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# ON A SUBFAMILY OF p-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

**S. R. Kulkarni, M. K. Aouf, S. B. Joshi**

**Abstract.** Sharp results concerning coefficients, distortion theorem and the radius of convexity for the class $P_p^*(\alpha,\beta,\xi)$ are determined. Furthermore it is shown that the class $P_p^*(\alpha,\beta,\xi)$ is closed under convex linear combinations. The extreme points of $P_p^*(\alpha,\beta,\xi)$ are also determined

## 1. Introduction

Let $S_p$ ($p$ a fixed integer greater than 0) denote the class of functions of the form $f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ that are holomorphic and $p$-valent in the unit disc $|z|<1$. Also let $T_p$ denote the subclass of $S_p$ consisting of functions that can be expressed in the form $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$. A function $f\in T_p$ is in $P_p^*(\alpha,\beta,\xi)$ if and only if

$$\left|\frac{f'(z)z^{1-p}-p}{2\xi(f'(z)z^{1-p}-\alpha)-(f'(z)z^{1-p}-p)}\right|<\beta,$$

$|z|<1$, for $0\leqslant\alpha<p/2\xi$, $0<\beta\leqslant 1$, $1/2<\xi\leqslant 1$.

Such type of study was carried out by Aouf [1] for $P_p^*(\alpha,\beta)$. We note that $P_1^*(\alpha)=P_1^*(0,\alpha,1)$ is precisely the class of functions in $E$ studied by Caplinger [2]. The class $P_1^*(\alpha,1,\beta)=P_1^*(\alpha,\beta)$ is the class of holomorphic functions discussed by Juneja-Mogra [4]. Gupta-Jain [3] studied the family of holomorphic univalent functions that have the form $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ and satisfy the condition

$$\left|\frac{f'(z)-1}{f'(z)+(1-2\alpha)}\right|<\beta,\quad 0\leqslant\alpha<1,\quad 0<\beta\leqslant 1.$$

Kulkarni [5] has studied above mentioned properties for the functions having Taylor series expansion of the type $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$

In this paper sharp results concerning coefficients, distortion theorem and the radius of convexity for the class $P_p^*(\alpha,\beta,\xi)$ are determined. Finally we prove that

*AMS Subject Classification (1980)*: Primary 30C45 the class $P_p^*(\alpha,\beta,\xi)$ is closed under the arithmetic mean and convex linear combinations.

## 2. Coefficient Theorem

THEOREM 1. *A function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *is in* $P_p^*(\alpha,\beta,\xi)$ *if and only if*

$$\sum_{n=1}^{\infty}(p+n)[1+\beta(2\xi-1)]|a_{p+n}|\leqslant 2\beta\xi(p-\alpha).$$

*The result is sharp, the extremal function being*

$$f(z)=z^p-\frac{2\beta\xi(p-\alpha)}{(p+n)(2\xi-1)\beta+1}z^{p+n}. \tag{1}$$

*Proof.* Let $|z|=1$. Then

$$\begin{aligned}&|f'(z)z^{1-p}-p|-\beta|2\xi(f'(z)z^{1-p}-\alpha)-(f'(z)z^{1-p}-p)|\\ &=\left|-\sum_{n=1}^{\infty}(p+n)|a_{p+n}|z^n\right|-\beta\left|2\xi(p-\alpha)-(2\xi-1)\sum_{n=1}^{\infty}(p+n)|a_{p+n}|z^n\right|\\ &\qquad\leqslant\sum_{n=1}^{\infty}(p+n)[1+(2\xi-1)\beta]|a_{p+n}|-2\beta\xi(p-\alpha)\leqslant 0\end{aligned}$$

by hypothesis. Hence, by the maximum modulus theorem $f\in P_p^*(\alpha,\beta,\xi)$.

For the converse we assume that

$$\begin{aligned}&\left|\frac{f'(z)z^{1-p}-p}{2\xi(f'(z)z^{1-p}-\alpha)-(f'(z)z^{1-p}-p)}\right|\\ &\qquad=\left|\frac{-\sum_{n=1}^{\infty}(p+n)|a_{p+n}|z^n}{2\xi(p-\alpha)-\sum_{n=1}^{\infty}(p+n)|a_{p+n}|(2\xi-1)z^n}\right|<\beta.\end{aligned}$$

Since $|\Re(z)|\leqslant|z|$ for all $z$ we have

$$\Re\left[\frac{\sum_{n=1}^{\infty}(p+n)|a_{p+n}|z^n}{2\xi(p-\alpha)-(2\xi-1)\sum_{n=1}^{\infty}(p+n)|a_{p+n}|z^n}\right]<\beta.$$

We select the values of $z$ on the real axis so that $f'(z)z^{1-p}$ is real. Simplifying the denominator in the above expression and letting $z\to 1$ through real values, we obtain

$$\sum_{n=1}^{\infty}(p+n)|a_{p+n}|\leqslant 2\beta\xi(p-\alpha)-(2\xi-1)\beta\sum_{n=1}^{\infty}(p+n)|a_{p+n}|,$$

and it results in the required condition.

The result is sharp for the function (1). ■

## 3. Distortion Theorem

THEOREM 2. *If $f \in P_p^*(\alpha, \beta\xi)$, then for $|z| = 1$,*

$$r^p - \frac{2\beta\xi(p-\alpha)}{(p+1)[1+\beta(2\xi-1)]} r^{p+1} \leqslant |f(z)| \leqslant r^p + \frac{2\beta\xi(p-\alpha)}{(p+1)[1+\beta(2\xi-1)]} r^{p+1}, \quad (2)$$

*and*

$$pr^{p-1} - \frac{2\beta\xi(p-\alpha)}{1+\beta(2\xi-1)} r^p \leqslant |f'(z)| \leqslant pr^{p-1} + \frac{2\beta\xi(p-\alpha)}{1+\beta(2\xi-1)} r^p, \quad (3)$$

*Proof.* In view of theorem 1 we have

$$\sum_{n=1}^{\infty} |a_{p+n}| \leqslant \frac{2\beta\xi(p-\alpha)}{(p+1)[1+\beta(2\xi-1)]}.$$

Hence

$$|f(z)| \leqslant r^p + \sum_{n=1}^{\infty} |a_{p+n}| r^{p+n} \leqslant r^p + \frac{2\beta\xi(p-\alpha)}{(p+1)[1+\beta(2\xi-1)]} r^{p+1}$$

and

$$|f(z)| \geqslant r^p - \sum_{n=1}^{\infty} |a_{p+n}| r^{p+n} \geqslant r^p - \frac{2\beta\xi(p-\alpha)}{(p+1)[1+\beta(2\xi-1)]} r^{p+1}.$$

In the same way we have

$$|f'(z)| \leqslant pr^{p-1} + \sum_{n=1}^{\infty} (p+n)|a_{p+n}| r^{p+n-1} \leqslant pr^{p-1} + \frac{2\beta\xi(p-\alpha)}{1+\beta(2\xi-1)} r^p$$

and

$$|f'(z)| \geqslant pr^{p-1} - \sum_{n=1}^{\infty} (p+n)|a_{p+n}| r^{p+n-1} \geqslant pr^{p-1} - \frac{2\beta\xi(p-\alpha)}{1+\beta(2\xi-1)} r^p.$$

This completes the proof of the theorem. ■

The above bounds are sharp. Equalities are attended for the following function

$$f(z) = z^p - \frac{2\beta\xi(p-\alpha)}{(p+1)(2\xi-1)\beta+1} z^{p+1}, \quad z = \pm r. \quad (4)$$

THEOREM 3. *Let $f \in P - p^*(\alpha, \beta, \xi)$. Then the disc $|z| < 1$ is mapped onto a domain that contains the disc*

$$|w| < \frac{(p+1) + \beta[(2\xi-1) - p + 2\xi\alpha]}{(p+1)[1+\beta(2\xi-1)]}.$$

*The result is sharp with extremal function (4).*

*Proof.* The result follows upon letting $r \to 1$ in (2). ■

THEOREM 4. *$f \in P_p^*(\alpha,\beta,\xi)$, then $f$ is convex in the disc $|z| < r = r(p,\alpha,\beta,\xi)$, where*

$$r(p,\alpha,\beta,\xi) = \inf_{n\in\mathbf{N}} \left\{ \frac{p^2[1+\beta(2\xi-1)]}{(p+n)2\beta\xi(p-\alpha)} \right\}^{1/n}.$$

*The result is sharp, the extremal function being of the form (1).*

*Proof.* It is enough to show that

$$\left| \left(1+\frac{zf''(z)}{f'(z)}\right) - p \right| \leqslant p \quad \text{for } |z|<1.$$

First we note that

$$\left| \left(1+\frac{zf''(z)}{f'(z)}\right) - p \right| = \left| \frac{zf''(z)+(1-p)f'(z)}{f'(z)} \right| \leqslant \frac{\sum_{n=1}^{\infty} n(p+n)|a_{p+n}||z|^n}{p-\sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^n}.$$

Thus, the result follows if

$$\sum_{n=1}^{\infty} n(p+n)|a_{p+n}||z|^n \leqslant p\left\{ p - \sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^n \right\},$$

or, equivalently, $\sum_{n=1}^{\infty}(\frac{p+n}{p})^2|a_{p+n}||z|^n \leqslant 1$.

But, in view of Theorem 1, we have

$$\sum_{n=1}^{\infty}(p+n)[1+\beta(2\xi-1)]|a_{p+n}| \leqslant 2\beta\xi(p-\alpha).$$

Thus $f$ is convex if

$$\left(\frac{p+n}{p}\right)^2 |z|^n \leqslant \frac{(p+n)[1+\beta(2\xi-1)]}{2\beta\xi(p-\alpha)}, \quad n=1,2,3,\ldots,$$

i.e.

$$|z| \leqslant \left\{ \frac{p^2[1+\beta(2\xi-1)]}{(p+n)2\beta\xi(p-\alpha)} \right\}^{1/n}, \quad n=1,2,3,\ldots,$$

which completes the proof. ■

## 4. Closure Theorem

Next two results respectively show that the family $P_p^*(\alpha,\beta,\xi)$ is closed under taking "arithmetic mean" and "convex linear combinations".

THEOREM 5. *If $f(z) = z^p - \sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ and $g(z) = z^p - \sum_{n=1}^{\infty}|b_{p+n}|z^{p+n}$ are in $P_p^*(\alpha,\beta,\xi)$, then $h(z) = z^p - \frac{1}{2}\sum_{n=1}^{\infty}|a_{p+n}+b_{p+n}|z^{p+n}$ is also in $P_p^*(\alpha,\beta,\xi)$.*

*Proof.* $f$ and $g$ both being members of $P_p^*(\alpha,\beta,\xi)$, we have in accordance with Theorem 1

$$\sum_{n=1}^{\infty}(p+n)[1+\beta(2\xi-1)]|a_{p+n}| \leqslant 2\beta\xi(p-\alpha) \tag{5}$$

$$\text{and} \quad \sum_{n=1}^{\infty}(p+n)[1+\beta(2\xi-1)]|b_{p+n}| \leqslant 2\beta\xi(p-\alpha). \tag{6}$$

To show that $h$ is a member of $P_p^*(\alpha,\beta,\xi)$ it is enough to show that

$$\frac{1}{2}\sum_{n=1}^{\infty}(p+n)[1+\beta(2\xi-1)]|a_{p+n}+b_{p+n}| \leqslant 2\beta\xi(p-\alpha).$$

This is exactly an immediate consequence of (5) and (6). ■

THEOREM 6. *Let* $f_p(z) = z^p$ *and*

$$f_{p+n}(z) = z^p - \frac{2\beta\xi(p-\alpha)}{(p+n)(2\xi-1)\beta+1} z^{p+n}, \quad n = 1,2,3,\dots .$$

*Then* $f \in P_p^*(\alpha,\beta,\xi)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=0}^{\infty}\lambda_{p+n}f_{p+n}(z)$, *where* $\lambda_{p+n} \geqslant 0$ *and* $\sum_{n=0}^{\infty}\lambda_{p+n} = 1$.

*Proof.* The proof of this theorem follows along the same lines as the proof of Theorem 3.3 in Kulkarni [5]. The details are omiited. ■

COROLLARY. *The extreme points of* $P_p^*(\alpha,\beta,\xi)$ *are the functions* $f_p(z) = z^p$ *and* $f_{p+n}(z) = z^p - \dfrac{2\beta\xi(p-\alpha)}{(p+n)(2\xi-1)\beta+1} z^{p+n}$, $n = 1,\ 2,\ \dots$ .

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Dept. of Mathematics, Willingdon College, Sangli – 416 415, India

Dept. of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt

Dept. of Mathematics, Walchand College of Eng., Sangli – 416 415, India