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A NOTE ON GENERALIZED RECURRENT RIEMANNIAN MANIFOLD

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Abstract. Generalized recurrent Riemannian manifold is a Riemannian manifold whose curvature tensor satisfies the condition (1). In this paper we prove: If the associated 1-form satisfies the condition (3), where $\gamma \neq 1 \neq 2$, or, in the case $\gamma \neq \text{const}$, $\gamma_s A^s \neq 0$, generalized recurrent Riemannian manifold reduces to a recurrent one.

Let us consider a non-flat Riemannian manifold (M, g) of dimension $n \geq 2$, whose curvature tensor satisfies

$$\nabla_r R_{ijkl} = 2A_r R_{ijkl} + A_i R_{rjkl} + A_j R_{irkl} + A_k R_{ijrl} + A_l R_{ijkr}, \tag{1}$$

where A is a non-zero 1-form and ∇ denotes the operator of covariant differentiation with respect to the metric g. If besides (1) the curvature tensor satisfies

$$A_r R_{ijkl} + A_k R_{ijlr} + A_l R_{ijrk} = 0, \tag{2}$$

the condition (1) reduces to

$$\nabla_r R_{ijkl} = 4A_r R_{ijkl},$$

i.e. (M, g) is a recurrent manifold. Conversely, every recurrent manifold also satisfies the condition of the form (1). This is the reason for calling the manifold satisfying (1), *generalized recurrent Riemannian manifold.* The 1-form A will be called its *associated* 1*-form.*

This type of manifold was introduced by M. C. Chaki [1] and called pseudo-symmetric manifold by him. We avoid this name because it is used for other type of manifolds (see for ex. [3]).

In [4] we proved that if the associated 1-form satisfies

$$\nabla_j A_i = \alpha g_{ij} + \gamma A_i A_j, \quad A_t A^t = 0,$$

where α and γ are some functions, generalized recurrent Riemannian manifold is a manifold of quasi-constant curvature. We also proved that if $\alpha = 0$, i.e. if

$$\nabla_j A_i = \gamma A_i A_j, \tag{3}$$

then $A_t A^t = 0$.

The object of this paper is to investigate generalized recurrent Riemannian manifold satisfying (3). In fact, we shall prove

THEOREM. *If the associated* 1*-form satisfies (3) where* $\gamma \neq 1 \neq 2$ *or, in the case* $\gamma \neq$ const, $\gamma_s A^s \neq 0$, $\gamma_s = \partial\gamma/\partial x^s$, *generalized recurrent Riemannian manifold reduces to a recurrent one.*

Proof. First, we note that for every generalized recurrent Riemannian space, the associated 1-form is a gradient and the following is satisfied [4]

$$\begin{aligned}\nabla_r\nabla_s R_{ijkl} - \nabla_s\nabla_r R_{ijkl} &= A_{ri}R_{sjkl} + A_{rj}R_{iskl} + A_{rk}R_{ijsl} + A_{rl}R_{ijks} \\ &\quad - A_{si}R_{rjkl} - A_{sj}R_{irkl} - A_{sk}R_{ijrl} - A_{sl}R_{ijkr}, \end{aligned} \tag{4}$$

where we have put $A_{ri} = \nabla_r A_i - A_r A_i$.

Using the Ricci identity

$$\nabla_r\nabla_s R_{ijkl} - \nabla_s\nabla_r R_{ijkl} = -R_{ajkl}R^a{}_{isr} - R_{iakl}R^a{}_{jsr} - R_{ijal}R^a{}_{ksr} - R_{ijka}R^a{}_{lsr},$$

the relation (4) can be written in the form

$$\begin{aligned} R_{ajkl}R^a{}_{isr} &+ R_{iakl}R^a{}_{jsr} + R_{ijal}R^a{}_{ksr} + R_{ijka}R^a{}_{lsr} \\ &= A_{si}R_{rjkl} + A_{sj}R_{irkl} + A_{sk}R_{ijrl} + A_{sl}R_{ijkr} \\ &\quad - A_{ri}R_{sjkl} - A_{rj}R_{iskl} - A_{rk}R_{ijsl} - A_{rl}R_{ijks}. \end{aligned} \tag{5}$$

Applying the operator ∇_m and using (1) and (5), we get, after some calculations,

$$\begin{aligned} &B_{sim}R_{rjkl} + B_{sjm}R_{irkl} + B_{skm}R_{ijrl} + B_{slm}R_{ijkr} \\ &\quad - B_{rim}R_{sjkl} - B_{rjm}R_{iskl} - B_{rkm}R_{ijsl} - B_{rlm}R_{ijks} \\ &\quad + A_aR^a{}_{isr}R_{mjkl} + A_aR^a{}_{jsr}R_{imkl} + A_aR^a{}_{ksr}R_{ijml} + A_aR^a{}_{lsr}R_{ijkm} \\ &\quad + A_aR^a{}_{jkl}R_{misr} - A_aR^a{}_{ikl}R_{mjsr} + A_aR^a{}_{lij}R_{mksr} - A_aR^a{}_{kij}R_{mlsr} = 0, \end{aligned} \tag{6}$$

where we have put $B_{ijm} = 2A_mA_{ij} + A_iA_{mj} + A_jA_{im} - \nabla_m A_{ij}$.

But, in view of (3) we have

$$B_{ijm} = -\gamma_m A_iA_j - 2(\gamma-1)(\gamma-2)A_iA_jA_m \tag{7}$$

and

$$\nabla_k\nabla_j A_i - \nabla_j\nabla_k A_i = (\gamma_k A_j - \gamma_j A_k)A_i,$$

or, using the Ricci identity,

$$A_aR^a{}_{ikj} = (\gamma_k A_j - \gamma_j A_k)A_i. \tag{8}$$

In the sequel we shall consider the cases $\gamma \neq$ const and $\gamma =$ const separately.

The case $\gamma \neq$ const, $\gamma_s A^s \neq 0$. First, we note that $A^i B_{ijm} = A^i B_{jim} = 0$ because of (7) and $A_t A^t = 0$. Also, $A^a A^k R_{aikj} = \gamma_t A^t A_i A_j$, because of (8). Thus, transvecting (6) with A^s, we get

$$\begin{gathered} - B_{rim} A_a R^a{}_{jkl} + B_{rjm} A_a R^a{}_{ikl} - B_{rkm} A_a R^a{}_{lij} + B_{rlm} A_a R^a{}_{kij} \\ + \gamma_t A^t A_r [A_i R_{mjkl} + A_j R_{imkl} + A_k R_{ijml} + A_l R_{ijkm}] \\ + A_a R^a{}_{jkl} A_s R^s{}_{rmi} - A_a R^a{}_{ikl} A_s R^s{}_{rmj} + A_a R^a{}_{lij} A_s R^s{}_{rmk} - A_a R^a{}_{kij} A_s R^s{}_{rml} = 0. \end{gathered}$$

Substituting (7) and (8), we have

$$\begin{aligned} A_r \big\{ & \gamma_t A^t [A_i R_{mjkl} + A_j R_{imkl} + A_k R_{ijml} + A_l R_{ijkm}] \\ & + A_i (\gamma_k \gamma_j A_l A_m - \gamma_l \gamma_j A_k A_m) + A_j (\gamma_l \gamma_i A_k A_m - \gamma_k \gamma_i A_l A_m) \\ & + A_l (\gamma_j \gamma_k A_i A_m - \gamma_i \gamma_k A_j A_m) + A_k (\gamma_i \gamma_l A_j A_m - \gamma_j \gamma_l A_i A_m) \big\} = 0. \end{aligned}$$

Let ϑ be a vector such that $A_r \vartheta^r = 1$. Then, transvecting the preceding relation by ϑ^r, we get

$$\begin{aligned} & A_i (\gamma_s A^s R_{mjkl} + \gamma_k \gamma_j A_l A_m + \gamma_l \gamma_m A_k A_j - \gamma_l \gamma_j A_k A_m - \gamma_k \gamma_m A_l A_j) \\ & + A_j (\gamma_s A^s R_{imkl} + \gamma_l \gamma_i A_k A_m + \gamma_k \gamma_m A_l A_i - \gamma_k \gamma_i A_l A_m - \gamma_l \gamma_m A_k A_i) \\ & + A_k (\gamma_s A^s R_{ijml} + \gamma_i \gamma_l A_j A_m + \gamma_j \gamma_m A_i A_l - \gamma_j \gamma_l A_i A_m - \gamma_i \gamma_m A_j A_l) \\ & + A_l (\gamma_s A^s R_{ijkm} + \gamma_j \gamma_k A_i A_m + \gamma_i \gamma_m A_j A_k - \gamma_i \gamma_k A_j A_m - \gamma_j \gamma_m A_i A_k) = 0. \end{aligned}$$

This can be written in the form

$$A_i T_{mjkl} + A_j T_{imkl} + A_k T_{ijml} + A_l T_{ijkm} = 0, \tag{9}$$

where we have put

$$T_{mjkl} = \gamma_s A^s R_{mjkl} + \gamma_k \gamma_j A_l A_m + \gamma_l \gamma_m A_k A_j - \gamma_l \gamma_j A_k A_m - \gamma_k \gamma_m A_l A_j.$$

We see that

$$T_{mjkl} = -T_{jmkl}, \quad T_{mjkl} = T_{klmj}, \quad T_{mjkl} + T_{mklj} + T_{mljk} = 0. \tag{10}$$

Thus, we can use the following lemma ([5], lemma 4).

If A_i and T_{mjkl} are numbers satisfying (9) and (10), then either each A_i is zero or each T_{mjkl} is zero.

According to our assumption, A is non-zero 1-form. Thus $T_{mjkl} = 0$, i.e.

$$\gamma_s A^s R_{mjkl} = \gamma_l \gamma_j A_k A_m + \gamma_k \gamma_m A_l A_j - \gamma_k \gamma_j A_l A_m - \gamma_l \gamma_m A_k A_j,$$

from which we get

$$\gamma_s A^s (A_r R_{ijkl} + A_k R_{ijlr} + A_l R_{ijrk}) = 0.$$

Therefore, if $\gamma_s A^s \neq 0$, the condition (2) is satisfied. But this means that generalized recurrent Riemannian manifold reduces to recurrent one.

Case $\gamma = \text{const}$, $\gamma \neq 1 \neq 2$. In this case, (7) and (8) reduce to

$$B_{ijm} = -2(\gamma-1)(\gamma-2)A_iA_jA_m, \quad A_aR^a{}_{ikj} = 0$$

respectively and (6) becomes

$$A_m(A_sA_iR_{rjkl} + A_sA_jR_{irkl} + A_sA_kR_{ijrl} + A_sA_lR_{ijkr} - A_rA_iR_{sjkl} - A_rA_jR_{iskl} - A_rA_kR_{ijsl} - A_rA_lR_{ijks}) = 0.$$

Let ϑ be a vector such that $A_i\vartheta^i = 1$. Then, transvecting the preceding relation with ϑ^m, we find

$$A_sA_iR_{rjkl} + A_sA_jR_{irkl} + A_sA_kR_{ijrl} + A_sA_lR_{ijkr} - A_rA_iR_{sjkl} - A_rA_jR_{iskl} - A_rA_kR_{ijsl} - A_rA_lR_{ijks} = 0. \quad (11)$$

Now, we apply the method used in [2], Proposition 4.1. Let us put

$$\vartheta^iR_{ijkl} = C_{jkl}, \quad \vartheta^a\vartheta^bR_{aijb} = D_{ij}.$$

We see that

$$C_{jkl} = -C_{jlk}, \quad C_{jkl} + C_{klj} + C_{ljk} = 0, \quad D_{ij} = D_{ji}, \quad \vartheta^aD_{aj} = 0.$$

Transvecting (11) with $\vartheta^i\vartheta^s$, we get

$$R_{rjkl} + A_jC_{rkl} + A_kC_{jrl} + A_lC_{jkr} + A_rC_{jlk} + A_rA_kD_{jl} - A_rA_lD_{jk} = 0. \quad (12)$$

A cyclic permutation of r, k and l gives

$$A_kC_{jrl} + A_rC_{jlk} + A_lC_{jkr} = 0,$$

because of which (12) reduces to

$$R_{rjkl} + A_jC_{rkl} + A_rA_kD_{jl} - A_rA_lD_{jk} = 0. \quad (13)$$

Trnsvecting (13) with ϑ^l we get $C_{kjr} = A_rD_{jk} - A_jD_{rk}$. Substituting this into (13), we obtain

$$R_{rjkl} = A_rA_lD_{jk} - A_rA_kD_{jl} + A_jA_kD_{rl} - A_jA_lD_{rk}.$$

But this relation implies (2).

This completes the proof of the Theorem. ∎

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