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A COUNTEREXAMPLE FOR ONE VARIANT OF MCINTOSH CLOSED GRAPH THEOREM

Lj. Čukić

Abstract. Counterexamples for two closed graph theorems from Köthe's monograph [5] are given.

In Köthe's monograph [5] the following two theorems ([5] §35.10.(1) and (2)) are "proved":

(1) *Let $E(t)$ be a sequentially complete locally convex space, t the Mackey topology, and let $E'(\beta(E',E))$ be complete. Let F be a semi-reflexive webbed space. Then every sequentially closed linear mapping A from E in F is continuous.*

(2) *Let E and F be (F)-spaces and A a weakly sequentially closed linear mapping from E' into F'. Then A is weakly continuous.*

The first of these theorems is a generalization of McIntosh closed graph theorem.

We shall prove here that both these theorems are incorrect, even if A is a sequentially continuous linear functional.

Both theorems are correct if we assume that the linear mapping A has a closed graph ([2]).

The notations we use here for weak and strong topology are as in [6]. Let us remark that in [5] by a sequentailly closed mapping it is assumed a mapping with a sequentially closed graph and by a weak continuity of a mapping $A\colon E' \to F'$ it is assumed its $\sigma(E',E)$-$\sigma(F',F)$ continuity.

EXAMPLE 1. Let T be a P-space which is not realcomplete (see [3], 9.L. or [1], Example 2.6-1), $E = C_b(T)$ space of all bounded continuous real-valued functions on T and t the strongest of all locally convex topologies on E which coincide with compact-open topology on the set $\{\, x \in E \;:\; \sup_T |x(s)| \le 1 \,\}$ (i.e. t is the strict topology [7]). Then the locally convex space $E(t)$ satisfies all conditions from (1) ([7], Theorems 2.1. and 2.2.). Let $p \in \mho T - T$, where $\mho T$ is the realcompletion of the space T, and $Ax = \bar{x}(p)$, where $\bar{x}$ is the (unique) continuous extension of $x \in E$ on $\mho T$.

The linear functional A is sequentially continuous on $E(t)$, but it is not continuous. In fact, if $x_n \to x$ in $E(t)$, then $x_n \to x$ pointwise on T. Then also $\bar{x}_n(p) \to \bar{x}(p)$, because there exists $s \in T$ so that $\bar{x}(p) = x(s)$ and $\bar{x}_n(p) = x_n(s)$ for all n (see [8], 2.5.(c)) and so A is sequentially continuous. The mapping A is not continuous because $p \in \mho T - T$ ([8], 2.4.(a)).

EXAMPLE 2. Let T be any infinite compact extremally disconnected space (for example, the Stone-Čech compactification of discrete space $\mathbf{N}$ of positive integers) and let E be the space of all continuous real-valued functions on T, with supremum norm. Then E is a Banach space and $E \neq E''$ ([1], 2.8-2). If $A \in E'' \setminus E$, then A is not a $\sigma(E', E)$-continuous linear functional on E', but it is $\sigma(E', E)$-sequentially continuous.

In fact, if a sequence (x_n) from E' $\sigma(E', E)$-converges to zero, then it $\sigma(E', E'')$-converges to zero ([4], Theorem 9), and so the sequence (Ax_n) converges to zero.

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Department of Mathematics, Faculty of Civil Engineering, Bul. Revolucije 73, 11000 Beograd, Yugoslavia