МАТЕМАТИЧКИ ВЕСНИК
46 (1994), 51–56

Dedicated to the memory
of Professor Đ. Kurepa

REMARK ON SOME ABSTRACT DISTANCE SPACES

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Abstract. Some terminological questions are discussed in connection with two essentially different classes of abstract distance spaces.

1. When one is dealing with pseudo-distance spaces (espaces pseudo-distanciés) introduced by G. Kurepa in [1934] and pseudo-metric spaces considered, for example, by L. Collatz in [1964], one may be led to some misunderstanding, so that a clarification is necessary. That is what I shall try to do here. In his paper [1934] G. Kurepa defined pseudo-distance spaces literally thus (page 1564):

> "Un ensemble quelconque E est dit *espaces pseudo-distancié* ou espace de la classe (Δ) si l'ensemble de couples de points de E peut être considéré comme un ensemble ordonné $\mathfrak{M} = \{(x,y)\}$, x, $y \in E$ tel que: 1° (x,x) est le premier élément ζ de $\mathfrak{M}$ pour tout $x \in E$, et *vice versa*, si $(x,y) = \zeta$, alors $x \equiv y$; 2° $(x,y) = (y,x)$ pour chaque x et y de E; 3° il existe une fonction univoque $\eta = \varphi(\xi)$, ξ, $\eta \in \mathfrak{M}$ telle que $\varphi(\xi) \to \zeta$ si $\xi \to \zeta$; si $(x,y) < \varepsilon$, $(y,z) < \varepsilon$ alors $(x,z) \leqslant \varphi(\varepsilon)$ pour chaque x, y, z de E; 4° pour qu'un point a de E soit point d'accumulation d'un ensemble $F \subset E$, il faut et il suffit qu'il existe un ensemble $G \subset F$ tel que $(a,y) \to \zeta$ si y parcourt G[1].
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> Si $\varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_\nu > \cdots \to \zeta$, $\nu < \omega_\alpha$, ω_α étant réguliér, des signes ε_ν jouent le rôle des nombres arbitrairement petits et les nombres ω_α, $\aleph_\alpha$ le rôle des nombres ω_0, $\aleph_0$ dans le cas des espaces distanciés. On a ainsi des classes (Δ^0), (Δ'), ... , (Δ^α), ... , ω_α étant régulier, qui n'ont en commun que des certains espaces pseudo-distanciés composés des points isolés. En particulier, la classe (Δ^0) continet la classe $(\mathfrak{D})$ de M. Fréchet[2].
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> D'une manière générale, en partant d'une certaine classe d'espaces abstraits et en leur attribuant un rôle actif de paramétrisation, on arrive à de nouvelles classes d'espaces abstraits. Dans le cas des espaces distanciés

[1] Évidemment, plusiers couples de points de E peuvent coincider dans $\mathfrak{M}$.

[2] Rendioconti Palermo, 22, 1906, p. 18; Espaces abstraits, Paris, 1928, p. 61 et 218. Si (Δ^0)–$(\mathfrak{D}) \neq 0$, la classe (Δ^0) fournirait une réponse affirmative à deux questions de M. Fréchet (Espaces abstraits, p. 190 et 193)

et pseudo-distanciés, le rôle des espaces actifs est joué respectivement par des ensembles de nombres réels non négatifs et par des ensembles ordonnés limités du côté gauche."

In his paper [1946] M. Fréchet rediscovered these spaces using the same ideas of G. Kurepa and called them "espaces écartisés". That is why today pseudo-distance spaces are called also Kurepa-Fréchet spaces.

2. Here I need a more maniable definition of these spaces and so I shall deduce the equivalent definition from a general setting as developed by a procedure in my book "Introduction to general topology" [1963] or [1989].

Let M be a set partially ordered by order relation $<$ with a first element ζ and let f be a single-valued function mapping the direct product $E \times E$ of a given set E into M. We introduce the following conditions (od = first letters of "ordered distance"):

(od_1) The set $s = M \setminus \zeta$ is not empty.

(od_2) For any two elements λ, $\mu \in s$ there exists an element $\nu \in s$ such that $\nu \leqslant \lambda$ and $\nu \leqslant \mu$.

(od_3) For any a, $b \in E$ from $a = b$ it follows that $f(a,b) = \zeta$, that is $f(a,a) = \zeta$ for each $a \in E$.

(od_4) If $f(a,b) = \zeta$ then $a = b$, for any a, $b \in E$.

(od_5) For every $\lambda \in s$ there is a $\mu \in s$ such that from $f(a,b) < \mu$ it follows $f(b,a) < \lambda$, for any a, $b \in E$.

(od_6) For every $\lambda \in s$ there exist two elements μ, ν in s such that from $f(a,b) < \mu$ and $f(b,c) < \nu$ it follows that $f(a,c) < \lambda$ for any a, b, $c \in E$.

It is easy to see that the conjunction of the conditions (od_2) and (od_6) implies the following one:

(od_6') For every $\lambda \in s$ there is a $\mu \in s$ such that from $f(a,b) < \mu$ and $f(b,c) < \mu$ it follows that $f(a,c) < \lambda$ for any a, b, $c \in E$.

(od_7) The element ζ is the first element of M but the set $s = M \setminus \zeta$ has no first element.

(od_8) The function f is symmetrical, that is $f(b,a) = f(a,b)$ for any a, $b \in E$.

(od_9) For every $\lambda \in s$ there is $\mu \in s$ such that for any three elements a, b, c in E from $f(a,b)$ non $\geqslant \mu$ and $f(b,c)$ non $\geqslant \mu$ it follows that $f(a,c)$ non $\geqslant \lambda$. Here x non $\geqslant y$ means that either x is not comparable with y or, if comparable, then one must have $x < y$, for any x, $y \in s$.

(od_{10}) The order relation $<$ is total ordering of M.

The function f is called an abstract-distance function and $f(a,b) \in M$ is called an abstract-distance between the points a, b in E. The sets M or s are sometimes called the scales on which the abstract-distance function takes its values.

The abstract-distance function f we call the pseudo-distance or Kurepa-Fréchet distance if the following conditions are satisfied: (od_{10}), (od_8), (od_7), (od_6'), (od_4), (od_3), (od_2) and (od_1).

The abstract-distance function f we call A. Appert's partially ordered distance if the first six conditions (od_1)–(od_6) are fulfilled.

The abstract-distance function f we call J. Colmez' partially ordered distance of first type if the following conditions are satisfied: (od_1), (od_2), (od_3), (od_4), (od_7), (od_8) and (od_9).

The abstract-distance function f we call J. Colmez' partially ordered distance of second type if the following conditions are fulfilled: (od_1), (od_2), (od_3), (od_4), (od_6'), (od_7) and (od_8).

We notice that in every of these definitions the conditions (od_1) and (od_3) are supposed. Under these two conditions at least and using the abstract distance f, we can now define a neighbourhood space on the set E and thus introduce a topological structure on E.

Except for Colmez' partially ordered distance of first type, in all other cases we define firstly a family of subsets on M in the following manner:

$$X_\xi = \{\, \lambda \,:\, \lambda \in M \text{ and } \lambda < \xi \,\}, \quad \xi \in s = M \setminus \zeta. \tag{2.1}$$

By means of these sets and using the abstract-distance function f, we define the neighbourhood bases on E thus:

$$W_\xi(a) = \{\, b \,:\, b \in E \text{ and } f(a,b) \in X_\xi \,\}, \quad \xi \in s, a \in E. \tag{2.2}$$

Let $\mathcal{U}$ be the family of all parts $V \subset E \times E$ which contain the set $f^{-1}(X_\xi)$ for some $\xi \in s$. Using the family $\mathcal{U}$ we can also define the neighbourhood base

$$V(a) = \{\, b \,:\, b \in E \text{ and } (a,b) \in V \,\}, \quad V \in \mathcal{U}, a \in E. \tag{2.3}$$

Evidently, the bases (2.2) and (2.3) are equivalent.

In the case of Colmez' partially ordered distance of first type, instead of the order relation $<$ one has to put non $\geqslant$ (in (2.1) and later on).

In this manner topological and corresponding uniform structures on E have to be introduced. We shortly say that the set E is topologized by the corresponding order distance f. Now the following theorems are valid:

Theorem 2.1 (A.Appert). *Topologizing the set E by the partially ordered distance of A. Appert one gets the uniform spaces of A. Weil.*

Theorem 2.2 (J.Colmez). *The class of spaces obtained by topologizing the set E with J. Colmez' partially ordered distance of first kind is topologically equivalent to the class of A. Weil's uniform spaces.*

Theorem 2.3 (J.Colmez). *The class of spaces obtained by topologizing the set E with J. Colmez' partially ordered distance of second kind is topologically equivalent to the class of A. Weil's uniform spaces which satisfy the following condition:*

$(J_\mathcal{U})$ *Let $\mathcal{B}_\mathcal{U}$ be a base of the uniformity $\mathcal{U}$. The intersection of all those elements $V \in \mathcal{B}_\mathcal{U}$ which contain a given ordered pair $(a,b) \in E \times E$, $a \neq b$, is also an element of the uniformity $\mathcal{U}$.*

THEOREM 2.4 *The class of spaces obtained by topologizing the set E with the pseudo-distance or Kurepa-Fréchet distance, that is the class of pseudo-distantial or Kurepa-Fréchet spaces, is topologically equivalent to the A. Weil's uniform spaces the uniformity of which is totally ordered by inclusion $\subset$.*

Recall that two classes of spaces are topologically equivalent if any space of one class is homeomorphic to some space of the other class and conversely. All theorems are proved in my book [1963] or [1989]. Moreover, necessary reference is also given there.

Here we are particularly interested in some properties of pseudo-distantial spaces. Firstly, it is easy to state that pseudo-distantial spaces verify the J. Colmez' property $(J_{\mathcal{U}})$ because their uniformity is totally ordered by inclusion $\subset$. To the other hand, P. Papić has thouroughly studied the pseudo-distantial spaces (for example in [1953] and [1954]). He developed a theory of so-called R-spaces which possess a ramified neighbourhood base, in which he proved that every R-space is totally disconnected. Also, he proved that all not metrizable pseudo-distantial spaces are R-spaces. Therefore, not metrizable pseudo-distantial spaces are totally disconnected. Therefore we deduce the following

THEOREM 2.5. *Not metrizable pseudo-distantial or Kurepa-Fréchet spaces do not possess the fixed point property.*

In fact, any permutation of such a space has no fixed point.

3. To the other hand, in his paper [1956] J. Schröder has introduced the "metric spaces relative to ... " in the following manner:

> "Es sei $\mathfrak{N}$ ein *halbgeordnetes lineares System* von Elementen $\rho, \sigma, \dots$, d.h. eine Menge, in der $\rho + \sigma$ und $\alpha \cdot \rho$ ($\alpha, \beta, \dots$ reelle Zahlen) mit den gewohnten Rechenregeln erklärt und gewisse Elemente ρ als positiv ($\geqslant o =$ Nullelement von $\mathfrak{N}$) definiert sind, wobei $\rho \geqslant o$, $-\rho \geqslant o$ (gleichzeitig) nur für $\rho = o$ gilt und $\rho + \sigma \geqslant o$ aus $\rho, \sigma \geqslant o$, sowie $\alpha\rho \geqslant o$ aus $\alpha \geqslant o$, $\rho \geqslant o$ folgt. $\rho \geqslant \sigma$ oder $\sigma \leqslant \rho$ bedeutet $\rho - \sigma \geqslant o$
>
> Wir nennen $\mathfrak{N}$ einen *halbgeordneten linearen Raum*, wenn gewissen (konvergent genannten) Folgen $\rho_n \in \mathfrak{N}$ ($n = 1, 2, 3, \dots$) ein (eindeutig bestimmtes) Grenzelement $\lim \rho_n \in \mathfrak{N}$ zugeordnet ist und dabei folgende Bedingungen erfüllt sind[2]:
>
> a) aus $\rho_n = \rho$ für $n = 1, 2, 3, \dots$ mit einem festen Element ρ folgt $\lim \rho_n = \rho$,
>
> b) aus $\lim \rho_n = \rho$ folgt $\lim \rho_{k_n} = \rho$ für jede monotone Folge natürlicher Zahlen k_n ($n = 1, 2, 3, \dots$),
>
> c) aus $\lim \rho_n = \rho$ und $\lim \sigma_n = \sigma$ folgt $\lim(\rho_n + \sigma_n) = \rho + \sigma$,
>
> d) aus $\lim \alpha_n = \alpha$ und $\lim \rho_n = \rho$ folgt $\lim \alpha_n \rho_n = \alpha\rho$,
>
> e) aus $o \leqslant \rho_n \leqslant \sigma_n$ für $n = 1, 2, 3, \dots$ und $\lim \sigma_n = o$ folgt $\lim \rho_n = o$,
>
> f) aus $\rho_n \geqslant 0$ für $n = 1, 2, 3, \dots$ und $\lim \rho_n = \rho$ folgt $\rho \geqslant o$."

[2] The reference to the paper [1906] of Fréchet where the conditions a, b are used

"Eine Menge $\mathfrak{R}$ von Elementen u, v, ... heiße ein *bezüglich* $\mathfrak{N}$ *metrischer Raum*, falls jedem Paar von Elementen u, $v \in \mathfrak{R}$ ein Element $\rho(u,v) \in \mathfrak{N}$ als deren Abstand zugeordnet ist, derart, daß gilt:

a) $\rho(u,v) = o$ genau dann, wenn $u = v$,

b) $\rho(u,v) \leqslant \rho(u,w) + \rho(v,w)$ für jedes Elementetripel u, v, w.

Es folgt $\rho(u,v) = \rho(v,u)$ und $\rho(u,v) \geqslant o$. Konvergenz und (eindeutiges) Grenzelement einer Folge u_n ($n = 1, 2, 3, \dots$) werden in $\mathfrak{R}$ dadurch definiert, daß $u = \lim u_n$ mit $\lim \rho(u, u_n) = o$ äquivalent ist."

L. Colatz in [1964] called these spaces "pseudo-metric" spaces (pseudometrische Räume) in honor of G. Kurepa. Nevertheless, from J. Schröder's definition above one may immediately conclude that the scale over which the "pseudo-metric" takes its values is much richer in mathematical structures than it is in the case of pseudo-distance. Moreover, instead of topology, as a primitive term the notion of convergence is introduced in the linear system $\mathfrak{N}$ as well as in the support $\mathfrak{R}$ of "pseudo-metric" space.

Unfortunately, under the same name of pseudo-metric spaces the usual metric spaces but without the condition that $d(x,y) = 0$ implies $x = y$ are defined and applied (for an extensive use of these spaces see, for example, Kelley in [1955]). In my book [1963] I also called these spaces pseudo-metric spaces. Moreover, quite recently, in their paper [1993] the authors P. A. Künzi, M. Mršević, I. L. Reilly and M. K. Vamanamurthy discussed several topological properties of quasi-pseudo-metric spaces.

From the partial ordering of the linear system $\mathfrak{N}$ over which the "pseudo-metric" space $\mathfrak{R}$ is defined, one may easily see that in the list of (od)-conditions in the part 2. of this paper, the conditions (od_1), (od_3) and (od_8) are surely fulfilled. So by using the neighbourhood bases (2.2) or (2.3) to introduce some "topology" in $\mathfrak{R}$, not much is obtained, because by this procedure only A. Appert's hypouniform spaces can be produced (see my book [1963] or [1989], page 125).

Again, let us try to topologize the support $\mathfrak{R}$ by the closure operator τ defined as follows. Let τ be the map of the power set $P(\mathfrak{R})$ into $P(\mathfrak{R})$ defined thus:

$$\tau(A) = A \cup \{\, a \;:\; a \in \mathfrak{R} \text{ and } a = \lim a_n \text{ of a sequence } a_n \in A,\ n = 1, 2, \dots \,\},$$

$A \neq \emptyset$, $A \in P(\mathfrak{R})$, where $\emptyset$ is the empty set. If $A = \emptyset$, put $\tau(\emptyset) = \emptyset$. Evidently, $A \subset \tau(A)$ and it is easy to prove that $\tau(A \cup B) = \tau(A) \cup \tau(B)$. But, unless the metric space $\mathfrak{R}$ relative to $\mathfrak{N}$ degenerates into an usual metric space (that is, when $\mathfrak{N}$ = system of real numbers), generally $\tau(\tau(A)) = \tau(A)$ does not hold. Such occasions do occur in case of composed mathematical structures. So the problem of what properties the closure operator τ possesses is still open. In particular, in the sense to find a natural number k such that $\tau^{k+1}(A) = \tau^k(A)$ for $A \in P(\mathfrak{R})$.

J. Schröder introduced metric spaces relative to $\mathfrak{N}$ because of their usefulness in numerical analysis. In the Colmez' book [1964], for instance, several interesting examples have been discussed. Of the important properties of J. Schröder's metric spaces I shall here mention only one.

A sequence $a_n \in \mathfrak{R}$, $n = 1, 2, \ldots$ is called a Cauchy-sequence if $\rho(a_m, a_n)$ converges to the nullelement o of $\mathfrak{N}$, when $m \to \infty$ and $n \to \infty$. A convergent sequence is a Cauchy-sequence, but conversely may not be true. A metric space $\mathfrak{R}$ over $\mathfrak{N}$ is called Cauchy-complete, or simply complete, if every Cauchy-sequence is convergent. A similar definition is valid in the case of a subspace $\mathfrak{D}$ of $\mathfrak{R}$. Now on $\mathfrak{R}$ as well as on $\mathfrak{N}$ some mappings, called operators, T resp. S are defined, and, under a little bit involved conditions, on a Cauchy-complete $\mathfrak{D}$ there is an iteration procedure $a_{n+1} = Ta_n$ leading to the equation $Ta = a$, $a \in \mathfrak{D}$, that is, the operator T has at least one fixed point. Thus, in contrasrt to pseudo-distantial spaces, there are metric spaces over $\mathfrak{N}$ possessing the fixed point property. For more details and proofs the reader may consult either J. Schröder in [1956] or L. Collatz in [1964].

In the end, let us remark that, anyhow, the Schröder's metric spaces $\mathfrak{R}$ relative to $\mathfrak{N}$ may be put in the class of abstract spaces obtained by giving "le rôle actif de paramétrisation" to the space $\mathfrak{N}$, in the sense of G. Kurepa's definition at the end of his text cited above.

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(received 21.04.1994.)

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