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TWO ESTIMATES FOR INTEGRAL MEANS OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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Abstract. In this note two estimates for integral means of analytic functions in the unit disc are obtained. It is also shown that both estimates are sharp.

If f is an analytic function in the unit disc $D = \{\, z \,:\, |z| < 1 \,\}$, as usual, integral mean of order p, $0 < p < \infty$, of function f, is defined by

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < r < 1.$$

To prove our main result we need two Littlewood-Paley's theorems.

Theorem 1. [2, p. 315] *Let $0 < p \leqslant 2$. Then there exists a constant $C > 0$ such that*

$$M_p^p(r,f) \leqslant C \int_0^1 (1-\rho)^{p-1} M_p^p(r\rho, f') \, d\rho, \tag{1}$$

for every f analytic in D.

Theorem 2. [2, p. 316] *Let $2 \leqslant p < \infty$. There exists a constant $C > 0$ such that*

$$M_p^2(r,f) \leqslant C \int_0^1 (1-\rho) M_p^2(r\rho, f') \, d\rho, \tag{2}$$

for every f analytic in D.

The following Lemma will also be needed.

Lemma 1. [2] *If $0 < p < \infty$, then there exists a constant $C > 0$ such that*

$$\int_0^1 (1-\rho)^{p-1} (1-r\rho)^{-p} \, d\rho \leqslant C \log \frac{1}{1-r}, \quad 0 < r < 1.$$

THEOREM. *Let* $M_p(r,f') = O((1-r)^{-1})$. *Then*

a) if $0 < p \leqslant 2$, *then* $M_p^p(r,f) = O(\log(1-r)^{-1})$;

b) if $2 \leqslant p < \infty$, *then* $M_p^2(r,f) = O(\log(1-r)^{-1})$.

Both estimates are sharp.

Note that from the same assumption $M_p(r,f') = O((1-r)^{-1})$ two different conclusions follows:

$$M_p(r,f) = O\left(\log^{1/p}\frac{1}{1-r}\right), \quad \text{if } 0 < p \leqslant 2,$$
$$M_p(r,f) = O\left(\log^{1/2}\frac{1}{1-r}\right), \quad \text{if } 2 \leqslant p < \infty.$$

Proof. a) Using (1) we find that

$$M_p^p(r,f) \leqslant C\int_0^1 (1-\rho)^{p-1}(1-r\rho)^{-p}\,d\rho.$$

Now a) follows by Lemma 1.

We use C to denote a positive constant not necessarily the same on any two occurrences.

b) By assumption,

$$M_p(r\rho,f') \leqslant \frac{C}{1-r\rho}. \tag{3}$$

From (2) and (3) we conclude that

$$M_p^2(r,f) \leqslant C\int_0^1 (1-\rho)(1-r\rho)^{-2}\,d\rho.$$

Using Lemma 1 ($p=2$) and (3) we find that $M_p^2(r,f) \leqslant C\log((1-r)^{-1})$.

We show that both estimates are sharp in the sense that there exists f, analytic in D, such that $M_p(r,f') \leqslant C(1-r)^{-1}$, $0 < p \leqslant 2$, and

$$C^{-1}\log\frac{1}{1-r} \leqslant M_p^p(r,f) \leqslant C\log\frac{1}{1-r}, \tag{4}$$

and that there exists g, analytic in D, such that $M_p(r,g') \leqslant C(1-r)^{-1}$, $2 \leqslant p < \infty$, and

$$C^{-1}\log\frac{1}{1-r} \leqslant M_p^2(r,f) \leqslant C\log\frac{1}{1-r}. \tag{5}$$

We show that the functions f and g defined by $f(z) = (1-z)^{-1/p}$ and $g(z) = \sum_{n=0}^{\infty} z^{2^n}$ satisfy conditions mentioned above. Using Lemma [1, p. 65] we find that

$$M_p(r,f') \leqslant C\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|1-re^{it}|^{p+1}}\right)^{1/p} \leqslant \frac{C}{1-r}.$$

On the other hand, $M_p^p(r,f) = (2\pi)^{-1}\int_0^{2\pi} dt/|1-re^{it}|$. It is easily seen that there exists a constant $C>0$ so that

$$C^{-1}\log\frac{1}{1-r} \leqslant \int_0^{2\pi}\frac{dt}{|1-re^{it}|} \leqslant C\log\frac{1}{1-r}.$$

Hence, we obtain (4).

Using inequality $M_p(r,g') \leqslant M_\infty(r,g')$, where $M_\infty(r,g') = \sup\limits_{0\leqslant t\leqslant 2\pi}|g'(re^{it})|$, we find that

$$M_p(r,g') \leqslant \sum_{n=0}^{\infty} 2^n r^{2^n-1} \leqslant C\frac{1}{1-r}$$

(for the last inequality see [1, p. 66]). In [2, p. 316] it is proved that (5) is valid.

This completes the proof of the Theorem. ■

REFERENCES

[**1**] P. Duran, *Theory of H^p spaces*, Academic Press, New York 1970.

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