МАТЕМАТИЧКИ ВЕСНИК
47 (1995), 19–22

UDC 515.126.4
оригинални научни рад
research paper

BANACH SPACES OVER TOPOLOGICAL SEMIFIELDS AND COMMON FIXED POINTS

Slobodan Č. Nešić

Abstract. In this paper we shall prove the existence of a common fixed point of two mappings of a Banach space over a topological semifield.

INTRODUCTION. The notion of topological semifield has been introduced by M. Antonovskiĭ, V. Boltyanskiĭ and T. Sarymsakov in [1]. Let E be a topological semifield and K the set of all its positive elements. Take any two elements x, y in E. If $y - x$ is in $\overline{K}$ (in K), this is denoted by $x \ll y$ ($x < y$). As proved in [1], every topological semifield E contains a subsemifield, called the axis of E, isomorphic to the field $\mathbf{R}$ of real numbers. Consequently, by identifying the axis and $\mathbf{R}$, each topological semifield can be regarded as a topological linear space over the field $\mathbf{R}$.

The ordered triple (X, d, E) is called a metric space over the topological semifield if there exists a mapping $d\colon X \times X \to \overline{K}$ satisfying the usual axioms for a metric (see [1] and [3]).

Linear spaces considered in this paper are defined over the field $\mathbf{R}$. Let X be a linear space. The ordered triple $(X, \|\ \|, E)$ is called a feeble normed space over the topological semifield if there exists a mapping $\|\ \|\colon X \to \overline{K}$ satisfying the usual axioms for a norm (see [1] and [2]).

Let $(X, \|\ \|, E)$ be a feeble normed space over the topological semifield E and let $d(x, y) = \|x - y\|$ for all x, y in X. The space $(X, \|\ \|, E)$ is said to be a Banach space over the topological semifield E if (X, d, E) is a sequentially complete metric space over the topological semifield E.

We shall prove the following theorem.

Key words and phrases: Banach space over a topological semifield, common fixed point, Cauchy sequence and sequentially complete metric space

AMS Subject Classifications (1991): 47H10, 54H25

Theorem 1. *Let X be a Banach space over the topological semifield E and $S, T\colon X \to X$ two maps. If there exist real numbers b, c, q and t such that*

$$0 \leqslant qt + |b|(1-t) - c < b + c, \quad 0 < t < 1, \tag{1}$$

$$b\,\|Sx - Ty\| + c(\|x - Sx\| + \|y - Ty\|) \ll q\|x - y\| \tag{2}$$

for all x, y in X, then the sequence $\{x_n\}$, the members of which are

$$\begin{aligned} x_{2n+1} &= (1-t)x_{2n} + t\,Sx_{2n}, \\ x_{2n+2} &= (1-t)x_{2n+1} + t\,Tx_{2n+1}, \quad x_0 \in X, \quad n = 0, 1, 2, \dots \end{aligned} \tag{3}$$

converges to the common fixed point of S and T in X.

Proof. Let x_0 in X be an arbitrary point. Using the sequence (3) we have

$$\begin{aligned} \|x_{2n+1} - x_{2n}\| &= t\,\|Sx_{2n} - x_{2n}\|, \\ \|x_{2n+2} - x_{2n+1}\| &= t\,\|Tx_{2n+1} - x_{2n+1}\| \end{aligned} \tag{4}$$

and hence

$$\begin{aligned} \|x_{2n+1} - Sx_{2n}\| &= \|(1-t)x_{2n} + t\,Sx_{2n} - Sx_{2n}\| \\ &= (1-t)\|x_{2n} - Sx_{2n}\| = \frac{1-t}{t}\,\|x_{2n} - x_{2n+1}\|, \\ \|x_{2n+2} - Tx_{2n+1}\| &= \|(1-t)x_{2n+1} + t\,Tx_{2n+1} - Tx_{2n+1}\| \\ &= (1-t)\|x_{2n+1} - Tx_{2n+1}\| = \frac{1-t}{t}\,\|x_{2n+1} - x_{2n+2}\|. \end{aligned}$$

Then the inequalities

$$\|x_{2n+1} - Tx_{2n+1}\| - \|x_{2n+1} - Sx_{2n}\| \ll \|Sx_{2n} - Tx_{2n+1}\| \tag{5}$$

and

$$\|x_{2n+2} - Sx_{2n+2}\| - \|x_{2n+2} - Tx_{2n+1}\| \ll \|Sx_{2n+2} - Tx_{2n+1}\| \tag{6}$$

respectively become

$$\frac{1}{t}\|x_{2n+2} - x_{2n+1}\| - \frac{1-t}{t}\,\|x_{2n} - x_{2n+1}\| \ll \|Sx_{2n} - Tx_{2n+1}\| \tag{7}$$

and

$$\frac{1}{t}\,\|x_{2n+3} - x_{2n+2}\| - \frac{1-t}{t}\,\|x_{2n+1} - x_{2n+2}\| \ll \|Sx_{2n+2} - Tx_{2n+1}\|. \tag{8}$$

Let $b \geqslant 0$. If we put in (2) $x = x_{2n}$ and $y = x_{2n+1}$, then from (4), (5) and (7) we get

$$\begin{aligned} \frac{b}{t}\,\|x_{2n+2} - x_{2n+1}\| - |b|\frac{1-t}{t}\,\|x_{2n} - x_{2n+1}\| + \frac{c}{t}\,\|x_{2n+1} - x_{2n}\| \\ + \frac{c}{t}\,\|x_{2n+2} - x_{2n+1}\| \ll q\|x_{2n} - x_{2n+1}\| \end{aligned} \tag{9}$$

and hence

$$\|x_{2n+2} - x_{2n+1}\| \ll k\,\|x_{2n+1} - x_{2n}\|, \tag{10}$$

where $k = (qt + (1-t)|b| - c)/(b+c)$ and $b = |b|$. Now, if we put in (2) $x = x_{2n+2}$ and $y = x_{2n+1}$, and use (4), (6) and (8), we get

$$\begin{aligned} \frac{b}{t}\,\|x_{2n+3} - x_{2n+2}\| - |b|\frac{1-t}{t}\,\|x_{2n+2} - x_{2n+2}\| + \frac{c}{t}\,\|x_{2n+3} - x_{2n+2}\| \\ + \frac{c}{t}\,\|x_{2n+2} - x_{2n+1}\| \ll q\|x_{2n+2} - x_{2n+1}\| \end{aligned} \tag{11}$$

and hence

$$\|x_{2n+3} - x_{2n+2}\| \ll k\,\|x_{2n+2} - x_{2n+1}\|. \tag{12}$$

Now, if $b < 0$, then we use the inequalities

$$\|x_{2n+1} - Tx_{2n+1}\| + \|x_{2n+1} - Sx_{2n}\| \gg \|Sx_{2n} - Tx_{2n+1}\| \tag{13}$$

and

$$\|x_{2n+2} - Sx_{2n+2}\| + \|x_{2n+2} - Tx_{2n+1}\| \gg \|Sx_{2n+2} - Tx_{2n+1}\|. \tag{14}$$

If in (2) we put $x = x_{2n}$ and $y = x_{2n+1}$, then by (13) we obtain (9), since then $-|b| = b$. If in (2) we put $x = x_{2n+2}$ and $y = x_{2n+1}$, then by (14) we obtain (11).

From (10) and (12) then we obtain

$$\|x_n - x_{n+1}\| \ll k\,\|x_{n-1} - x_n\|,$$

which implies

$$\|x_n - x_{n+1}\| \ll k^n\|x_0 - x_1\|.$$

Since (1) implies $0 \leqslant k < 1$, it follows that $\{x_n\}$ is a Cauchy sequence in X. Because X is a Banach space over the topological semifield E, we deduce that $\{x_n\}$ converges to a point u in X. Then from (4) we have $\lim_n Tx_{2n+1} = u$.

If in (2) we put $x = u$ and $y = x_{2n+1}$, then by (4) we have

$$b\,\|Su - Tx_{2n+1}\| + c\|u - Su\| + \frac{c}{t}\,\|x_{2n+2} - x_{2n+1}\| \ll q\|u - x_{2n+1}\|.$$

Letting $n \to +\infty$, we get $(b+c)\|Su - u\| \ll 0$. Then, as $b + c > 0$, it follows that $Su = u$. Hence, u is a fixed point for S. Similarly, $Tu = u$. Thus, u is a common fixed point of S and T. This completes the proof. ∎

Remark. In case $S = T$, $b = 0$ and $c = 1$ in Theorem 1, we obtain the Theorem 1 of Nešić [4].

REFERENCES

[**1**] M. Ya. Antonovskiĭ, V. G. Boltyanskiĭ and T. A. Sarymsakov, *Topological semifields*, Tashkent 1960.

[**2**] S. Kasahara, *On formulations of topological linear spaces by topological semifields*, Math. Sem. Notes, **1** (1973), 11–29

[**3**] Z. Mamuzić, *Some remarks on abstract distance in general topology*, $E\Lambda EY\Theta EPIA$ **2** (1979), 433–446

[**4**] S. Nešić, *On fixed point theorems in Banach spaces over topological semifields*, Math. Balkanica (to appear)

(received 26.04.1995.)

Studentska 14
11000 Beograd
Yugoslavia