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ON THE SOLUTIONS OF THE FUNCTIONAL EQUATION $x(t) + A(t)x(f(t)) = F(t)$ WHEN THE FUNCTION F SATISFIES SPECIAL CONDITIONS

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Abstract. The results of this paper are concerned with the soluiton $x(t)$ of the functional equation $x(t) + A(t)x(f(t)) = F(t)$. Using regular summability methods T, we derive some necessary and also some sufficient conditions for the T-sum $x(t)$ of the series $\sum_{i=0}^{\infty}(-1)^i F(f^i(t))$ to be a solution of the above mentioned equation under the specific conditions for $F(t)$.

1. Introduction

We consider here the linear functional equation

$$x(t) + A(t)x(f(t)) = F(t) \tag{1.1}$$

on a given set $S \subset \mathbf{R}$, where $f\colon S \to S$, $F\colon S \to \mathbf{R}$ and $A\colon S \to \mathbf{R}$ are given functions, $x\colon S \to \mathbf{R}$ denotes the unknown function and $\mathbf{R}$ is the set of real numbers. Throughout the paper we assume that the functions A and f satisfy the condition $A(t) = A(f(t))$. We define

$$f^0(t) = t, \qquad f^{i+1}(t) = f(f^i(t)), \quad i = 0, 1, 2, \dots$$

Equations of this type have been considered by many mathematicians. H. Steinhaus [10] discussed the equation $\varphi(x) + \varphi(x^2) = x$ and G. H. Hardy [4] considered the equation $\varphi(x) + \varphi(x^\alpha) = x$ $(\alpha > 0)$, where φ denotes the unknown function. R. Raclis [9] discusses the equation $\varphi(x) + \varphi(f(x)) = F(x)$ for complex x and finds meromorphic solutions. Under certain hypothesis in regard to the function f, N. M. Gersevanoff [2] solves the equation $x(t) + A(t)x(f(t)) = F(t)$ and Ghermanescu [3] solves the equation

$$A_0\varphi + A_1\varphi(f) + A_2\varphi(f(f)) + \cdots + A_n\varphi(f(\dots)\dots) = F(x),$$

where φ denotes the unknown function, and f and F are given functions. Euqation (1.1) is a direct generalization of the equation $\varphi(x) + \varphi(f(x)) = F(x)$ considered by M. Kuczma in [6]. The basic method used in that paper is the iteration of the function f, and solutions are given in the form of the sums of a convergent series of functions. M. Bajraktarević [1] has also considered solutions of the equation $\varphi(x) + \varphi(f(x)) = F(x)$ under assumption that the series of functions given by M. Kuczma is divergent, but summable by some regular method of summability. M. Malenica [7], [8], using method of M. Bajraktarević has examined solutions of the same equation. Papers by M. Kuczma [6], M. Bajraktarević [1] and M. Malenica [8] are of special interest to us.

In this paper, considering the series

$$\sum_{i=0}^{\infty}(-1)^i F(f^i(t)) \tag{1.2}$$

and using regular summability methods T we derive some sufficient and also some necessary conditions for T-sum $x(t)$ of the series (1.2) to be a solution of equation (1.1). The function $F(t)$ has a special form. Whenever we say that (1.2) is T-summable with sum $x(t)$ we assume that $T = (a_{kn})$ is a regular sequence matrix transformation, i.e. its elements a_{kn} satisfy the following conditions

(i) $a_{kn} \to 0$ $(k \to \infty,\ n \geqslant 0)$,

(ii) $\sum_{i=0}^{n} |a_{ki}| \leqslant K$ $(n \geqslant 0,\ k \geqslant 0,\ K \text{ fixed})$,

(iii) $\sum_{n=0}^{\infty} a_{kn} = A_k \to 1$ $(k \to \infty)$.

2. Results

Iti is interesting to consider the equation (1.1) in the case when the form of F causes the series (1.2) to diverge and to look for the conclusions on $T = (a_{kn})$ that assure that the series (1.2) is T-summable to the function $x(t)$ so that the relations

$$Ts_n(t) \to x(t), \qquad x(t) + A(t)x(f(t)) = F(t), \tag{2.1}$$

are valid. The sum $s_n(t)$ is a partial sum of order n of the series (1.2). If we take F to be a function satisfying

$$F(f(t)) = F(t), \tag{2.2}$$

and for $T = (a_{kn})$ regular transformation with the property

$$\sum_{m=0}^{\infty} a_{k,2m} \to \frac{1}{1 + A(t)}, \quad k \to \infty \qquad (A(t) \neq -1),$$

we get the following theorem.

THEOREM 2.1 *Suppose $F(t)$ satisfies* (2.2) *and suppose $Ts_n(t) \to x(t)$, where T is a regular transformation.*

(a) *For $x(t)$ to be a solution of the equation* (1.1), *it is sufficient that T satisfies condition* (2.3). *In this case the solution has the form $x(t) = \dfrac{F(t)}{1 + A(t)}$.*

(b) *For* $x(t)$ *to be a solution of the equation* (1.1), *it is necessary that* T *satisfies condition* (2.3) *with* $F(t) \not\equiv 0$.

(c) *If* $F(t) \equiv 0$, *then* $x(t) \equiv 0$ *is always a solution of the equation* (1.1).

Proof. For the function F satisfying (2.2) we have

$$\sum_{i=0}^{n}(-1)^i F(f^i(t)) = \sum_{i=0}^{n}(-1)^i F(t) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ F(t), & \text{if } n \text{ is even}; \end{cases}$$
$$s'_k(t) = \sum_{n=0}^{\infty} a_{kn} s_n(t) = \sum_{m=0}^{\infty} a_{k,2m+1} 0 + \sum_{m=0}^{\infty} a_{k,2m} F(t),$$

i.e.

$$s'_k(t) = F(t) \sum_{m=0}^{\infty} a_{k,2m}. \tag{2.4}$$

(a) Let $T = (a_{kn})$ be the transformation satisfying (2.3). If we let $k \to \infty$ in (2.4) we get

$$x(t) = \lim_{k\to\infty} s'_k(t) = \frac{F(t)}{1 + A(t)}. \tag{2.5}$$

It is easy to verify that if $x(t)$ has the form given in (2.5), then it is a solution of the equation (1.1).

(b) Let $x(t)$ be a solution of the equation (1.1). Then by (2.4), the following relations hold.

$$\begin{aligned} G_k(t) &= s'_k(t) + A(t) s'_k(f(t)) - F(t) \\ &= F(t) \sum_{m=0}^{\infty} a_{k,2m} + A(t) F(t) \sum_{m=0}^{\infty} a_{k,2m} - F(t) \to 0 \quad (k \to \infty), \\ F(t)(1 + A(t)) &\sum_{m=0}^{\infty} a_{k,2m} - F(t) = G_k(t) \to 0 \quad (k \to \infty), \\ F(t)(1 + F(t)) &\sum_{m=0}^{\infty} a_{k,2m} = F(t) + G_k(t). \end{aligned}$$

If $F(t) \not\equiv 0$, when $k \to \infty$, we get $\lim\limits_{k\to\infty} \sum\limits_{m=0}^{\infty} a_{k,2m} = \dfrac{1}{1 + A(t)}$, i.e. $T = (a_{kn})$ is a regular transformation with the property (2.3).

(c) If $F(t) \not\equiv 0$, then $x(t) \equiv 0$ is always a solution of the equation (1.1). ■

It is easy to verify that C_1-method, E-method and E_r-method, $r \geqslant 1$, given in [5] are examples of methods satisfying property (2.3).

Remark 2.1. Obviously, if $A(t) = -1$, the equation (1.1) has the solution $x(t) = \frac{1}{2} F(t)$, under the condition $F(f(t)) = -F(t)$.

At the end, let $F(t)$ has the following properties

$$F(f^p(t)) = 0, \qquad F(f^i(t)) \neq 0, \quad i = 1, 2, \dots, p-1. \tag{2.6}$$

We look for the solution $x(t)$ of the equation (1.1) in the form

$$x(t) = a + \sum_{i=0}^{p-1} b_i F(f^i(t)). \tag{2.7}$$

In connection with that, the following theorem is valid.

THEOREM 2.2. *In order that the equation* (1.1), *in which the function* $F(t)$ *has properties* (2.6), *has a solution* $x(t)$ *of the form* (2.7), *it is sufficient that*

$$a = 0, \qquad b_i = (-1)^i (A(t))^i, \quad i = 0, 1, \dots, p-1, \tag{2.8}$$

i.e. that the function $x(t)$ *has the form*

$$x(t) = \sum_{i=0}^{p-1} (-1)^i (A(t))^i F(f^i(t)). \tag{2.9}$$

Proof. From (2.9) and (2.6) we have

$$\begin{aligned} x(t) + A(t)x(f(t)) &= \sum_{i=0}^{p-1} (-1)^i (A(t))^i F(f^i(t)) + A(t) \sum_{i=0}^{p-1} (-1)^i (A(f(t))^i F(f^{i+1}(t)) \\ &= F(t) + \sum_{i=1}^{p-1} (-1)^i (A(t))^i F(f^i(t)) \\ &- \sum_{i=1}^{p-1} (-1)^i (A(t))^i F(f^i(t)) - (-1)^p (A(t))^p F(f^p(t)) = F(t) \quad \blacksquare \end{aligned}$$

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