МАТЕМАТИЧКИ ВЕСНИК
46 (1994), 89–92

UDC 517.965
оригинални научни рад
research paper

ON LOCALLY SUBADDITIVE FUNCTIONS

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Abstract. We define locally (on $C \subset \mathbf{R}^2$) subadditive functions f, $f\colon C \to \mathbf{R}$, by

$$f(x_1 + x_2, y_1 + y_2) \leqslant f(x_1, y_1) + f(x_2, y_2), \quad (x_1, y_1), (x_2, y_2) \in C,$$

where C is some cone in $\mathbf{R}^2$. The purpose of the paper is to find explicit form of such functions.

A function $f\colon \mathbf{R} \to \mathbf{R}$ is said to be additive if it satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y), \quad x, y \in \mathbf{R}.$$

Under some smoothing restrictions (measurability or Baire property) the only form of additive functions, as is well known, is that of cx.

Two-dimensional case of Cauchy equation, i.e.

$$f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_2, y_2), \quad (x_1, y_1), (x_2, y_2) \in \mathbf{R}^2,$$

in a similar way has the solution $f(x, y) = c_1 x + c_2 y$.

We define *locally* (on $C \subset \mathbf{R}^2$) *subadditive functions* f, $f\colon C \to \mathbf{R}$, by

$$f(x_1 + x_2, y_1 + y_2) \leqslant f(x_1, y_1) + f(x_2, y_2), \quad (x_1, y_1), (x_2, y_2) \in C, \tag{1}$$

where C is some cone in $\mathbf{R}^2$. A non-empty convex subset C of $\mathbf{R}^n$ is called a cone if $\lambda C \subset C$ for all $\lambda \geqslant 0$. For the definition of a cone in an arbitrary vector space see [2]. We shall denote the class of all such functions on C by LS_C. In these considerations we also admit sets C which are cones without the point 0.

Our task in this paper is to "solve" functional inequality (1), i.e. to give an explicit form of $f \in LS_C$.

We begin with the following results:

PROPOSITION 1. *If $f_k \in LS_{C_k}$, $k = 1, 2, \ldots, n$ then:*

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = f \in LS_C, \tag{2}$$

where $C = \bigcap_{k=1}^n C_k$ and c_1, c_2, $\ldots$, c_n are arbitrary positive constants.

Proof follows immediately from definition (1) of locally subadditive functions and the fact that intersection of any family of cones is a cone. ■

PROPOSITION 2. *If $g(t)$ is a convex function defined for $t \in (a,b)$, then*

$$x \cdot g(y/x) = f(x,y) \in LS_C,$$

where $C = \{\,(x,y) \mid a < y/x < b, x > 0\,\}$ is a subset of $\mathbf{R}^2$.

Proof. It is clear that C is a non-empty convex subset of $\mathbf{R}^2$ with $\lambda C \subset C$ for all $\lambda > 0$. From this it follows that $C + C \subset C$. Now, according to definition of a convex function $g(t)$, $t \in (a,b)$

$$g(pr+qs) \leqslant pg(r) + qg(s) \tag{3}$$

for each r, $s \in (a,b)$ and each p, $q \geqslant 0$, $p+q=1$, and since (x_1,y_1), $(x_2,y_2) \in C$ implies that $(x_1+x_2, y_1+y_2) = (x_1,y_1)+(x_2,y_2) \in C + C \subset C$, we have

$$\begin{aligned}
f(x_1+x_2,y_1+y_2) &= (x_1+x_2)g\left(\frac{y_1+y_2}{x_1+x_2}\right)\\
&= (x_1+x_2)g\left(\frac{x_1}{x_1+x_2}\cdot\frac{y_1}{x_1}+\frac{x_2}{x_1+x_2}\cdot\frac{y_2}{x_2}\right)\\
&\leqslant (x_1+x_2)\left(\frac{x_1}{x_1+x_2}g\left(\frac{y_1}{x_1}\right)+\frac{x_2}{x_1+x_2}g\left(\frac{y_2}{x_2}\right)\right)\\
&= x_1 g\left(\frac{y_1}{x_1}\right)+x_2 g\left(\frac{y_2}{x_2}\right) = f(x_1,y_1)+f(x_2,y_2),
\end{aligned}$$

i.e. $f \in LS_C$. ■

REMARK 1. Since $0 \notin C$, the subset C from the proposition 2 is not a cone, but this is permitted, by our previous convention.

REMARK 2. We can conclude that a system of functions $g_i(t)$, convex for $t \in (a,b)$, produces a system of subadditive functions $f_i(x,y)$ over $C \subset \mathbf{R}^2$ (denoted as $g(t) \rightrightarrows f(x,y)$), so, according to propostition 1, we obtain a solution of (1) in the form

$$f(x,y) = \sum_{i=1}^{n} c_i f_i(x,y), \quad c_i > 0, \quad (x,y) \in C.$$

Conversely to proposition 2, we have the following

PROPOSITION 2′. *If the function $f \in LS_C$, where C is the same subset of $\mathbf{R}^2$ as in the proposition 2 and $f(\alpha x, \alpha y) = \alpha f(x,y)$ for every $\alpha \in \mathbf{R}^+$, then $f(x,y) = x \cdot g(y/x)$, where $g(t)$ is a convex function.*

Proof. The function $g(y) = f(1,y)$ is convex. Indeed, for $p \geqslant 0$, $q \geqslant 0$, $p+q=1$:

$$\begin{aligned}
g(py_1+qy_2) &= f(1,py_1+qy_2) = f(p+q,py_1+qy_2) \leqslant f(p,py_1)+f(q,qy_2)\\
&= pf(1,py_1)+qf(1,y_2) = pg(y_1)+qg(y_2).
\end{aligned}$$

Now, for $\alpha = 1/x$ we obtain

$$\frac{1}{x} \cdot f(x,y) = f\left(\frac{1}{x} \cdot x, \frac{1}{x} \cdot y\right) = f\left(1, \frac{y}{x}\right) = g\left(\frac{y}{x}\right),$$

i.e. $f(x,y) = xg(y/x)$. This prove the proposition. ■

Remark 3. A method of obtaining the function from LS_C is following: If $\sup_{(x,y)}(f(x+a, y+b) - f(x,y)) = g(a,b)$, then $g \in LS_C$, where $f\colon C \to \mathbf{R}$.

Proof. Since

$$\begin{aligned} g(a_1+b_1, a_2+b_2) &= \sup_{(x,y)} (f(x+a_1+b_1, y+a_2+b_2) - f(x,y)) \\ &= \sup_{(x,y)} (f(x+a_1+b_1, y+a_2+b_2) - f(x+b_1, y+b_2) + f(x+b_1, y+b_2) - f(x,y)) \\ &\leqslant \sup_{(x,y)} (f(x+a_1+b_1, y+a_2+b_2) - f(x+b_1, y+b_2)) + \sup_{(x,y)} (f(x+b_1, y+b_2) - f(x,y)) \\ &= g(a_1,a_2) + g(b_1,b_2), \end{aligned}$$

then $g \in LS_C$. ■

Another property of subadditive functions is the following

Proposition 3. *If $f \in LS_C$ then*

$$f\left(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i\right) \leqslant \sum_{i=1}^n f(x_i, y_i) \quad \textit{for } (x_i, y_i) \in C,\ i = 1, 2, \dots, n.$$

Proof. This is easy to prove by induction on n, since from $(x_i, y_i) \in C$, $i = 1, 2, \dots, n$ it follows that

$$\begin{aligned} \left(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i\right) = \sum_{i=1}^n (x_i, y_i) &\in \underbrace{C + C + \cdots + C}_{n} \\ &\subset \underbrace{C + C + \cdots + C}_{n-1} \subset \cdots \subset C + C \subset C. \quad ■ \end{aligned}$$

Propositions 2 and 3 are the source for obtaining several kinds of two-parameter inequalities. We illustrate this with some examples.

Example 1. Since $\ln t \rightrightarrows -x\ln(y/x)$, $x, y > 0$, using proposition 3 and putting $x_i = b_i$, $y_i = a_i b_i$, $i = 1, 2, \dots, n$, we obtain generalized arithmetic-geometric inequality

$$\prod_{i=1}^n a_i^{b_i} \leqslant \left(\sum_{i=1}^n a_i b_i \Big/ \sum_{i=1}^n b_i\right)^{\sum_{i=1}^n b_i}, \quad a_i, b_i > 0,$$

i.e. putting $b_i \Big/ \sum_{i=1}^n b_i = p_i$, $i = 1, 2, \dots, n$:

$$\prod_{i=1}^n a_i^{p_i} \leqslant \sum_{i=1}^n a_i p_i, \quad p_i, a_i > 0,\ \sum_{i=1}^n p_i = 1.$$

EXAMPLE 2. Since $t^r \rightrightarrows \begin{cases} -x(y/x)^r, & \text{for } r \in (0,1), \\ x(y/x)^r, & \text{for } r \in \mathbf{R} \setminus [0,1], \end{cases}$ $x,\ y > 0$, putting $x_i = b_i^q$, $y_i = a_i^p$ and $r = 1/p$, $1 - r = 1/q$ in proposition 3, we obtain Hölder's inequality

$$\sum_{i=1}^n a_i b_i \leqslant \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1;\ p, q > 1, \text{ and}$$
$$\sum_{i=1}^n a_i b_i \geqslant \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1;\ p < 1, \text{ or } q < 1.$$

EXAMPLE 3. Since $\ln \sin t \rightrightarrows -x \ln \sin y/x$, using proposition 3 with $x_i = 1$, $i = 1, 2, \ldots, n$ we have

$$\prod_{i=1}^n \sin y_i \leqslant \sin^n \left(\frac{1}{n} \sum_{i=1}^n y_i\right), \quad y_i \in (0, \pi).$$

For the n-dimensional case of locally subadditive functions we give the following definition: A function $f \in LS_C$ if

$$f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \leqslant f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n) \tag{4}$$

for each $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in C$, where C is a cone in $\mathbf{R}^n$.

Now we have the following

PROPOSITION 4. *A function $g(t)$, convex for $t \in (a, b)$, produces a locally subadditive function $f(\,\cdot\,)$ on $C \subset \mathbf{R}^n$ by*

$$f(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^n A_i x_i\right) g\left(\sum_{i=1}^n B_i x_i \Big/ \sum_{i=1}^n A_i x_i\right),$$

where

$$C = \left\{ (x_1, x_2, \ldots, x_n) \;\middle|\; \sum_{i=1}^n A_i x_i > 0, a < \sum_{i=1}^n B_i x_i \Big/ \sum_{i=1}^n A_i x_i < b \right\},$$

and A_i, B_i are arbitrary constants, not all equal to zero.

Proof is similar to that of proposition 2. Since $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in C$ imply that $(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in C$, putting

$$p = \frac{\sum_{i=1}^n A_i x_i}{\sum_{i=1}^n A_i (x_i + y_i)}, \quad q = \frac{\sum_{i=1}^n A_i y_i}{\sum_{i=1}^n A_i (x_i + y_i)}, \quad r = \frac{\sum_{i=1}^n B_i x_i}{\sum_{i=1}^n A_i x_i}, \quad s = \frac{\sum_{i=1}^n B_i y_i}{\sum_{i=1}^n A_i y_i}$$

in (3), we obtain (4), i.e. that $f \in LS_C$. ■

It is obvious that propositions 1 and 3 could be easily transformed to $\mathbf{R}^n$.

Authors are indebted to Prof. D. Adamović for help in preparing this article.

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(received 07.09.1994.)

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