde{Z}_N(\lambda)$ as follows:

$$\widetilde{\xi}_N(\lambda) = \sqrt{NB_N}\left[\widehat{f}_N(\lambda B_N) - E\widehat{f}_N(\lambda B_N)\right], \quad |\lambda| \le \pi B_N^{-1},$$
$$\widetilde{Z}_N(\lambda) = \sqrt{NB_N}\left[\widehat{f}_N(\lambda B_N) - f(\lambda B_N)\right], \quad |\lambda| \le \pi B_N^{-1},$$

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and let $Z(\lambda)$, $\lambda \in \mathbf{R}$, be the Gaussian random process with the mean $EZ(\lambda) = 0$ and the covariance function

$$R(\lambda, \mu) = EZ(\lambda)Z(\mu) = 2\pi f^2(0) \int_{-\infty}^{\infty} \varphi(x-\lambda)[\varphi(x-\mu) + \varphi(x+\mu)]\,dx.$$

The cumulant spectral density of order n of the sequence $X(t)$, $t \in D$, is defined as follows:

$$f_n(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) = \frac{1}{(2\pi)^{n-1}} \sum_{t_1, t_2, \dots, t_{n-1}} S_n(t_1, \dots, t_{n-1}, 0) \exp\Big(-\sum_{j=1}^{n-1} \lambda_j t_j\Big),$$

where $S_n(t_1, \dots, t_{n-1}, 0)$ denote the n–th cumulant of the sequence $X(t), t \in D$. (The function f_n is defined if the series on the right–hand side of (1) converges.) Of course, the spectral density f is the cumulant spectral density of second order.

We shall also use the following denotations:

$C = C[a,b]$—the space of all real continuous funstions defined on $[a,b]$, where $-\infty < a < b < +\infty$, with the uniform metric $\rho(x,y) = \sup\limits_{a \le t \le b} |x(t) - y(t)|$;

$\mathcal{C}$—the class of Borel sets in C;

$Lip_H(1,\infty)$—the class of real functions g which satisfy the condition

$$(\forall x, y)\, |g(x) - g(y)| \le H|x-y|;$$

P_N, Q_N and P—the probability measures generated by the random processes $\widetilde{\xi}_N(\lambda)$, $a \le \lambda \le b$, $\widetilde{Z}_N(\lambda)$, $a \le \lambda \le b$, and $Z(\lambda)$, $a \le \lambda \le b$, respectively, on the space $(C, \mathcal{C})$.

2. Weak convergence of probability measure Q_N

The spectral density estimate of Grenander-Rosenblatt type have been considered in many papers. See, for example, [6] for references. A very important result is that the spectral density estimate has asymptoticaly normal distribution [3,6]. In the papers [4,5] the asymptotic properties of the random process $\widetilde{\xi}_N(\lambda)$ have been investigated. Specially, the weak convergence of the mesure P_N has been proved. Here we shall prove the following two theorems:

Theorem 1. *Let all cumulant spectral densities of the random process $X(t)$, $t \in D$, are bounded and let the spectral density function f be continuously-differentiable. Then, the finite-dimensional distributions of the random process $\widetilde{Z}_N(\lambda)$, $\lambda \in \mathbf{R}$, converge weakly to those of the Gaussian process $Z(\lambda)$, $\lambda \in \mathbf{R}$.*

Theorem 2. *Let the sequence $X(t)$ be Gaussian and $f' \in Lip_H(1,\infty)$. Then, Q_N converges weakly to P, when $N \to \infty$.*

3. Proofs

In order to prove these two theorems we need several lemmas:

LEMMA 1. *Let the first derivative of the spectral density be bounded, i.e.* $\|f'\| = C < +\infty$. *Then, the following inequality is valid:*

$$\sup_{-\pi\le\lambda\le\pi} |f(\lambda) - E\widehat{f}_N(\lambda)| \le C\int_{-\pi}^{\pi} |x\varphi_N(x)|\,dx + o\Big(\int_{-\pi}^{\pi} |x\varphi_N(x)|\,dx\Big).$$

Proof. Let us denote $I_N(x) = \dfrac{1}{2\pi N}\Big|\sum\limits_{t=1}^{N} X(t)e^{-itx}\Big|^2$. Then we have

$$\begin{aligned}
E\widehat{f}_N(\lambda) - f(\lambda) &= E\int_{-\pi}^{\pi} \varphi_N(x) I_N(x+\lambda)\,dx - f(\lambda)\\
&= \frac{1}{2\pi N}\int_{-\pi}^{\pi} \varphi_N(x)\, E\Big(\sum_{t=1}^{N} X(t)e^{-it(x+\lambda)}\sum_{s=1}^{N} X(s)e^{is(x+\lambda)}\Big)\,dx - f(\lambda)\\
&= \frac{1}{2\pi N}\int_{-\pi}^{\pi} \varphi_N(x)\,\sum_{t=1}^{N}\sum_{s=1}^{N} EX(t)X(s)e^{i(x+\lambda)(s-t)}\,dx - f(\lambda).
\end{aligned}$$

Using the equalities $EX(t) = 0$ and $EX(t)X(s) = \int_{-\pi}^{\pi} e^{i\lambda(t-s)} f(\lambda)\,d\lambda$ we obtain

$$\begin{aligned}
E\widehat{f}_N(\lambda) - f(\lambda) &= \frac{1}{2\pi N}\int_{-\pi}^{\pi} \varphi_N(x)\sum_{t=1}^{N}\sum_{s=1}^{N}\int_{-\pi}^{\pi} f(u)e^{iu(t-s)}e^{i(x+\lambda)(s-t)}du\,dx - f(\lambda)\\
&= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi_N(x)f(u)\frac{1}{2\pi N}\sum_{t=1}^{N}\sum_{s=1}^{N} e^{i(u-x-\lambda)(t-s)}du\,dx - f(\lambda).
\end{aligned}$$

Using the equality $\sum\limits_{t=1}^{N} e^{itx} = \dfrac{\sin(Nx/2)}{\sin(x/2)}e^{i(N+1)x/2}$, we get

$$\frac{1}{2\pi N}\sum_{t=1}^{N}\sum_{s=1}^{N} e^{ix(t-s)} = \frac{1}{2\pi N}\frac{\sin^2\frac{Nx}{2}}{\sin^2\frac{x}{2}} = \Phi_N(x),$$

where Φ_N is Fejér's kernel and

$$E\widehat{f}_N(\lambda) - f(\lambda) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi_N(x)\Phi_N(x+\lambda-u)f(u)\,du\,dx - f(\lambda).$$

Using the periodicity of the functions φ_N, Φ_N and f and the fact that the functions φ_N and Φ_N are even, we get

$$\begin{aligned}
E\widehat{f}_N(\lambda) - f(\lambda) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (f(\lambda+u) - f(\lambda))\varphi_N(u)\Phi_N(x)\,dx\\
&\quad + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (f(\lambda+u) - f(\lambda))(\varphi_N(x+u) - \varphi_N(u))\Phi_N(x)\,du\,dx\\
&\le C\int_{-\pi}^{\pi} |u\varphi_N(u)|\,du + o\Big(\int_{-\pi}^{\pi} |x\varphi_N(x)|\,dx\Big). \qquad \blacksquare
\end{aligned}$$

LEMMA 2. [4] *Let $X(t)$, $t \in D$, be a stationary random sequence with the mean $EX(t) = 0$, the spectral density f with $\|f'\| = C < +\infty$ and the spectral density of order four for which we have $\sup|f_4(u_1, u_2, u_3, u_4)| = C_1 < +\infty$. For every $\lambda, \mu \in [-\pi, \pi]$ the covariance of the random variable $\widehat{f}_N(\lambda)$ and $\widehat{f}_N(\mu)$ is given by*

$$\begin{aligned} cov(\widehat{f}_N(\lambda), \widehat{f}_N(\mu)) &= \frac{2\pi}{N} f(\lambda) f(\mu) \int_{-\pi}^{\pi} \varphi_N(x-\lambda)[\varphi(x-\mu) + \varphi_N(x+\mu)]dx \\ &+ |\lambda - \mu| O\left(\frac{\ln^2 N}{N} \int_{-\pi}^{\pi} \varphi_N^2(x)dx\right) + o\left(\frac{1}{N}\int_{-\pi}^{\pi} \varphi_N^2(x)dx\right) \\ &+ O\left(\frac{\ln^4 N}{N^2} \int_{-\pi}^{\pi} x^2 \varphi_N^2(x)dx \int_{-\pi}^{\pi} \varphi_N^2(x)dx\right), \end{aligned}$$

when $N \to \infty$, uniformly for $\lambda, \mu \in [-\pi, \pi]$.

LEMMA 3. *Let the conditions of Theorem 1 be satisfied. Then, for every λ and μ the following equalities are valid:*

$$\lim_{N\to\infty} cov(\widetilde{\xi}_N(\lambda), \widetilde{\xi}_N(\mu)) = \lim_{N\to\infty} cov(\widetilde{Z}_N(\lambda), \widetilde{Z}_N(\mu)) = R(\lambda, \mu). \tag{1}$$

Proof. The first of the equalities (1), as a consequence of Lemma 2, was proved in [4]. Using the fact that $B_N = N^{-\varepsilon}$, where $\frac{1}{3} < \varepsilon < \frac{1}{2}$, and Lemma 1 we get

$$\begin{aligned} &\sup_{|\lambda|\le \pi B_N^{-1}} |\widetilde{\xi}_N(\lambda) - \widetilde{Z}_N(\lambda)| = \sup_{-\pi\le\lambda\le\pi} \sqrt{NB_N}|E\widehat{f}_N(\lambda) - f(\lambda)| \\ &= O\biggl(\sqrt{NB_N}\int_{-\pi}^{\pi} |x\varphi_N(x)|\,dx\biggr) = O\biggl(\sqrt{NB_N}\int_{-\pi B_N^{-1}}^{\pi B_N^{-1}} B_N|t\varphi(t)|\,dt\biggr) \\ &= O(N^{1/2}B_N^{3/2}) = o(1), \quad N\to\infty, \end{aligned}$$

and the second of the equalities (1) follows. ■

LEMMA 4. [1] *Let all cumulant spectral densities of the random sequence $X(t)$, $t \in D$, be bounded. For the cumulants of the random process $\widehat{f}_N(\lambda)$, $-\pi \le \lambda \le \pi$, the following inequality is valid:*

$$|S_N(\widehat{f}_N(\lambda_1), \widehat{f}_N(\lambda_2), \ldots, \widehat{f}_N(\lambda_n))| \le \frac{K_n}{(NB_N)^{n-1}}.$$

The constant K_n does not depend on the particular choice of points $\lambda_1, \ldots, \lambda_n$.

Proof of Theorem 1. It follows from Lemmas 1, 3 and 4 that all cumulants of the random process $\widetilde{Z}_N(\lambda)$ converge to those of the process $Z(\lambda)$ and this completes the proof. ■

LEMMA 5. [5] *Let the sequence $X(t)$, $t \in D$, be Gaussian and its spectral density f bounded. Then, there exist constants $K > 0$, $\eta > 0$ and $\varepsilon > 0$ such that the inequality $E|\widetilde{\xi}_N(\lambda) - \widetilde{\xi}_N(\mu)|^\eta \le K|\lambda - \mu|^{1+\varepsilon}$ holds for every N, λ and μ.*

LEMMA 6. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be continuously-differentiable function and* $f' \in Lip_H(1,\infty)$. *Then, for every* $\lambda, \mu \in \mathbf{R}$ *the following inequality is valid:*

$$\Delta_f(\lambda,\mu,u) := |f(\lambda+u) - f(\lambda) - f(\mu+u) + f(\mu)| \le 2H|u| \cdot |\lambda-\mu|.$$

Proof. From the fact that the function f is differentiable it follows that the following equalities are valid:

$$f(\lambda+u) - f(\lambda) = uf'(\lambda+\vartheta_1 u), \quad 0 \le \vartheta_1 \le 1, \tag{2}$$

$$f(\mu+u) - f(\mu) = uf'(\mu+\vartheta_2 u), \quad 0 \le \vartheta_2 \le 1, \tag{3}$$

$$f(\lambda) - f(\mu) = (\lambda-\mu)f'(\mu+\vartheta_3(\lambda-\mu)), \quad 0 \le \vartheta_3 \le 1, \tag{4}$$

$$f(\lambda+u) - f(\mu+u) = (\lambda-\mu)f'(\mu+u+\vartheta_4(\lambda-\mu)), \quad 0 \le \vartheta_4 \le 1. \tag{5}$$

Using the fact that $f' \in Lip_H(1,\infty)$ and the equalities (2)–(5) we get

$$\begin{aligned}
\Delta_f(\lambda,\mu,u) &= |u| \cdot |f'(\lambda+\vartheta_1 u) - f'(\mu+\vartheta_2 u)| \\
&\le H|u| \cdot |\lambda-\mu+(\vartheta_1-\vartheta_2)u| \le H|u| \cdot |\lambda-\mu| + H|u|^2, \\
\Delta_f(\lambda,\mu,u) &= |\lambda-\mu| \cdot |f'(\mu+\vartheta_3(\lambda-\mu) - f'(\mu+u+\vartheta_4(\lambda-\mu))| \\
&\le H|\lambda-\mu| \cdot |u+(\vartheta_4-\vartheta_3)(\lambda-\mu)| \le H|u| \cdot |\lambda-\mu| + H|\lambda-\mu|^2.
\end{aligned}$$

From these inequalities it follows that

$$\begin{aligned}
\Delta_f(\lambda,\mu,u) &\le H|u| \cdot |\lambda-\mu| + \min\{Hu^2, H|\lambda-\mu|^2\} \\
&\le H|u| \cdot |\lambda-\mu| + H|u| \cdot |\lambda-\mu| = 2H|u| \cdot |\lambda-\mu|. \quad \blacksquare
\end{aligned}$$

LEMMA 7. [6] *For every integrable function* h, *with* $|h(x)| \le H|x|$ *and Fejér's kernel* Φ_N *the following equality is valid:*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(x)\Phi_N(x)(\varphi_N(x+z) - \varphi_N(x))\,dx\,dz = o\left(\int_{-\pi}^{\pi} |x\varphi_N(x)|\,dx\right).$$

DEFINITION. For the (random) function $\xi(\lambda)$, $\lambda \in \Lambda \subset \mathbf{R}$ we define the modulus of continuity $\omega_\xi(\delta)$ in the following way:

$$\omega_\xi(\delta) = \sup_{|\lambda-\mu|<\delta,\, \lambda,\mu\in\Lambda} |\xi(\lambda) - \xi(\mu)|.$$

LEMMA 8. *Let* $f' \in Lip_H(1,\infty)$. *Then, we have*

$$\omega_{\widetilde{Z}_N}(\delta) \le \omega_{\widetilde{\xi}_N}(\delta) + \omega(\delta), \tag{6}$$

where $\omega(\delta) \downarrow 0$ *when* $\delta \downarrow 0$ *and the function* $\omega(\cdot)$ *does not depend on* N.

Proof. Since

$$\sup_{|\lambda-\mu|<\delta} |\widetilde{Z}_N(\lambda) - \widetilde{Z}_N(\mu)| \le \sup_{|\lambda-\mu|<\delta} |\widetilde{\xi}_N(\lambda) - \widetilde{\xi}_N(\mu)| + \sup_{|\lambda-\mu|<\delta} |\widetilde{Z}_N(\lambda) - \widetilde{\xi}_N(\lambda) - (\widetilde{Z}_N(\mu) - \widetilde{\xi}_N(\mu))|,$$

we obtain that $\omega_{\widetilde{Z}_N}(\delta) \le \omega_{\widetilde{\xi}_N}(\delta) + \omega_{\widetilde{Z}_N - \widetilde{\xi}_N}(\delta)$. It is sufficient to prove that $\omega_{\widetilde{Z}_N - \widetilde{\xi}_N}(\delta)$ is bounded and tends to 0 when δ tends to 0.

$$\begin{aligned}
\omega_{\widetilde{Z}_N - \widetilde{\xi}_N}(\delta) &= \sup_{|\lambda-\mu|<\delta} \sqrt{NB_N} |f(\lambda) - E\widehat{f}_N(\lambda) - f(\mu) + E\widehat{f}_N(\mu)| \\
&= \sup_{|\lambda-\mu|<\delta} \sqrt{NB_N} \Bigg| \int_{-\pi}^{\pi} (f(\lambda+u) - f(\lambda) - f(\mu+u) + f(\mu))\varphi_N(u)\,du \\
&+ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (f(\lambda+u) - f(\lambda) - f(\mu+u) + f(\mu))(\varphi_N(u+x) - \varphi_N(u))\Phi_N(x)\,du\,dx \Bigg| \\
&\equiv \sup_{|\lambda-\mu|<\delta} \sqrt{NB_N}|A_1 + A_2|.
\end{aligned}$$

If $|\lambda - \mu| \le \delta$ then using Lemma 6 we get

$$\begin{aligned}
\sqrt{NB_N}|A_1| &\le 2H|\lambda-\mu|\sqrt{NB_N} \int_{-\pi}^{\pi} |u\varphi_N(u)|\,du \\
&\le 2H\delta N^{1/2} B_N^{3/2} \int_{-\infty}^{+\infty} |x\varphi(x)|\,dx \le C'_N \delta \downarrow 0, \quad \delta \downarrow 0,
\end{aligned}$$

Note that the sequence $C'_N = 2HN^{1/2}B_N^{3/2}\int_{-\infty}^{+\infty} |x\varphi(x)|\,dx$, $N = 1,2,3,\dots$ is bounded. Using Lemmas 6 and 7 and the inequality $|\lambda-\mu| \le \delta$ we obtain $\sqrt{NB_N}|A_2| \le C''_N\delta$, where

$$C''_N = \sqrt{NB_N}\, o\bigg(\int_{-\pi}^{\pi} |x\varphi_N(x)|\,dx\bigg) = o(1), \quad N \to \infty..$$

Hence, for $\omega(\delta) = \delta \cdot \sup_N (C'_N + C''_N)$ the inequality (6) is valid. ∎

Proof of Theorem 2. The sequence (Q_N) is *tight* if and only if the following two conditions hold [2]:

(a) For each $\eta > 0$, there exists a constant a such that

$$P\{|\widetilde{Z}_N(0)| > a\} \le \eta, \quad \text{for } N \ge 1;$$

(b) For each $\varepsilon > 0$ and $\eta > 0$, there exist a constant $\delta \in (0,1)$ and an integer n_0, such that

$$P\{\omega_{\widetilde{Z}_N}(\delta) \ge \varepsilon\} \le \eta, \quad \text{for } N \ge n_0.$$

The condition (a) follows from the fact that $\widetilde{Z}_N(\lambda)$ has asimptotically normal distribution, and the conditions (b) follows from Lemmas 5 and 8. Since the sequence (Q_N) is tight and Theorems 1 holds, it follows that Q_N converge weakly to P. ■

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