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CONVERGENCE OF FINITE-DIFFERENCE SCHEMES FOR POISSON'S EQUATION WITH BOUNDARY CONDITION OF THE THIRD KIND

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Abstract. In this paper we study the convergence of finite-difference schemes to generalized solutions of the third boundary-value problem for Poisson's equation on the unit square. Using the generalized Bramble-Hilbert lemma, we obtain error estimates in discrete H^1 Sobolev norm compatible, in some cases, with the smoothness of the data.

The outline of the paper is as follows. In section 1 notational conventions are presented. The stability theorem is proved in section 2. In section 3 we prove estimates of the energy of the operator Δ_h. Finally, in section 4, we derive our main results.

1. Preliminaries and notation

Consider the third boundary-value problem for Poisson's equation on the unit square $\Omega = (0,1)^2$:

$$\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + \sigma u = 0 \quad \text{on } \partial\Omega \tag{1}$$

where $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\cos(x,n) + \frac{\partial u}{\partial y}\cos(y,n)$, n the unit outward normal to $\partial\Omega$, and $\sigma(x,y)$ is continuous function on $\partial\Omega$ such that $\sigma(x,y) \geq \sigma_0 > 0$, $\sigma_0 = \text{const}$.

We suppose that, for $f \in H^0(\Omega)$, our problem (1) has a unique solution in $H^2(\Omega)$ and, provided $f \in H^{s-2}(\Omega)$, $u \in H^s(\Omega)$ for $2 \leq s \leq 4$ (see [1,4]).

Problem (1) is discretised on the uniform mesh with step-size $h : \overline{\Omega}_h = \{(ih, jh) : i,j = 0,1,2,...,N; Nh = 1\}$. We define $\Omega_h = \Omega \cap \overline{\Omega}_h$ and $\partial\Omega_h = \partial\Omega \cap \overline{\Omega}_h$. In $\partial\Omega_h$ we distinguish between two kinds of meshpoints: $\partial\Omega_h^2 = \partial\Omega_h \setminus \partial\Omega_h^1$ and $\partial\Omega_h^1 = \{(0,0),(1,0),(0,1),(1,1)\}$.

For a function U defined on $\overline{\Omega}_h$, the following notation will be used:
$U_{ij} = U(x_i, y_j)$, $x_i = ih$, $y_j = jh$, $\quad i,j = 1,2,...,N$ and

$$\Delta_x^- U_{ij} = \frac{U_{ij} - U_{i-1,j}}{h}, \quad \Delta_x^+ U_{ij} = \Delta_x^- U_{i+1,j},$$
$$\Delta_y^- U_{ij} = \frac{U_{ij} - U_{i,j-1}}{h}, \quad \Delta_y^+ U_{ij} = \Delta_y^- U_{i,j+1}.$$

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In the linear space of functions defined on $\overline{\Omega}_h$ let

$$[U,V] = h^2 \sum_{i,j=1}^{N-1} U_{ij}V_{ij} + \frac{h^2}{2}\sum_{i=1}^{N-1}(U_{i0}V_{i0} + U_{0i}V_{0i} + U_{Ni}V_{Ni} + U_{iN}V_{iN}) + \\ + \frac{h^2}{4}(U_{00}V_{00} + U_{N0}V_{N0} + U_{0N}V_{0N} + U_{NN}V_{NN})$$

be the scalar product and $|\,[U]\,| = \sqrt{[U,U]}$ the corresponding norm. The discrete H^1 norm $|\,[\,\cdot\,]\,|_{1,h}$ is defined by $|\,[U]\,|_{1,h} = \sqrt{|\,[U]\,|^2 + |U|^2_{1,h}}$, where $|\cdot|_{1,h}$ is the discrete H^1 seminorm:

$$|U|_{1,h} = \sqrt{\big|\big|\Delta_x^- U\big]\big|_x^2 + \big|\big|\Delta_y^- U\big]\big|_y^2}, \qquad ||U]|_x = \sqrt{(U,U]_x}, \qquad ||U]|_y = \sqrt{(U,U]_y},$$

$$(U,V]_x = h^2 \sum_{i=1}^{N}\sum_{j=0}^{N} U_{ij}V_{ij} \qquad \text{and} \qquad (U,V]_y = h^2 \sum_{i=0}^{N}\sum_{j=1}^{N} U_{ij}V_{ij}.$$

Let T_i, $\overline{T}_i$ and $\overline{\overline{T}}_i$ $(i = 1,2)$ denote the mollifiers defined by

$$T_1 f(x,y) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+th,y)\,dt \quad , \quad T_2 f(x,y) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x,y+th)\,dt,$$

$$\overline{T}_1 f(0,y) = 2\int_0^{\frac{1}{2}} f(th,y)\,dt \qquad , \qquad \overline{T}_2 f(x,0) = 2\int_0^{\frac{1}{2}} f(x,th)\,dt,$$

$$\overline{\overline{T}}_1 f(1,y) = 2\int_{-\frac{1}{2}}^{0} f(1+th,y)\,dt \ , \quad \overline{\overline{T}}_2 f(x,1) = 2\int_{-\frac{1}{2}}^{0} f(x,1+th)\,dt.$$

We approximate problem (1) by the finite-difference scheme

$$\Delta_h U = F \quad \text{in } \overline{\Omega}_h, \tag{2}$$

where $\Delta_h U = \Delta_{h,x}U + \Delta_{h,y}U$,

$$\Delta_{h,x}U = \begin{cases} \frac{2}{h}(\Delta_x^+ U - \sigma U)\ , & i = 0 \quad , \qquad j = 0,1,2,\dots,N \\ \Delta_x^+\Delta_x^- U \quad , & i = 1,2,\dots,N-1, \quad j = 0,1,2,\dots,N\ , \\ -\frac{2}{h}(\Delta_x^- U + \sigma U), & i = N \quad , \qquad j = 0,1,2,\dots,N \end{cases}$$

$$F_{ij} = T_1T_2 f_{ij},\ F_{0j} = \overline{T}_1 T_2 f_{0j}, (i,j = 1,2,\dots,N-1),\ F_{00} = \overline{T}_1\overline{T}_2 f_{00},$$

and $\Delta_{h,y}U$, F_{Nj}, F_{i0}, F_{iN}, F_{N0}, F_{0N}, F_{NN} defined analogously.

2. Stability of the scheme

To begin, let us prove two lemmas.

LEMMA 1. *Let U, V denote mesh-functions on $\overline{\Omega}_h$. Then* $[\Delta_h U, V] = [U, \Delta_h V]$.

Proof. Using summation by parts it is easy to prove that

$$[\Delta_{h,x}U,V] = -h^2\sum_{j=1}^{N-1}\sum_{i=1}^{N}\Delta_x^- U_{ij}\Delta_x^- V_{ij} -$$
$$-\frac{h^2}{2}\sum_{i=1}^{N}\left(\Delta_x^- U_{i0}\Delta_x^- V_{i0} + \Delta_x^- U_{iN}\Delta_x^- V_{iN}\right) - h\sum_{i=1}^{N-1}\left(\sigma_{Ni}U_{Ni}V_{Ni} + \sigma_{0i}U_{0i}V_{0i}\right)$$
$$-\frac{h}{2}\left(\sigma_{00}U_{00}V_{00} + \sigma_{0N}U_{0N}V_{0N} + \sigma_{N0}U_{N0}V_{N0} + \sigma_{NN}U_{NN}V_{NN}\right) = [U,\Delta_{h,x}V]\,.$$

The operator $\Delta_{h,y}$ has the same property. Therefore,

$$[\Delta_h U,V] = [\Delta_{h,x}U,V] + [\Delta_{h,y}U,V] = [U,\Delta_{h,x}V] + [U,\Delta_{h,y}V] = [U,\Delta_h V]\,. \quad \blacksquare$$

LEMMA 2. *Let U denote mesh-function on $\overline{\Omega}_h$. Then $[\Delta_h U,U] \le -C|[U]|^2$, where $C = \min\{1, 2\sigma_0\}$.*

Proof. For fixed $j = 0,1,2,...,N$, using an inequality from [8] we get

$$\max\left\{U_{ij}^2 : 0 \le i \le N\right\} \le 2h\sum_{i=1}^{N}\left(\Delta_x^- U_{ij}\right)^2 + U_{0j}^2 + U_{Nj}^2.$$

This yields the following inequality:

$$h\sum_{i=1}^{N-1}U_{ij}^2 + \frac{h}{2}\left(U_{0j}^2 + U_{Nj}^2\right) \le 2h\sum_{i=1}^{N}\left(\Delta_x^- U_{ij}\right)^2 + U_{0j}^2 + U_{Nj}^2.$$

Now let us prove that $[\Delta_{h,x}U,U] \le -\frac{C}{2}|[U]|^2$. Summing by parts and using last inequality, we obtain:

$$[\Delta_{h,x}U,U] = -h^2\sum_{j=1}^{N-1}\sum_{i=1}^{N}\left(\Delta_x^- U_{ij}\right)^2 - \frac{h^2}{2}\sum_{i=1}^{N}\left[\left(\Delta_x^- U_{i0}\right)^2 + \left(\Delta_x^- U_{iN}\right)^2\right] -$$
$$-h\sum_{j=1}^{N-1}\left(\sigma_{Nj}U_{Nj}^2 + \sigma_{0j}U_{0j}^2\right) - \frac{h}{2}\left(\sigma_{00}U_{00}^2 + \sigma_{0N}U_{0N}^2 + \sigma_{N0}U_{N0}^2 + \sigma_{NN}U_{NN}^2\right) \le$$
$$-h\sum_{j=1}^{N-1}\left[h\sum_{i=1}^{N}\left(\Delta_x^- U_{ij}\right)^2 + \sigma_0 U_{Nj}^2 + \sigma_0 U_{0j}^2\right] - \frac{h}{2}\left[h\sum_{i=1}^{N}\left(\Delta_x^- U_{i0}\right)^2 + \sigma_0 U_{00}^2 + \sigma_0 U_{N0}^2\right]$$
$$-\frac{h}{2}\left[h\sum_{i=1}^{N}\left(\Delta_x^- U_{iN}\right)^2 + \sigma_0 U_{0N}^2 + \sigma_0 U_{NN}^2\right] \le -\frac{C}{2}|[U]|^2.$$

The inequality $[\Delta_{h,y}U,U] \le -\frac{C}{2}|[U]|^2$ can be proved analogously and we easily obtain Lemma 2. $\blacksquare$

THEOREM 1. *For any $f \in H^s(\Omega)$, $s \geq 0$, finite-difference scheme (2) has unique solution U. Moreover,*

$$\left|[U]\right|_{1,h} \leq \sqrt{2 + \frac{1}{C}}\,\left|[F]\right|, \tag{3}$$

where $C = \min\{1, 2\sigma_0\}$.

Proof. The existence and uniqueness of solutions follow from the fact that Δ_h is a self-adjoint and negative definite operator (Lemmas 1. and 2.). Further, (using Lemma 2.) we can prove stability in the norm $|[\,\cdot\,]|$: $\quad |[U]|^2 \leq \frac{1}{C}[-\Delta_h U, U] = \frac{1}{C}[-F, U] \leq \frac{1}{C}|[F]|\,|[U]|$ and thus $|[U]| \leq \frac{1}{C}|[F]|$. Summing by parts we can also prove that $[\Delta_{h,x}U, U] \leq -\frac{1}{2}\left|\left|\Delta_x^- U\right]\right|_x^2$, $[\Delta_{h,y}U, U] \leq -\frac{1}{2}\left|\left|\Delta_y^- U\right]\right|_y^2$, and $|U|_{1,h}^2 \leq 2\left[-\Delta_h U, U\right]$. Thence and using Lemma 2. we get (3). ∎

3. The estimates of energy norm $[-\Delta_h U, U]$

In this section we present three lemmas. Each of them will be used to obtain an appropriate error estimate for scheme (2).

LEMMA 3. *Let U denote a mesh-function on $\overline{\Omega}_h$ which is a solution of finite-difference scheme (2). Then*

$$[-\Delta_h U, U] \leq C_3 \left\{ h^2 \sum_{i,j=1}^{N-1} F_{ij}^2 + \frac{h^3}{4}\left[\sum_{i=0}^{N}\left(F_{i0}^2 + F_{iN}^2\right) + \sum_{i=1}^{N-1}\left(F_{0i}^2 + F_{Ni}^2\right)\right]\right\}, \tag{4}$$

where C_3 is a positive constant.

Proof. Using the ε-inequality: $|ab| \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$; $\quad a, b \in \mathbf{R}$, $\quad \varepsilon > 0$; in the identity $[-\Delta_h U, U] = [-F, U]$ as follows:

$$-h^2 \sum_{i,j=1}^{N-1} F_{ij}U_{ij} \leq \varepsilon h^2 \sum_{i,j=1}^{N-1} U_{ij}^2 + \frac{h^2}{4\varepsilon}\sum_{i,j=1}^{N-1} F_{ij}^2,$$

$$-\frac{h^2}{2}\sum_{i=1}^{N-1} F_{i0}U_{i0} \leq \varepsilon h \sum_{i=1}^{N-1} U_{i0}^2 + \frac{h^3}{16\varepsilon}\sum_{i=1}^{N-1} F_{i0}^2 \text{ and } -\frac{h^2}{4}F_{00}U_{00} \leq \frac{\varepsilon h}{4}U_{00}^2 + \frac{h^3}{16\varepsilon}F_{00}^2$$

we obtain the following inequality:

$$[-\Delta_h U, U] \leq \varepsilon \mathbb{U} + \frac{1}{4\varepsilon}\left\{ h^2 \sum_{i,j=1}^{N-1} F_{ij}^2 + \frac{h^3}{4}\sum_{i=0}^{N}\left(F_{i0}^2 + F_{iN}^2\right) + \frac{h^3}{4}\sum_{i=1}^{N-1}\left(F_{0i}^2 + F_{Ni}^2\right)\right\}$$

where $\mathbb{U} = \mathbb{U}(U, h)$, more precisely

$$\mathbb{U} = h^2 \sum_{i,j=1}^{N-1} U_{ij}^2 + h \sum_{i=1}^{N-1}\left(U_{0i}^2 + U_{i0}^2 + U_{Ni}^2 + U_{iN}^2\right) + \frac{h}{4}\left(U_{00}^2 + U_{0N}^2 + U_{N0}^2 + U_{NN}^2\right).$$

It is easy to prove that $\mathbb{U}(U,h) \le \left(\frac{1}{C}+\frac{1}{\sigma_0}\right)[-\Delta_h U, U]$, where $C=\min\{1,2\sigma_0\}$ and $\sigma \ge \sigma_0 > 0$. Thus we get (4) where $C_3 = \left[4\varepsilon - 4\varepsilon^2\left(C^{-1}+\sigma_0{}^{-1}\right)\right]^{-1}$ and the value of ε can be chosen so that $1 > \varepsilon\left(C^{-1}+\sigma_0{}^{-1}\right)$, the optimal choice being

$$\varepsilon = C\sigma_0(2\sigma_0+2C)^{-1} = \begin{cases} 1+\sigma_0{}^{-1}, & \sigma_0 \ge \frac{1}{2}, \\ \frac{3}{2}\sigma_0{}^{-1}, & 0<\sigma_0<\frac{1}{2}. \end{cases} \qquad \blacksquare$$

Using the same technique, we can prove the following two lemmas. Their proofs are omitted.

LEMMA 4. *Let U denote a mesh-function on $\overline{\Omega}_h$ which is the solution of finite-difference scheme (2). If we substitute F_{ij} by $F_{ij} = \Delta_x^+\xi_{1,ij} + \Delta_y^+\xi_{2,ij}, (i,j = 1,2,...,N-1), F_{i0} = \Delta_x^+\xi_{1,i0} + \eta_{2,i0},\ F_{0i} = \eta_{1,0i} + \Delta_y^+\xi_{2,0i},\ F_{Ni} = \eta_{1,Ni} + \Delta_y^+\xi_{2,Ni},\ F_{iN} = \Delta_x^+\xi_{1,iN} + \eta_{2,iN},\ (i=1,2,...,N-1), F_{00} = \eta_{1,00}+\eta_{2,00},\ F_{0N} = \eta_{1,0N}+\eta_{2,0N},\ F_{N0} = \eta_{1,N0}+\eta_{2,N0}$ and $F_{NN} = \eta_{1,NN}+\eta_{2,NN}$ then*

$$[-\Delta_h U, U] \le C_4\left\{h^2\sum_{j=0}^{N}\sum_{i=1}^{N}\left(\xi_{1,ij}^2+\xi_{2,ji}^2\right) + h\sum_{i=1}^{N}\left(\xi_{1,Ni}^2+\xi_{1,1i}^2+\xi_{2,iN}^2+\xi_{2,i1}^2\right)\right.$$
$$\left. + h^3\sum_{i=0}^{N}\left(\eta_{1,i0}^2+\eta_{1,iN}^2+\eta_{2,0i}^2+\eta_{2,Ni}^2\right)\right\}.$$

LEMMA 5. *Under the same assumptions as in Lemma 4, and defining $\alpha_{0i} = \xi_{1,1i} - \frac{h}{2}\eta_{1,0i},\ \alpha_{Ni} = -\xi_{1,Ni} - \frac{h}{2}\eta_{1,Ni},\ \beta_{i0} = \xi_{2,i1} - \frac{h}{2}\eta_{2,i0}$ and $\beta_{iN} = -\xi_{2,iN} - \frac{h}{2}\eta_{2,iN},\ (i=0,1,2,...,N)$, the following inequality holds:*

$$[-\Delta_h U, U] \le C_5\left\{h^2\sum_{j=0}^{N}\sum_{i=1}^{N}\left(\xi_{1,ij}^2+\xi_{2,ji}^2\right) + h\sum_{i=0}^{N}\left(\alpha_{0i}^2+\alpha_{Ni}^2+\beta_{i0}^2+\beta_{iN}^2\right)\right\}.$$

4. Convergence of the finite-difference scheme

Before stating our main results we quote the following theorem which is a variant of the well-known Dupont-Scott approximation theorem (see [2]).

THEOREM 2. *Let E be a bounded connected domain in $\mathbf{R}^2$ satisfying the cone condition and $\mathcal{A}(u)$ a bounded linear functional on $H^s(E)$ $(s = \{s\}+\alpha,\ \{s\} \ge 0$ is integer and $\{s\} < s,\ 0<\alpha\le 1)$ such that $P_{\{s\}} \subset \mathrm{Kernel}\,(\mathcal{A}(u))$, where $P_{\{s\}}$ denotes the set of polynomials of degree $\le \{s\}$. Then, for any $u \in H^s(E)$, $|\mathcal{A}(u)| \le C|u|_{H^s(E)}$, where $C = C(E,s)$ is a positive constant independent of u and $|\cdot|_{H^s(E)}$ is the highest seminorm of $H^s(E)$.*

The derivations of all error estimates below are based on the above theorem.

THEOREM 3. *Suppose that* $u \in H^s(\Omega)$, $2 \le s \le 4$, *is the solution of problem (1) and* U *is the solution of the finite-difference scheme (2). Then*

$$|[U-u]|_{1,h} \le Ch^{s-2}\|u\|_{H^s(\Omega)} = O(h^{s-2}).$$

Proof. Let us define the global error as $z = U - u$. Then $\Delta_h z_{ij} = \Delta_h U_{ij} - \Delta_h u_{ij} = F_{ij} - \Delta_h u_{ij} = \varphi_{ij}$. We shall consider three cases:

i) If $(ih, jh) \in \Omega_h$, then

$$\varphi_{ij} = T_2 \Delta_x^- \frac{\partial u}{\partial x}\left(ih + \frac{h}{2}, jh\right) + T_1 \Delta_y^- \frac{\partial u}{\partial x}\left(ih, jh + \frac{h}{2}\right) - \Delta_h u_{ij}.$$

Using Theorem 2 and standard technique based on Theorem 2 as in [3], [5] or [9], we obtain $|\varphi_{ij}| \le Ch^{s-3}|u|_{H^s(e_{ij})}$, $2 < s \le 4$, where $e_{ij} = \{(x,y) : ih - h \le x \le ih + h;\ jh - h \le y \le jh + h\}$.

ii) If $(ih, jh) \in \partial\Omega_h^2$, for example $(0, jh)$, then

$$\varphi_{0j} = \frac{2}{h} T_2 \left[\frac{\partial u}{\partial x}\left(\frac{h}{2}, jh\right) - \frac{\partial u}{\partial x}(0, jh)\right] + \overline{T}_1 \Delta_y^- \frac{\partial u}{\partial y}\left(0, jh + \frac{h}{2}\right) - \\ - \frac{1}{h^2}\left[u_{0,j+1} + u_{0,j-1} + 2u_{1,j} - 2h\frac{\partial u}{\partial x}(0, jh) - 4u_{0j}\right].$$

In the same way, except that $2 < s \le 3$, we obtain $|\varphi_{0j}| \le Ch^{s-3}|u|_{H^s(e_{0j})}$, where $e_{0j} = \{(x,y) :\ 0 \le x \le h;\ jh - h \le y \le jh + h\}$.

iii) If $(ih, jh) \in \partial\Omega_h^1$, for example $(0, 0)$, then

$$\varphi_{00} = \frac{2}{h}\overline{T}_2 \left[\frac{\partial u}{\partial x}\left(\frac{h}{2}, 0\right) - \frac{\partial u}{\partial x}(0,0)\right] + \frac{2}{h}\overline{T}_1 \left[\frac{\partial u}{\partial y}\left(0, \frac{h}{2}\right) - \frac{\partial u}{\partial y}(0,0)\right] - \\ - \frac{2}{h^2}\left[u_{10} - u_{00} - h\frac{\partial u}{\partial x}(0,0) + u_{01} - u_{00} - h\frac{\partial u}{\partial y}(0,0)\right]$$

and we obtain, provided $2 < s \le 3$, $|\varphi_{00}| \le Ch^{s-3}|u|_{H^s(e_{00})}$, where $e_{00} = \{(x,y) :\ 0 \le x, y \le h\}$.

However, we can obtain $|[U]|_{1,h}^2 \le C\,[\Delta_h U, U]$. Thence, using (4), we get

$$|[z]|_{1,h}^2 \le C\left\{h^2 \sum_{i,j=1}^{N-1} \varphi_{ij}^2 + \frac{h^3}{4}\sum_{i=0}^{N}\left(\varphi_{i0}^2 + \varphi_{iN}^2\right) + \frac{h^3}{4}\sum_{i=1}^{N-1}\left(\varphi_{0i}^2 + \varphi_{Ni}^2\right)\right\}.$$

Now, for $2 < s \le 3$, it is easy to prove that $|[z]|_{1,h} \le Ch^{s-2}|u|_{H^s(\Omega)}$. If $3 < s \le 4$, then

$$h^2 \sum_{i,j=1}^{N-1} \varphi_{ij}^2 \le Ch^{2s-4}|u|_{H^s(\Omega)}^2. \tag{5}$$

On the other hand, if $u \in H^s(\Omega)$, then $u \in H^3(\Omega)$ and

$$\frac{h^3}{4}\left[\sum_{i=0}^{N}\left(\varphi_{i0}^2+\varphi_{iN}^2\right)+\sum_{i=1}^{N-1}\left(\varphi_{0i}^2+\varphi_{Ni}^2\right)\right] \le Ch^3|u|_{H^3(\partial_h\Omega)}, \tag{6}$$

where $\partial_h\Omega$ is the boundary strip of width h. But, according to [7]

$$|u|_{H^3(\partial_h\Omega)} \le C\|u\|_{H^s(\Omega)} \cdot \begin{cases} h^{s-3}, & 3 < s < \frac{7}{2} \\ \sqrt{h}|\ln h|, & s = \frac{7}{2} \\ \sqrt{h}\ , & \frac{7}{2} < s \le 4. \end{cases} \tag{7}$$

Using (5), (6) and (7) we obtain $|[z]|_{1,h} \le Ch^{s-2}\|u\|_{H^s(\Omega)}$ and that completes the proof. ∎

THEOREM 4. *Suppose that* $u \in H^s(\Omega)$, $\frac{3}{2} < s \le 3$, *is the solution of problem (1) where* $\sigma \in M\left(H^{s-1}(0,1)\right)$ *(see [6]) and* U *is the solution of (2). Then*

$$|[U-u]|_{1,h} = \begin{cases} O(h^{s-1})\ , & \frac{3}{2} < s < \frac{5}{2} \\ O(h\sqrt{h}|lnh|), & s = \frac{5}{2} \\ O(h\sqrt{h})\ , & \frac{5}{2} < s \le 3. \end{cases}$$

Proof. This theorem is similar to the previous one. Therefore we begin the proof as before. Naturally, this time we shall use Lemma 5 and thus we have to derive the following:

i) If $i = 1,2,...,N$; $j = 1,2,...N-1$ and $\frac{3}{2} < s \le 3$, then

$$\xi_{1,ij} = T_2\left[\frac{\partial u}{\partial x}\left(ih-\frac{h}{2}, jh\right)\right] - \Delta_x^- u_{ij} \quad \text{and} \quad |\xi_{1,ij}| \le Ch^{s-2}|u|_{H^s(e_{ij})},$$

where $e_{ij} = \left\{(x,y) :\ ih-h \le x \le ih,\ jh-\frac{h}{2} \le y \le jh+\frac{h}{2}\right\}$.

ii) If $i = 1,2,...,N$; $j = 0$ or $j = N$ and $\frac{3}{2} < s \le 2$, then

$$\xi_{1,i0} = \overline{T}_2\left[\frac{\partial u}{\partial x}\left(ih-\frac{h}{2}, 0\right)\right] - \Delta_x^- u_{i0}, \quad \xi_{1,iN} = \overline{\overline{T}}_2\left[\frac{\partial u}{\partial x}\left(ih-\frac{h}{2}, 1\right)\right] - \Delta_x^- u_{iN}$$

$$\text{and} \qquad |\xi_{1,ij}| \le Ch^{s-2}|u|_{H^s(e_{ij})}$$

where $e_{i0} = \left\{(x,y) :\ ih-h \le x \le ih,\ 0 \le y \le \frac{h}{2}\right\}$
or $e_{iN} = \left\{(x,y) :\ ih-h \le x \le ih,\ 1-\frac{h}{2} \le y \le 1\right\}$.
(Analogous results can be obtained for ξ_2.)

iii) If $j = 1,2,...,N-1$, then $\alpha_{0j} = T_2(\sigma_{0j}u_{0j}) - \sigma_{0j}u_{0j}$ and

$$|\alpha_{0j}| \le Ch^{s-\frac{3}{2}}|\sigma u|_{H^{s-1}(d_{0j})}, \qquad 1 \le s \le 3,$$

where $d_{0j} = \left[jh-\frac{h}{2}, jh+\frac{h}{2}\right]$. (The same results can be obtained for α_{Nj}, β_{jN} and β_{j0}.)

iv) If $i=0$ and $j=0$, then $\alpha_{00}=\overline{T}_2(\sigma_{00}u_{00})-\sigma_{00}u_{00}$ and
$$|\alpha_{00}|\le Ch^{s-\frac{3}{2}}|\sigma u|_{H^{s-1}(d_{00})},\qquad 1\le s\le 2,$$
where $d_{00}=\left[0,\frac{h}{2}\right]$. (The same results can be obtained for α_{ij} and β_{ij} where $(ih,jh)\in\partial\Omega_h^1$.)

Thence:
$$h^2\sum_{i=1}^{N}\sum_{j=1}^{N-1}\left(\xi_{1,ij}^2+\xi_{2,ji}^2\right)\le Ch^{2s-2}|u|^2_{H^s(\Omega)},\quad \frac{3}{2}<s\le 3,$$
$$h^2\sum_{i=1}^{N}\left(\xi_{1,i0}^2+\xi_{1,iN}^2+\xi_{2,0i}^2+\xi_{2,Ni}^2\right)\le C\|u\|^2_{H^s(\Omega)}\cdot\begin{cases}h^s\;, & \frac{3}{2}<s<\frac{5}{2}\\ h^3\ln^2 h, & s=\frac{5}{2}\\ h^3\;, & \frac{5}{2}<s\le 3\end{cases},$$
$$h\sum_{j=1}^{N-1}\alpha_{0j}^2\le Ch^{2s-2}|\sigma u|^2_{H^{s-1}(0,1)}\,,\quad 1\le s\le 3,$$
$$h\alpha_{00}^2\le C\|\sigma u\|^2_{H^{s-1}(0,1)}\cdot\begin{cases}h^{2s-2}\;, & 1\le s\le\frac{5}{2}\\ h^3\ln^2 h, & s=\frac{5}{2}\\ h^3\;, & \frac{5}{2}<s\le 3\end{cases}$$
and
$$|\sigma u|_{H^{s-1}(0,1)}\le\|\sigma u\|_{H^{s-1}(0,1)}\le C\|\sigma\|_{M(H^{s-1}(0,1))}\|u\|_{H^{s-1}(0,1)}\le$$
$$\le C\|\sigma\|_{M(H^{s-1}(0,1))}\|u\|_{H^s(\Omega)}.$$
Now using Lemma 5, we easily complete the proof of the theorem. ■

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