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ON THE EXPANSION THEOREM FOR A CERTAIN BOUNDARY VALUE PROBLEM FOR A FUNCTIONAL DIFFERENTIAL EQUATION

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Abstract. The boundary value problem

$$-y'' + q(x)y = \lambda y + \int_0^\pi y\, d\sigma(x), \quad y(0) = y(\pi) = 0,$$

is concerned, where $q \in C[0,\pi]$ and σ is a function of bounded variation. It is proved that the system of eigenfunctions of the given problem is complete and minimal in $L^2(0,\pi)$, and also that functions of a certain class can be expanded into uniformly convergent series with respect to the mentioned system.

Introduction

In [1] and [2] problems concerning the asymptotics of spectra as well as determining regularized traces of the following two boundary problems were examined

$$-y'' + q(x)y = \lambda y + y\left(\frac{\pi}{2}\right), \quad y(0) = y(\pi) = 0; \tag{1}$$

$$-y'' + q(x)y = \lambda y + \sum_{k=1}^{n-1} \alpha_k y\left(\frac{k\pi}{n}\right) + \alpha_n \int_0^\pi y(t)\, dt, \quad y(0) = y(\pi) = 0, \tag{2}$$

where q is a sufficiently smooth function with complex values.

In [3] we see that, under certain conditions, the system of eigenfunctions of the boundary problem (1) is a Riesz base of $L^2(0,\pi)$ and also that any function $f \in C^2[0,\pi]$ for which $f(0) = f(\pi) = 0$ can be expanded into a uniformly convergent series with respect to a system of eigenfunctions of the boundary problem (1).

In this paper we examine the following boundary problem

$$-y'' + q(x)y = \lambda y + \int_0^\pi y\, d\sigma(x), \quad y(0) = y(\pi) = 0, \tag{3}$$

where q is a real continuous function and σ is a function of bounded variation on $[0,\pi]$. We prove a similar expansion theorem.

1. Preliminaries

Let L_0 be the differential operator generated by the differential expression $l_0(y) = -y'' + q(x)y$ and boundary conditions $y(0) = y(\pi) = 0$. Note that L_0 is a selfadjoint operator. Let L be the operator generated by the differentially-integral expression $l(y) = -y'' + q(x)y - \int_0^\pi y\, d\sigma(x)$ with the same boundary conditions. We say that $\lambda = \lambda_0$ is an eigenvalue of the boundary problem (3) if there is a function $y_0 \in C^2[0,\pi]$ for which $y_0 \not\equiv 0$ and

$$-y_0'' + q(x)y_0 = \lambda_0 y_0 + \int_0^\pi y_0\, d\sigma(x), \quad y_0(0) = y_0(\pi) = 0.$$

An eigenvalue is simple if there is one and only one corresponding eigenfunction (up to a multiplicative constant).

The Green's function of the operator $L_0 - \lambda$ is given by [4]

$$G(x,\xi,\lambda) = \frac{H(x,\xi,\lambda)}{\Delta(\lambda)}, \tag{4}$$

where Δ is the characteristic determinant of the boundary problem

$$-y'' + q(x)y = \lambda y, \quad y(0) = y(\pi) = 0. \tag{5}$$

For any fixed x and ξ from $[0,\pi]$, the function $H(x,\xi,\lambda)$ is entire. The function Δ is also entire and its roots are eigenvalues of the problem (5).

Then, the function $\Delta_1(\lambda) = \Delta(\lambda) - \int_0^\pi\int_0^\pi H(x,\xi,\lambda)\, d\sigma(x)\, d\xi$ is also entire.

We suppose:

1° No eigenvalue of the problem (5) is an eigenvalue of the problem (3).

2° Roots $\lambda_1, \lambda_2, \ldots$ of the function Δ_1 are simple and $\Delta(\lambda_n) \neq 0$.

3° Boundary problems $-y'' + q(x)y = 0$, $y(0) = y(\pi) = 0$ and $-y'' + q(x)y = \int_0^\pi y\, d\sigma(x)$, $y(0) = y(\pi) = 0$ have only trivial solutions.

2. Main results

LEMMA. *If the conditions* 1° *and* 2° *are satisfied then the eigenvalues of the boundary problem (3) are simple and they are roots of the function* Δ_1.

Proof. Let λ_0 be an eigenvalue of the boundary problem (3). It means that there is a function $y_0 \in C^2[0,\pi]$ for which

$$-y_0'' + q(x)y_0 = \lambda_0 y_0 + \int_0^\pi y_0\, d\sigma(x), \quad y_0(0) = y_0(\pi) = 0 \tag{6}$$

and $y_0 \not\equiv 0$ on $[0,\pi]$.

From 1° we see that $\Delta(\lambda_0) \neq 0$. Applying the Green's function of the operator $L_0 - \lambda_0$ to (6) we get

$$y_0(x) = \int_0^\pi G(x,\xi,\lambda_0)\, d\xi \int_0^\pi y_0\, d\sigma(x). \tag{7}$$

From (7) we have $y_0(x) = K\int_0^\pi G(x,\xi,\lambda_0)\,d\xi$ (for some constant K). Also from (7) we see that $\int_0^\pi y_0\,d\sigma(x) \not\equiv 0$ (otherwise we have $y_0 \equiv 0$ on $[0,\pi]$).

Integrating (7) with respect to the function σ we have

$$\int_0^\pi y_0\,d\sigma(x) = \int_0^\pi\int_0^\pi G(x,\xi,\lambda_0)\,d\xi\,d\sigma(x)\int_0^\pi y_0\,d\sigma(x). \tag{8}$$

From (8), knowing that $\int_0^\pi y_0\,d\sigma \neq 0$, we get

$$1 - \int_0^\pi\int_0^\pi G(x,\xi,\lambda_0)\,d\xi\,d\sigma(x) = 0$$

and (because of $\Delta(\lambda_0) \neq 0$ and (4)) $\Delta_1(\lambda_0) = 0$.

Let now λ_0 be a root of the function Δ_1. Then, from 2°, we have $\Delta(\lambda_0) \neq 0$. Let us examine the function

$$y_0(x) = \int_0^\pi G(x,\xi,\lambda_0)\,d\xi. \tag{9}$$

As $\Delta_1(\lambda_0) = 0$ we have

$$1 - \int_0^\pi\int_0^\pi G(x,\xi,\lambda_0)\,d\xi\,d\sigma(x) = 0. \tag{10}$$

From (9) we get (using characteristics of the Green's function)

$$-y_0'' + q(x)y_0 = \lambda_0 y_0 + 1, \quad y_0(0) = y_0(\pi) = 0. \tag{11}$$

Integrating (9) with respect to the function σ (and knowing (10)) we get

$$\int_0^\pi y_0\,d\sigma(x) = 1. \tag{12}$$

From (11) and (12) we conclude that λ_0 is an eigenvalue and y_0 an eigenfunction of the boundary problem (3).

Lemma is proved. ∎

The operator L_0 is selfadjoint, so from $(L_0 - \lambda)^* = L_0 - \bar\lambda$ we get (see [4])

$$\overline{G(x,\xi,\bar\lambda)} = G(\xi,x,\lambda) \tag{13}$$

if $\lambda \in \rho(L_0)$. Let us denote by $G_0(x,\xi)$ the Green's function of the operator L_0, i.e. $G_0(x,\xi) = G(x,\xi,0)$. From 3° we get $\Delta(0) \neq 0$ and $\Delta_1(0) \neq 0$.

Solving the equation

$$-y'' + q(x)y - \int_0^\pi y\,d\sigma(x) = f(x), \quad y(0) = y(\pi) = 0$$

we get

$$y(x) = (L^{-1}f)(x) = \int_0^\pi G_0(x,\xi)f(\xi)\,d\xi + \int_0^\pi G_0(x,\xi)\,d\xi\,\frac{\int_0^\pi\int_0^\pi G_0(x,\xi)f(\xi)\,d\xi\,d\sigma(x)}{1-\int_0^\pi\int_0^\pi G_0(x,\xi)\,d\xi\,d\sigma(x)}. \tag{14}$$

Now, we put

$$c = 1 - \int_0^\pi \int_0^\pi G_0(x,\xi)\,d\xi\,d\sigma(x), \quad \varphi(\xi) = \frac{1}{\bar{c}} \int_0^\pi G_0(\xi,x)\,\overline{d\sigma(x)}.$$

So (14) becomes

$$L^{-1}f(x) = \int_0^\pi G_0(x,\xi)f(\xi)\,d\xi + (f,\varphi)\int_0^\pi G_0(x,\xi)\,d\xi, \tag{15}$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(0,\pi)$. As $L_0^{-1}f(x) = \int_0^\pi G_0(x,\xi)f(\xi)\,d\xi$, from (15) we get

$$L^{-1}f = L_0^{-1}f + (f,\varphi)L_0^{-1}1. \tag{16}$$

Let us now define linear operators A and A_0 by

$$\begin{aligned} Af(x) &= \int_0^\pi G_0(x,\xi)f(\xi)\,d\xi + (f,\varphi)\int_0^\pi G_0(x,\xi)\,d\xi \qquad (A = L^{-1}), \\ A_0f(x) &= \int_0^\pi G_0(x,\xi)f(\xi)\,d\xi \qquad (A_0 = L_0^{-1}). \end{aligned} \tag{17}$$

From (15), (16) and (17) we get

$$A = A_0 + (\cdot,\varphi)A_0 1. \tag{18}$$

We define an operator $S\colon L^2(0,\pi) \to L^2(0,\pi)$ by $Sf(x) = (f,\varphi)\cdot 1$. Then we have (from (18))

$$A = A_0(I+S). \tag{19}$$

The operators A, A_0, S act on $L^2(0,\pi)$. The operator A has the eigenvalues that are reciprocal to the eigenvalues of the operator L; therefore

$$\sigma(A) = \left\{ \frac{1}{\lambda_n} \;:\; \Delta_1(\lambda_n) = 0 \right\} \cup \{0\}$$

(because $A \in \mathfrak{S}_\infty$, $\mathfrak{S}_\infty$ – the set of compact operators) and the eigenvectors are equal to the eigenfunctions of the operator L. Similarly, the operator A_0 has the eigenvalues that are reciprocal to the eigenvalues of L_0 and the eigenvectors are equal to the eigenfunctions of the operator L_0.

Since q is a real, continuous function on $[0,\pi]$, the eigenvalues of the operator L_0 have the following asymptotics

$$\mu_n = n^2 + O(1).$$

From this we see that the asymptotics of eigenvalues of the operator A_0 ($= A_0^*$) is

$$\mu_n^{-1} = n^{-2}(1 + O(n^{-2}))$$

and so we conclude that $A_0 \in \mathfrak{S}_1$ (nuclear operator).

As $y_n(x) = \int_0^\pi G(x,\xi,\lambda_n)\,d\xi$ are eigenfunctions of the operator L corresponding to the eigenvalues $\{\lambda_n\}$, they are the eigenvectors of the operator A corresponding to the eigenvalues $\{\lambda_n^{-1}\}$.

THEOREM 1. *If the conditions* 1°, 2° *and* 3° *are satisfied, then the system of eigenfunctions of the boundary problem (3) is complete and minimal in* $L^2(0,\pi)$.

Proof. Because of the previous results it is enough to prove that the system of eigenvectors of the operator A is complete in $L^2(0,\pi)$.

Since the operator S is compact (its rank is one) and the operator A_0 is selfadjoint and nuclear, by the Keldysh's theorem (see [5]), for proving that our system is complete it is enough to prove that $\operatorname{Ker} A = \{0\}$.

As $Af = 0$ then, from (19), we have

$$A_0(I+S)f = 0. \tag{20}$$

As the operator A_0 is 1–1 (because the operator L_0 is 1–1), we get $(I+S)f = 0$ i.e.

$$f + (f,\varphi)1 = 0. \tag{21}$$

From this we get

$$(f,\varphi)\cdot(1+(1,\varphi)) = 0. \tag{22}$$

It is easy to check that $1+(1,\varphi) \neq 0$ (if 3° is satisfied). Then, from (22), we get $(f,\varphi) = 0$ and so, from (21), we conclude that $f = 0$.

That proves the completeness of the system $\{y_n(x)\}_1^\infty$.

For the proof of minimality, it is enough to construct a system biorthogonal to the system $\{y_n\}_1^\infty$.

Since $Ay_n = \lambda_n^{-1}y_n$ and all eigenvalues are simple, if one denotes by $z_n(x)$ the eigenvectors of the adjoint operator A^* corresponding to the eigenvalues $(\overline{\lambda_n})^{-1}$, then $A^*z_n = (\overline{\lambda_n})^{-1}z_n$ (all eigenvalues $(\overline{\lambda_n})^{-1}$ are simple). It is easy to check that

$$(z_n, y_n) \neq 0, \quad n \in N; \qquad (z_n, y_m) = 0, \quad m \neq n.$$

Because of this, we will suppose that the system $\{z_n\}_1^\infty$ is chosen in such a way that $(y_n, z_m) = \delta_{nm}$. This system $\{z_n\}_1^\infty$ is biorthogonal to the system $\{y_n\}_1^\infty$. That proves the minimality. ∎

Since the operator A is compact, with eigenvalues that are simple, $(I-\lambda A)^{-1}$ is a meromorphic operator function with simple poles in points λ_n (see [5]). Moreover, the principal part of the Laurent expansion in the neighborhood of the point $\lambda = \lambda_n$ is $-\lambda_n \frac{(\cdot,z_n)y_n}{\lambda-\lambda_n}$, i.e.

$$(I-\lambda A)^{-1} = -\lambda_n \frac{(\cdot,z_n)y_n}{\lambda-\lambda_n} + G_1(\lambda), \tag{23}$$

where the function G_1 is holomorphic in the neighborhood of the point $\lambda = \lambda_n$.

As $Ay_n = \lambda_n^{-1}y_n$, from (23) we get

$$\operatorname*{Res}_{\lambda=\lambda_n} A(I-\lambda A)^{-1} = -(\cdot,z_n)y_n. \tag{24}$$

As before, applying the Green's function to the equation

$$-y'' + q(x)y - \lambda y - \int_0^\pi y\,d\sigma(x) = f, \quad y(0) = y(\pi) = 0$$

we get

$$(L-\lambda)^{-1}f = \int_0^\pi G(x,\xi,\lambda)f(\xi)\,d\xi + \int_0^\pi G(x,\xi,\lambda)\,d\xi\,\frac{\int_0^\pi\int_0^\pi G(x,\xi,\lambda)f(\xi)\,d\xi\,d\sigma(x)}{1-\int_0^\pi\int_0^\pi G(x,\xi,\lambda)\,d\xi\,d\sigma(x)}. \quad (25)$$

We also note that

$$(L-\lambda)^{-1}f = A(I-\lambda A)^{-1}f \quad (f \in C^2[0,\pi]). \quad (26)$$

Now, from (25) and (26), we have (for $f \in C^2[0,\pi]$)

$$A(I-\lambda A)^{-1}f = \int_0^\pi G(x,\xi,\lambda)f(\xi)\,d\xi + \int_0^\pi G(x,\xi,\lambda)\,d\xi\,\frac{\int_0^\pi\int_0^\pi G(x,\xi,\lambda)f(\xi)\,d\xi\,d\sigma(x)}{1-\int_0^\pi\int_0^\pi G(x,\xi,\lambda)\,d\xi\,d\sigma(x)}. \quad (27)$$

There is a sequence of circles Γ_k (whose centers are in the point $\lambda = 0$ and radii $R_k \to +\infty$ $(k \to \infty)$) so that, on Γ_k, we have

$$|G(x,\xi,\lambda)| \leqslant \frac{M}{\sqrt{|\lambda_k|}} = \frac{M}{\sqrt{R_k}}, \quad (28)$$

where the constant M does not depend on k and $x, \xi \in [0,\pi]$ (see [4]). Let us examine the integral

$$\frac{1}{2\pi i}\int_{\Gamma_k}\frac{A(I-\lambda A)^{-1}f}{\lambda}\,d\lambda,$$

where f is a fixed, continuous function (on $[0,\pi]$). By the Cauchy's residue theorem we get

$$\frac{1}{2\pi i}\int_{\Gamma_k}\frac{A(I-\lambda A)^{-1}f}{\lambda}\,d\lambda = \operatorname*{Res}_{\lambda=0}\frac{A(I-\lambda A)^{-1}f}{\lambda} + \sum_{|\lambda_n|<R_k}\operatorname*{Res}_{\lambda=\lambda_n}\frac{A(I-\lambda A)^{-1}f}{\lambda}. \quad (29)$$

As $\operatorname{Res}_{\lambda=0}\frac{A(I-\lambda A)^{-1}f}{\lambda} = Af$ and (because of (24))

$$\operatorname*{Res}_{\lambda=\lambda_n}\frac{A(I-\lambda A)^{-1}f}{\lambda} = -\frac{1}{\lambda_n}(f,z_n)y_n,$$

we have (from (29))

$$\frac{1}{2\pi i}\int_{\Gamma_k}\frac{A(I-\lambda A)^{-1}f}{\lambda}\,d\lambda = Af - \sum_{|\lambda_n|<R_k}\frac{1}{\lambda_n}(f,z_n)y_n. \quad (30)$$

This can be written as

$$\frac{1}{2\pi i}\int_{\Gamma_k}\frac{A(I-\lambda A)^{-1}f}{\lambda}\,d\lambda = Af - \sum_{\nu=0}^{k}\Bigg(\sum_{R_\nu\leqslant|\lambda_n|<R_{\nu+1}}\frac{1}{\lambda_n}(f,z_n)y_n\Bigg), \quad (31)$$

where we take $R_0 = 0$.

From (31) we get

$$\left| Af(x) - \sum_{\nu=0}^{k} \left(\sum_{R_\nu \leqslant |\lambda_n| < R_{\nu+1}} \frac{1}{\lambda_n} (f, z_n) y_n(x) \right) \right| \leqslant \max_{\substack{\lambda \in \Gamma_k \\ x \in [0,\pi]}} |A(I - \lambda A)^{-1} f|. \tag{32}$$

From (27) and (28) we get $|A(I-\lambda A)^{-1}f| \leqslant C/\sqrt{R_k}$, where C is a constant that does not depend on k and $x \in [0, \pi]$. From this and (32) we have

$$Af - \sum_{\nu=0}^{k} \left(\sum_{R_\nu \leqslant |\lambda_n| < R_{\nu+1}} \frac{1}{\lambda_n} (f, z_n) y_n \right) \rightrightarrows 0$$

when $k \to \infty$, i.e.

$$Af(x) = \sum_{\nu=0}^{\infty} \left(\sum_{R_\nu \leqslant |\lambda_n| < R_{\nu+1}} \frac{1}{\lambda_n} (f, z_n) y_n \right). \tag{33}$$

We will put $Af = g$. Since $f \in C[0, \pi]$, for the function g we know that $g \in C^2[0, \pi]$, $g(0) = g(\pi) = 0$ and

$$-g'' + q(x)g - \int_0^\pi g \, d\sigma(x) = f.$$

Now, from (33) we get

$$g(x) = \sum_{\nu=0}^{\infty} \left(\sum_{R_\nu \leqslant |\lambda_n| < R_{\nu+1}} (f, A^* z_n) y_n(x) \right) = \sum_{\nu=0}^{\infty} \left(\sum_{R_\nu \leqslant |\lambda_n| < R_{\nu+1}} (Af, z_n) y_n(x) \right)$$

or

$$g(x) = \sum_{\nu=0}^{\infty} \left(\sum_{R_\nu \leqslant |\lambda_n| < R_{\nu+1}} (g, z_n) y_n(x) \right). \tag{34}$$

All this can be formulated as a theorem:

THEOREM 2. *If the conditions* 1°, 2° *and* 3° *are satisfied and if a function* $g \in C^2[0, \pi]$ *and* $g(0) = g(\pi) = 0$, *then it can be expanded into the uniformly convergent series (34) with respect to a system of eigenfunctions of the boundary value problem (3).*

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