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A SPECTRALITY CONDITION FOR INFINITESIMAL GENERATORS OF COSINE OPERATOR FUNCTIONS

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Abstract. We will give a necessary and sufficient condition for the infinitesimal generator of a strongly continuous cosine operator function $C(t)$, such that $\|C(t)\| \leq 1$ for all $t \in \mathbf{R}$ on a reflexive, strictly convex (complex) Banach space with a Gâteaux differentiable norm to be a spectral scalar type operator with the spectral family of hermitian bounded linear projectors.

1. Introduction

Strongly continuous semi-groups of spectral bounded operators on a Banach space and their infinitesimal generators in particular were considered by many authors (see e.g. [1], [2], [6]). In this paper we will consider the strongly continuous cosine operator function $C(t)$, such that $\|C(t)\| \leq 1$ for all $t \in \mathbf{R}$ on a reflexive strictly convex (complex) Banach space with a Gâteaux differentiable norm, and we will prove (Theorem 3.1) that a necessary and sufficient condition for its infinitesimal generator to be the spectral scalar type operator with the spectral family of hermitian bounded linear projectors is that all operators $C(t)$, $t \in \mathbf{R}$ are hermitian operators.

First, we recall some notations and basic notions. Let X be a complex Banach space, and let $\mathcal{B}(X)$ denote the complex Banach algebra of all bounded linear operators on X.

Definition 1.1. A function $C\colon \mathbf{R} \to \mathcal{B}(X)$ ($\mathbf{R} = (-\infty, +\infty)$) satisfying

a) $C(0) = I$ (I – the identity operator on X),

b) $C(t+s) + C(t-s) = 2C(t)C(s)$, $t, s \in \mathbf{R}$

is called a *cosine operator function* on X. It is *strongly continuous* if the vector-valued function $C(t)x$ is strongly continuous on $\mathbf{R}$ for each $x \in X$. If, in addition, there exists a constant M ($M \geqslant 1$) such that $\|C(t)\| \leq M$ for all $t \in \mathbf{R}$, then the strongly continuous cosine operator function $C(t)$ is said to be *bounded*.

Throughout this paper, $C(t)$ is a bounded strongly continuous cosine operator function such that $\|C(t)\| \leq 1$ for all $t \in \mathbf{R}$. The infinitesimal generator A of $C(t)$

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is defined by $Ax = \lim_{t\to 0} 2(C(t)x - x)/(t^2)$ for all $x \in X$ for which the last limit exists. It is known that A is a closed operator with dense domain $D(A)$ in X. The spectrum of operator A is a subset of $(-\infty, 0]$.

Let us consider the function $\langle\cdot,\cdot\rangle\colon X\times X \to \mathbf{R}$ defined by

$$\langle x, y\rangle := \|x\| \cdot \lim_{t\searrow 0} \frac{\|x+ty\| - \|x\|}{t}, \qquad x, y \in X. \tag{1.1}$$

Definition 1.2. The norm of (the normed linear space) X is said to be *Gâteaux differentiable* if $\lim\limits_{t\to 0} \dfrac{\|x+ty\| - \|x\|}{t}$ exists for every $x, y \in X$.

Definition 1.3. The normed linear space X is said to be *strictly convex* if $\|x+y\| = \|x\| + \|y\| \implies x = ay$ for some real $a \geqslant 0$ and all $x, y \in X$.

Definition 1.4. A densely defined closed linear operator B is said to be *hermitian* if

$$\langle x, \pm iBx\rangle = 0, \qquad x \in D(B), \tag{1.2}$$

and if the spectrum of B is real.

It can be shown that the last definition and the following one are equivalent if the norm of X is Gâteaux differentiable.

Definition 1.5. A densely defined linear operator $A\colon X \to X$ is said to be *hermitian* if iA is the infinitesimal generator of a group of isometries.

We will need the following theorem (proved in [5]).

Theorem 1.6. *Let X be a reflexive strictly convex complex Banach space with a Gâteaux differentiable norm, and let $C(t)$ be a bounded strongly continuous cosine operator function on X. If all operators $C(t)$, $t \in \mathbf{R}$ are hermitian, then the residual spectrum of the infinitesimal generator of $C(t)$ is an empty set.*

2. Family of operators F_a, $a \geq 0$

Family F_a, $a \geq 0$ was introduced in [10] as

$$F_a x := \lim_{a\searrow 0} F_{a,a} x, \qquad x \in X, \ \ a \geq 0, \tag{2.1}$$

where

$$F_{a,a}x := \frac{1}{\pi i}\int_0^a du \int_{\alpha+i0}^{\alpha+iu} [\lambda R(\lambda^2, A) + \bar\lambda R(\bar\lambda^2, A)]\, d\lambda, \qquad \lambda = \alpha + iy, \ \ i = \sqrt{-1}.$$

Here the resolvent of A is denoted by $R(\lambda^2, A)$, i.e., $R(\lambda^2, A) = (\lambda^2 I - A)^{-1} \in \mathcal{B}(X)$. This is a family of bounded linear operators for every bounded strongly continuous cosine operator function.

In [5] it is proved that the limit in (2.1) exists for $x \in X$ and $a \geq 0$, and that

$$F_a x = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin at}{t}\right)^2 C(2t)x\, dt = \frac{2a}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^2 C\left(\frac{2t}{a}\right) x\, dt. \tag{2.2}$$

Since $\|C(t)\| \leq 1$ for all $t \in \mathbf{R}$, (2.2) implies that $\|F_a\| \leq a$ for all $a \geq 0$ and that the function $a \mapsto F_a$ is strongly continuous on $[0, +\infty)$.

Let us note some further properties of operators F_a, $a \geq 0$ (proved in [5]):

i) $\lim\limits_{a\to+\infty} \dfrac{F_a x}{a} = x, \quad x \in X.$ (2.3)

ii) $F_a F_b x = F_b F_a x = 2\int_0^a F_u x\,du + (b-a)F_a x, \quad x \in X, \ \ 0 \leq a \leq b.$ (2.4)

iii) $AF_a x = F_a A x$ for all $x \in D(A)$ and (2.5)
$AF_a x = -a^2 F_a x + \int_0^a (6u - 2a) F_u x\,du = \int_0^a [(a-u)u^2]'_u \,dF_u x$ for all $x \in X$. (2.6)

iv) $C(t)F_a x = -\int_0^a [(a-u)\cos ut]'_u \,dF_u x =$
$= \cos at F_a x + \int_0^a [(a-u)\cos ut]''_{uu} F_u x\,du$ for $a \geq 0$, $t \in \mathbf{R}$ and $x \in X$. (2.7)

v) $R(\lambda^2, A)x = 2\displaystyle\int_0^\infty \frac{3u^2 - \lambda^2}{(u^2+\lambda^2)^3} F_u x\,du$ for all $x \in X$ and $\lambda \in \mathbf{C}$, $\operatorname{Re}\lambda > 0$. (2.8)

vi) Define $\tilde{X}_a := \overline{F_a(X)}$ for all $a \geq 0$. Then $\tilde{X}_a \subseteq \tilde{X}_b$ follows from $0 \leq a \leq b$. Moreover, subspaces $\tilde{X}_a$, $a \geq 0$ are invariant relative to A and $C(t)$ for all $t \in \mathbf{R}$.

From (2.3) it follows that $\bigcup_{a\geq 0} \tilde{X}_a$ is dense in X. The operator $f(A)$ is defined on $\bigcup_{a\geq 0} \tilde{X}_a$ by

$$f(A)F_a x := f(-a^2)F_a x + \int_0^a [(a-u)f(-u^2)]''_{uu} F_u x\,du, \quad x \in X, \ \ a \geq 0, \tag{2.9}$$

where $f(-u^2)$ is two times continuously differentiable for $u \geq 0$. (The last assumption ensures the existence of the integral in (2.9)). One readily shows that the definition (2.9) is correct, i.e. that $F_a x = F_a y$ for some $a, b \geq 0$, $x, y \in X$ implies $f(A)F_a x = f(A)F_b y$ (see [8]).

From (2.9) it follows that

$$f(A)F_a x = -\int_0^a [(a-u)f(-u^2)]'_u \,dF_u x, \qquad x \in X, \ \ a \geq 0. \tag{2.10}$$

(The last integral is an abstract Stieltjes integral.)

In order to prove the main result of this paper, we have to formulate and prove some facts. First, let us give the definition (2.1) of operators F_a, $a \geq 0$ in the following form

$$F_a x := \lim_{a\searrow 0} \int_0^a \hat{E}_{u,a} x\,du, \qquad x \in X,$$

where

$$\begin{aligned} \hat{E}_{u,a} x &:= \frac{1}{\pi i}\int_{\alpha+i0}^{\alpha+iu} [\lambda R(\lambda^2, A) + \bar{\lambda} R(\bar{\lambda}^2, A)]x\,d\lambda \\ &= \frac{2}{\pi}\int_0^\infty e^{-at}\frac{\sin ut}{t} C(t)x\,dt, \qquad \lambda = \alpha + iy, \end{aligned} \tag{2.11}$$

for each $u \geq 0$ and $a > 0$, $x \in X$.

Lemma 2.1. *$\lim_{\alpha\searrow 0} \hat{E}_{u,\alpha}(A + u^2 I)x$ exists for every $x \in D(A)$.*

Proof. Since $\|C(t)\| \leq 1$ for all $t \in \mathbf{R}$, by definition (2.11), it is easy to see that operators $\hat{E}_{u,\alpha}$ are bounded. Also, it is easy to see that the function $(u, \alpha) \mapsto \hat{E}_{u,\alpha} x$,

$x \in X$ is r times continuously differentiable on I (in the strong operator topology) for all $r = 1, 2, \ldots$. The set I consists of all (u, α) such that $0 < \varepsilon \le \alpha \le \bar{\alpha}$, $0 \le u \le a$, where $\varepsilon > 0$, $\bar{\alpha} > 0$ and $a > 0$ are arbitrary, but fixed. Also, it is easy to see that for each $x \in X$ and $(u, \alpha) \in I$

$$\frac{\partial \hat{E}_{u,\alpha}}{\partial \alpha} x = -\frac{2}{\pi} \int_0^{+\infty} e^{-at} \sin ut\, C(t)x\, dt, \tag{2.12}$$

$$\frac{\partial \hat{E}_{u,\alpha}}{\partial u} x = \frac{2}{\pi} \int_0^{+\infty} e^{-at} \cos ut\, C(t)x\, dt. \tag{2.13}$$

Using the relation $\dfrac{d^2}{dt^2} C(t)x = AC(t)x = C(t)Ax$, $x \in D(A)$, (2.12) and (2.13), we obtain

$$\frac{\partial \hat{E}_{u,\alpha}}{\partial u}(A + u^2 I)x = -\frac{2}{\pi}\alpha x + \alpha^2 \frac{\partial \hat{E}_{u,\alpha}}{\partial u} x - 2u\alpha \frac{\partial \hat{E}_{u,\alpha}}{\partial \alpha} x, \tag{2.14}$$

$$\frac{\partial \hat{E}_{u,\alpha}}{\partial \alpha}(A + u^2 I)x = -\frac{2}{\pi} u x + \alpha^2 \frac{\partial \hat{E}_{u,\alpha}}{\partial \alpha} x + 2u\alpha \frac{\partial \hat{E}_{u,\alpha}}{\partial u} x, \tag{2.15}$$

for $x \in D(A)$ and $(u, \alpha) \in I$.

From (2.12) and (2.13) for all $x \in X$ and $(u, \alpha) \in I$, we get

$$\left\| \alpha \frac{\partial \hat{E}_{u,\alpha}}{\partial \alpha} x \right\| \le M\|x\| \quad \text{and} \quad \left\| \alpha \frac{\partial \hat{E}_{u,\alpha}}{\partial u} x \right\| \le M\|x\|,$$

where $M = 2/\pi$. Therefore, from (2.15) we obtain

$$\left\| \frac{\partial \hat{E}_{u,\alpha}}{\partial \alpha}(A + u^2 I)x \right\| \le K\|x\|, \quad \text{where the constant } K = M(3a + \bar{\alpha}),$$

for $x \in D(A)$, $(u, \alpha) \in I$. But, then the relation $\hat{E}_{u,\beta}(A + u^2 I)x - \hat{E}_{u,\gamma}(A + u^2 I)x = \displaystyle\int_\beta^\gamma \frac{\partial \hat{E}_{u,\alpha}}{\partial \alpha}(A + u^2 I)x\, d\alpha$ implies $\|(\hat{E}_{u,\beta} - \hat{E}_{u,\gamma})(A + u^2 I)x\| \le K\|x\|\, |\gamma - \beta| \to 0$ $(\beta, \gamma \to 0)$ for all $(u, \beta), (u, \gamma) \in I$, $x \in D(A)$. Hence, $\lim_{\alpha \searrow 0} \hat{E}_{u,\alpha}(A + u^2 I)x$ exists for $x \in D(A)$. This proves the lemma. ∎

Using the relation $(F_b - F_a)^n = n \int_a^b (b-u)^{n-1}\, dF_u$, which is valid for $0 \le a \le b$ and for each $n = 1, 2, \ldots$ (proved in [5]), we easily get

$$e^{it(F_b - F_a)} = I + it \int_a^b e^{it(b-a)}\, dF_u = I - \int_a^b (e^{it(b-a)})'_u\, dF_u, \tag{2.16}$$

for all $0 \le a \le b$ and $t \in \mathbf{R}$.

Let $\psi(u)$ and $\varphi(u)$, $u \in (-\infty, 0]$ be two continuously differentiable (scalar-valued) functions, and let $0 \le a \le b \le c \le d$. Then

$$\int_a^b \varphi(u)\, dF_u \int_c^d \psi(v)\, dF_v = \int_c^d \psi(v)\, dv \int_a^b \varphi(u)\, dF_u, \tag{2.17}$$

where $\int_a^b \varphi(u)\,dF_u$ and $\int_c^d \psi(v)\,dF_v$ are Stieltjes integrals. ((2.17) formally follows from the identity $dF_u\,dF_v = dv\,dF_u$, $0 \le u \le v$.)

Lemma 2.2. *Let $\lambda(u)$ be a two times continuously differentiable function on $[0,a]$, where $a > 0$ is an arbitrary real number. Then*

$$e^{i\int_0^a \lambda(u)\,dF_u} F_a x = -\int_0^a \left[(a-u)e^{i\int_u^a \lambda(t)\,dt}\right]'_u dF_u x \tag{2.18}$$

for each $x \in X$.

Proof. By induction, using (2.16) and (2.17), it can be easily proved that

$$e^{i\sum\limits_{k=0}^{n-1} t_k(F_{u_{k+1}} - F_{u_k})} = I - \sum_{k=0}^{n-1} e^{i\sum\limits_{j=k+1}^{n-1} tj(u_{j+1}-u_j)} \int_{u_k}^{u_{k+1}} \left(e^{it_k(u_{k+1}-u)}\right)'_u dF_u \tag{2.19}$$

for each natural number n, where $u_0 < u_1 < \cdots < u_n$ and t_k $(k = 0, 1, \ldots, n-1)$ are arbitrary numbers. Here, we assume that $\sum_{j=k+1}^{n-1} t_j(u_{j+1} - u_j) = 0$ when $k+1 > n-1$.

Let now $u_0 < u_1 < \cdots < u_n$, $n \in \mathbf{N}$ be a division of the interval $[0,a]$. Then, by setting $t_k = \lambda(u_k)$ in (2.19) and by taking the limit as $\max_{0\le k\le n-1}(u_{k+1} - u_k) \to 0$ on both of its sides, we obtain

$$e^{i\int_0^a \lambda(u)\,dF_u} = I - \int_0^a \left(e^{i\int_u^a \lambda(t)\,dt}\right)'_u dF_u, \tag{2.20}$$

where the integrals in (2.20) converge in the uniform operator topology. From (2.20) and $dF_u F_a = F_u\,du + (a-u)dF_u$ for $0 \le u \le a$ (obtained from (2.4)), we get that relation (2.18) is valid for all $x \in X$, $a > 0$, which proves the lemma. ∎

The following lemma is proved in [5].

Lemma 2.3. *If $C(t)$ is a bounded strongly continuous cosine operator function with the infinitesimal generator A, and if all operators $C(t)$, $t \in \mathbf{R}$ are hermitian, then operators F_a (for each $a \ge 0$) are hermitian.*

Using Lemma 2.2 and Lemma 2.3, we are able to prove the following proposition which gives a property of the operator $f(A)$, defined by (2.10).

Proposition 2.4. *Let $C(t)$ be a bounded strongly continuous cosine function with the infinitesimal generator A, and let all operators $C(t)$, $t \in \mathbf{R}$ be hermitian. If $f(-u^2)$ is a real (two times continuously differentiable) function such that $|f(-u^2)| \le 1$ $(u \ge 0)$, then $\|f(A)\| \le 1$.*

Proof. First, let us remark that for every complex function $f(-v^2)$ which is two times continuously differentiable for $v \ge 0$, and such that $|f(-v^2)| = 1$ for all $v \ge 0$, a real function $\lambda(t)$ such that, for each $a > 0$

$$e^{i\int_u^a \lambda(t)\,dt} = e^{iC} \cdot f(-u^2), \qquad u \in [0,a], \;\; C - \text{a real constant},$$

can be found. Namely, it is sufficient to set $\lambda(u) = -[\varphi_2'(u)\varphi_1(u) - \varphi_2(u)\varphi_1'(u)]$, where $\varphi_1(u) = \operatorname{Re} f(-u^2)$, $\varphi_2(u) = \operatorname{Im} f(-u^2)$. By definition (2.10) of the operator $f(A)$ and (2.18) (Lemma 2.2), this implies that

$$f(A)F_a x = e^{-iC} \cdot e^{i\int_0^a \lambda(u)\, dF_u} F_a x, \tag{2.21}$$

for every $x \in X$, $a \geq 0$.

By assumption, all $C(t)$, $t \in \mathbf{R}$ are hermitian operators. Then, by Lemma 2.3, all operators F_a, $a \geq 0$ are hermitian. Therefore, by Definition 1.5 and (2.21), we conclude that $\|f(A)F_a x\| = \|F_a x\|$ for all $a \geq 0$ and all $x \in X$. This, and the property (2.3) of the operators F_a show that $f(A)$ is an isometric operator from X to X. Since we can repeat the same procedure for the function $1/f(-v^2)$, it follows that $f(A)$ is an isometric operator from X into X.

Now, let $f(-u^2)$ be a function satisfying the assumptions of the Lemma. Let $\varphi_\pm(-u^2) = f(-u^2) \pm i\sqrt{1 - f^2(-u^2)}$, $u \in [0, +\infty)$. Then, functions $\varphi_\pm(-u^2)$ are complex functions (two times continuously differentiable), and such that $|\varphi_\pm(-u^2)| = 1$, thus the operators $\varphi_\pm(A)$ are isometric operators from X into X (as we have just proved). This, and $f(A) = (\varphi_+(A) + \varphi_-(A))/2$ imply $\|f(A)\| \leq 1$, proving the proposition. ∎

Remark. By Proposition 2.4, Cayley transform U of the infinitesimal generator A of a bounded strongly continuous cosine operator function $C(t)$ ($U := (A + iI)(A - iI)^{-1}$) is an isometric operator from X into X, because $UF_a x = -\int_0^a \left[(a-u)\dfrac{u^2 - i}{u^2 + i}\right]_u' dF_u x$, for all $x \in X$ and $a \geq 0$, and $\left|\dfrac{\lambda - i}{\lambda + i}\right| = \left|\dfrac{\lambda + i}{\lambda - i}\right| = 1$ for all real λ.

Using the result of Proposition 2.4, we can now prove the following

Lemma 2.5. *If $C(t)$ is a bounded strongly continuous cosine operator function, and if all $C(t)$, $t \in \mathbf{R}$ are hermitian operators, then the operators $\hat{E}_{a,\alpha}$, $a \geq 0$, $\alpha > 0$, defined by* (2.11) *are uniformly bounded* ($\|\hat{E}_{a,\alpha}\| \geq 1$ *for all $a \geq 0$, and* $\alpha > 0$).

Proof. Let $a, b, \alpha > 0$ be arbitrary, but fixed. Then, by (2.7), we have

$$\begin{aligned} \hat{E}_{a,\alpha} F_b x &= \frac{2}{\pi} \int_0^\infty e^{-\alpha t} \frac{\sin at}{t} C(t) F_b x\, dt \\ &= -\frac{2}{\pi} \int_0^\infty e^{-\alpha t} \frac{\sin at}{t}\, dt \int_0^b [(b-u)\cos ut]_u'\, dF_u x \\ &= -\int_0^b \left[(b-u) \cdot \frac{2}{\pi} \int_0^\infty e^{-\alpha t} \frac{\sin at \cos ut}{t}\, dt\right]_u' dF_u x, \end{aligned} \tag{2.22}$$

for each $x \in X$. Set $f(-u^2) := \dfrac{2}{\pi} \displaystyle\int_0^\infty e^{-\alpha t} \frac{\sin at \cos ut}{t}\, dt$ for $u \in [0, b]$. Clearly, the function $f(-u^2)$ is infinitely differentiable.

Furthermore, since

$$I := \int_0^\infty e^{-\alpha t}\frac{\sin at\cos ut}{t}\,dt = \begin{cases} \frac{1}{2}\operatorname{arctg}\frac{a-u}{\alpha} + \frac{1}{2}\operatorname{arctg}\frac{a+u}{\alpha}, & \text{for } a > u, \\ -\frac{1}{2}\operatorname{arctg}\frac{u-a}{\alpha} + \frac{1}{2}\operatorname{arctg}\frac{a+u}{\alpha}, & \text{for } a < u, \end{cases}$$

and thus $|I| \le \pi/2$, we have $|f(-u^2)| \le 1$ for $u \in [0,b]$. By Proposition 2.4, (2.22) and (2.10), this implies that $\|\hat{E}_{a,\alpha}\| \le 1$ for each $a, \alpha > 0$, proving the lemma. ■

3. A spectral operator

The following theorem is proved in [9].

Theorem A. *Suppose X is a real Banach space. Then the "Riesz representation theorem" holds: Given $\delta \in X^*$ there exists $x_\delta \in X$ such that*

$$\|x_\delta\| = \|\delta\| \quad \text{and} \quad \langle x_\delta, y\rangle = \delta(y) \ \text{ for all } y \in X$$

if and only if X is reflexive with a Gâteaux differentiable norm. Furthermore, x_δ is unique (and the mapping $\delta \mapsto x_\delta$ is continuous from the norm topology on X^* to the weak topology on X) if and only if X is also strictly convex.

Recall, in this paper, X is a complex Banach space. Set

$$(x,y) := \langle x,y\rangle - i\langle x, iy\rangle. \tag{3.1}$$

Then the similar theorem (i.e., Theorem A with $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle$) holds. Hence, if X is a (complex) reflexive strictly convex Banach space with a Gâteaux differentiable norm, then for each fixed $x \in X$, (x,y) is a continuous linear functional in y and $|(x,y)| \le \|x\|\cdot\|y\|$, $x, y \in X$.

We will need the following theorem (proved in [5]).

Theorem B. *A necessary and sufficient condition for the real number $-a_0^2$ to be an eigenvalue of the infinitesimal generator A of a bounded strongly continuous cosine operator function $C(t)$ is that there exists a vector $x_0 \in D(A)$, $x_0 \ne 0$ such that*

$$F_a x_0 = \begin{cases} (a-a_0)x_0, & a > a_0, \\ 0, & 0 \le a \le a_0, \end{cases}$$

where F_a, $a \ge 0$ is the corresponding family defined by (2.1).

Such vector x_0 is said to be an *eigenvector belonging to the eigenvalue* $-a_0^2$.

Now we can prove the main result of this paper.

Theorem 3.1. *Let X be a reflexive strictly convex Banach space with a Gâteaux differentiable norm, and let $C(t)$ be a bounded strongly continuous cosine operator function on X. The infinitesimal generator A of $C(t)$ is a spectral scalar type operator with the spectral family of hermitian bounded linear projectors if and only if all $C(t)$, $t \in \mathbf{R}$ are hermitian operators.*

Proof. Let all $C(t)$, $t \in \mathbf{R}$ be hermitian operators. First, we will prove that for each $a \ge 0$, $\lim_{\alpha\searrow 0}\hat{E}_{a,\alpha}x$ exists for each $x \in X$ (the operators $\hat{E}_{a,\alpha}$ are defined by (2.11)), and that operators defined for each $a \ge 0$ by

$$\hat{E}_a x := \lim_{\alpha\searrow 0}\hat{E}_{a,\alpha}x, \qquad x \in X \tag{3.2}$$

are uniformly bounded and $\|\hat{E}_a\| \le 1$ (for all $a \ge 0$).

By Lemma 2.1, $\lim_{\alpha\searrow 0}\hat{E}_{a,\alpha}(A+a^2I)x$ exists for each $x\in D(A)$. Hence, the claim is valid in the case when $-a^2$ belongs to the resolvent set of A, because in this case the set of all $(A+a^2I)x$, $x\in D(A)$ is X. In the case when $-a^2$ belongs to the continuous spectrum of A, the set $(A+a^2I)[D(A)]$ is dense in X. Therefore, Lemmas 2.1 and 2.5 and the Banach-Steinhaus theorem imply that $\lim_{\alpha\searrow 0}\hat{E}_{a,\alpha}x$ exists for each $x\in X$, the operator $\hat{E}_a$ is bounded and $\|\hat{E}_a\|\le 1$. By Theorem 1.6, the residual spectrum of the operator A is empty. So, it only remains to be shown that $\lim_{\alpha\searrow 0}\hat{E}_{\lambda_0,\alpha}x$ $(=\hat{E}_{\lambda_0}x)$ exists for each $x\in X$ if $-\lambda_0^2$ is an eigenvalue of A.

Let the set of all eigenvectors belonging to the eigenvalue $-\lambda_0^2$ be denoted by L_{λ_0}, and let the set of all $x\in X$ for which $(y,x)=0$ (for all $y\in L_{\lambda_0}$) be denoted by $L_{\lambda_0}^{\perp}$. Clearly, $L_{\lambda_0}^{\perp}$ is a (closed) subspace of X.

By the Remark after Proposition 2.4, the Cayley transform U of the operator A is an isometric operator (from X into X). By the definition (3.1), it follows that

$$(Ux,Uy)=(x,y)\quad\text{for all } x,y\in X. \tag{3.3}$$

It is easy to see that $-\lambda_0^2$ is an eigenvalue of A if and only if $\dfrac{\lambda_0^2-i}{\lambda_0^2+i}$ is an eigenvalue of U, and that L_{λ_0} is the set of all eigenvectors belonging to this eigenvalue. From this and (3.3) it easily follows that the subspace $L_{\lambda_0}^{\perp}$ is invariant relative to the operator U, and thus it is invariant relative to operators A and $C(t)$, $t\in\mathbf{R}$.

Let us show that the set $L_{\lambda_0}+\mathcal{R}_{A+\lambda_0^2I}$ is dense in X (here, $\mathcal{R}_{A+\lambda_0^2I}$ denotes the range of $A+\lambda_0^2I$). If it is not dense, then there is $x_0\in X$, $x_0\neq 0$ such that

$$(x_0,y)=0\quad\text{for all } y\in L_{\lambda_0}+\mathcal{R}_{A+\lambda_0^2I}. \tag{3.4}$$

In particular, $\langle x_0,(A+\lambda_0^2I)x\rangle=0$ for each $x\in D(A)$. Thus, $\langle x_0,(A+\lambda_0^2I)F_ax\rangle=0$ for all $x\in X$ and $a\ge 0$, because $F_ax\in D(A)$. By (2.6)

$$(\lambda_0^2-a^2)\langle x_0,F_ax\rangle-\int_0^a(2a-6u)\langle x_0,F_ux\rangle\,du=0. \tag{3.5}$$

Set $\varphi(a):=\langle x_0,F_ax\rangle$ ($x\in X$ is fixed). Now (3.5) becomes $(\lambda_0^2-a^2)\varphi(a)-\int_0^a(2a-6u)\varphi(u)\,du=0$. From this we see that the function $\varphi(a)$ is infinitely differentiable at a for $a\neq\lambda_0^2$. So, differentiating the last equality two times at a, we get $(\lambda_0^2-a^2)\varphi''(a)=0$. It follows $\varphi(a)=0$ for $0\le a\le\lambda_0$ (because $\varphi(0)=\varphi'(0)=0$).

On the other hand, $\langle x_0,F_ax\rangle=(a-\lambda_0)\langle x_0,x\rangle$ for $a>\lambda_0$, because $\frac{F_a}{a}x\to x$ $(a\to\infty)$. So,

$$\langle x_0,F_ax\rangle=\begin{cases}0, & \text{for } 0\le a\le\lambda_0,\\ (a-\lambda_0)\langle x_0,x\rangle, & \text{for } a>\lambda_0.\end{cases} \tag{3.6}$$

Remark that $\langle x_0,F_a^nx\rangle=(a-\lambda_0^n)\langle x_0,x\rangle$ for $a>\lambda_0$, $n=1,2,\ldots$. For $n=0$ it is obvious, for $n=1,2,\ldots$, it is easy to prove by induction. Since

$$e^{itF_a}x=x+itF_ax-t^2\int_0^a e^{it(a-u)}F_ux\,du, \tag{3.7}$$

we have

$$\langle x_0, e^{itF_a}x\rangle = \langle x_0, x\rangle + t\langle x_0, iF_a x\rangle -$$
$$- t^2\int_0^a \cos t(a-u)\langle x_0, F_u x\rangle\, du - t^2\int_0^a \sin t(a-u)\langle x_0, iF_u x\rangle\, du, \qquad (3.8)$$

because $\langle x, y\rangle$ is continuous and linear in y for each $x \in X$.

Since operators F_a, $a \geq 0$ are hermitian (by Lemma 2.3), from (3.6) and (3.8) we get $\langle x_0, e^{itF_a}x_0\rangle = \|x_0\|^2$, and thus $e^{itF_a}x_0 = x_0$ for all $t \in \mathbf{R}$ and $0 \leq a \leq \lambda_0$, because all e^{itF_a} are isometric operators, and because X is strictly convex. Since $e^{itF_a}x_0 = x_0 + \dfrac{it}{1!}F_a x_0 + \dfrac{i^2t^2}{2!}F_a^2 x_0 + \cdots$, because F_a is a bounded operator, from the last equality, for $0 \leq a \leq \lambda_0$, $t \in \mathbf{R}$, we get $iF_a x_0 + \dfrac{i^2 t}{2}F_a^2 x_0 + \cdots = 0$, thus, by taking the limit as $t \to 0$,

$$F_a x_0 = 0 \quad \text{for } 0 \leq a \leq \lambda_0. \qquad (3.9)$$

From (3.8) and (3.6) for $a > \lambda_0$ we obtain

$$\langle x_0, e^{iyF_a}x_0\rangle = \|x_0\|^2 - t^2\int_{\lambda_0}^a \cos t(a-u)(u-\lambda_0)\|x_0\|^2\, du,$$

because F_a, $a \geq 0$ are hermitian. Since $\int_{\lambda_0}^a (u-\lambda_0)\cos t(a-u)\, du = \frac{1}{t^2}[1-\cos t(a-\lambda_0)]$, from the last equality it follows that

$$\langle x_0, e^{itF_a}x_0\rangle = \|x_0\|^2 \cos t(a-\lambda_0) \quad \text{for } a > \lambda_0 \text{ and } t \in \mathbf{R}.$$

Particularly for $t = \dfrac{2\pi}{a-\lambda_0}$ we have $\langle x_0, e^{i\frac{2\pi}{a-\lambda_0}F_a}x_0\rangle = \|x_0\|^2$. Hence

$$e^{i\frac{2\pi}{a-\lambda_0}F_a}x_0 = x_0 \quad \text{for } a > \lambda_0, \qquad (3.10)$$

because e^{itF_a} is an isometric operator for each $t \in \mathbf{R}$, and because X is strictly convex. By (3.7),

$$x_0 + i\frac{2\pi}{a-\lambda_0}F_a x_0 - \frac{(2\pi)^2}{(a-\lambda_0)^2}\int_0^a e^{i\frac{2\pi}{a-\lambda_0}(a-u)}F_u x_0\, du = x_0.$$

From this, we see that the function $a \mapsto F_a$ is differentiable at a for $a \neq \lambda_0$. Differentiating (3.10) at a we get

$$e^{i\frac{2\pi}{a-\lambda_0}F_a}\left[\frac{F_a'}{a-\lambda_0} - \frac{F_a}{(a-\lambda_0)^2}\right]x_0 = 0.$$

Thus, by (3.10), $\dfrac{F_a' x_0}{a-\lambda_0} - \dfrac{F_a x_0}{(a-\lambda_0)^2} = 0$.

Set $\varphi(a) := f(F_a x_0)$ ($f \in X^*$ is fixed). Then we have $\varphi'(a) = \dfrac{\varphi(a)}{a-\lambda_0}$. By (3.6), this implies $\varphi(a) = (a-\lambda_0)f(x_0)$ for $a > \lambda_0$. So, for $a > \lambda_0$ we have $f(F_a x_0) = (a-\lambda_0)f(x_0)$ for each $f \in X^*$, thus

$$F_a x_0 = (a-\lambda_0)x_0 \quad \text{for } a > \lambda_0. \qquad (3.11)$$

By Theorem B, from (3.9) and (3.11), it follows that x_0 is an eigenvector belonging to the eigenvalue $-\lambda_0^2$ of A. By (3.4), this implies $\langle x_0, x_0\rangle = 0$. Thus $x_0 = 0$, which is a contradiction, finishing the proof that the set $L_{\lambda_0} + \mathcal{R}_{A+\lambda_0^2 I}$ is dense in X.

It is easy to show that $\lim_{\alpha\searrow 0} \hat{E}_{\lambda_0,\alpha}x$ exists for each $x \in L_{\lambda_0}$. Since $L_{\lambda_0} + \mathcal{R}_{A+\lambda_0^2 I}$ is dense in X, Lemmas 2.1 and 2.5, and the Banach-Steinhaus theorem imply that in the case when $-\lambda_0^2$ is an eigenvalue of A, $\lim_{\alpha\searrow 0} \hat{E}_{\lambda_0,\alpha}x$ exists for each $x \in X$. Moreover, the operator $\hat{E}_{\lambda_0}$ is bounded and $\|\hat{E}_{\lambda_0}\| \le 1$. This completes the proof of the claim: For each $a \ge 0$, $\lim_{\alpha\searrow 0} \hat{E}_{a,\alpha}x$ exists for each $x \in X$, and operators $\hat{E}_a$, $a \ge 0$ (defined by (3.2)) are uniformly bounded and $\|\hat{E}_a\| \le 1$ for all $a \ge 0$.

By the Lemma proved in [7, pp. 384–390] from the existence of these operators, and because they are bounded, it follows that for each $a \ge 0$ there is a bounded projector E_a such that

$$E_a E_b = E_b E_a = E_a \quad \text{for } 0 \le a \le b. \tag{3.12}$$

Here, the operator E_a, $a \ge 0$ is defined by

$$\begin{aligned} &E_0 x := 0, && x \in X, \\ &E_a x := \lim_{\beta\searrow 0} \frac{2}{\pi} \int_0^a \frac{\beta}{(a-u)^2+\beta^2} \hat{E}_u x \, du, && x \in X, \quad a > 0. \end{aligned} \tag{3.13}$$

The existence of the limit in (3.13) for all $x \in X$ and $a > 0$ is proved in [10].

Clearly, $\|E_a\| \le 1$, for all $a \ge 0$, because $\|\hat{E}_a x\| \le \|x\|$ for all $u \in [0, a]$. Hence,

$$\langle x, E_a x\rangle \le \|x\|^2, \qquad x \in X, \quad a > 0. \tag{3.14}$$

Let us show $0 \le \langle x, E_a x\rangle$, $x \in X$, $\ a > 0$.

Since all operators $C(t)$, $t \in \mathbf{R}$ are hermitian, it is easy to see that operators E_a, $a \ge 0$ are hermitian; thus e^{itE_a} $(t \in \mathbf{R},\ a > 0)$ are isometries. On the other hand, because operators E_a are bounded, we can write

$$e^{itE_a} x = x + itE_a x + \frac{(it)^2}{2!} E_a^2 x + \cdots .$$

From this, since $E_a^2 = E_a$, $a \ge 0$, we get $e^{itE_a}x = x - E_a x + e^{it} E_a x$. Hence

$$\langle x, e^{itE_a} x\rangle = \langle x, x\rangle - \langle x, E_a x\rangle = \cos t \langle x, E_a x\rangle + \sin t \langle x, iE_a x\rangle.$$

So, $(1-\cos t)\langle x, E_a x\rangle = \|x\|^2 - \langle x, e^{itE_a}x\rangle$, because E_a are hermitian. Since $\|x\|^2 - \langle x, e^{itE_a}x\rangle \ge 0$ (all e^{itE_a} are isometries, thus $\langle x, e^{itE_a}x\rangle \le \|x\|^2$), from the last equality we have $\langle x, E_a x\rangle \ge 0$. This together with (3.14) proves

$$0 \le \langle x, E_a x\rangle \le \|x\|^2, \qquad x \in X, \quad a > 0. \tag{3.15}$$

In a similar way, it can be proved that for $0 \le a \le b$ and $x \in X$

$$\|(E_b - E_a)x\| \le \|x\|, \tag{3.16}$$

$$0 \le \langle x, (E_b - E_a)x\rangle \le \|x\|^2, \tag{3.17}$$

because operators E_a, $a \geq 0$ are bounded and $(E_b - E_a)^n = E_b - E_a$, $a < b$, $n = 1, 2, \ldots$. Furthermore, we note that $\lim_{a\to\infty} E_a x = x$, $x \in X$ (proved in [10]).

If we set $X_a := E_a(X)$, $a \geq 0$, then it is easy to see that X_a is a closed subspace of X, and that X_a is invariant relative to all operators A and $C(t)$, $t \in \mathbf{R}$, because A and $C(t)$ commute with E_a, $a \geq 0$. Set

$\Delta := (-b^2, -a^2)$ and $E_\Delta := E_b - E_a$ for $0 < a < b$,

$\Delta := (-b^2, 0]$ and $E_\Delta := E_b$ for $a = 0$, and

$\Delta := (-\infty, -a^2)$ and $E_\Delta := I - E_a$ for $b = +\infty$.

Using (3.12), it is easy to verify that

$E_{\Delta_1} E_{\Delta_2} = E_{\Delta_2} E_{\Delta_1} = E_{\Delta_1 \cap \Delta_2}$ for every two intervals $\Delta_1, \Delta_2 \subseteq (-\infty, 0]$.

Note that $E_\emptyset = 0$ denotes the operator such that $E_\emptyset x = 0$ for each $x \in X$.

Set $X_\Delta := E_\Delta(X)$ and $X'_\Delta := (I - E_\Delta)(X)$. Since $E_\Delta^2 = E_\Delta$, it follows that

$$x = E_\Delta x \quad \text{for each } x \in X_\Delta, \tag{3.18}$$

and that $E_\Delta x = 0$ for each $x \in X'_\Delta$. So, each $x \in X'_\Delta$ can be written in the form

$$x = x - E_\Delta x. \tag{3.19}$$

It is easy to verify that X_Δ and X'_Δ are closed subspaces of X, invariant relative to the operator A. Let us prove that

$$x_0 \perp X_\Delta \implies x_0 \in X'_\Delta. \tag{3.20}$$

If $x_0 \perp X_\Delta$, i.e. if $\langle x_0, x\rangle = 0$ for all $x \in X_\Delta$, then $\langle x_0, x_0\rangle = \langle x_0, x_0 - E_\Delta x_0 + E_\Delta x_0\rangle = \langle x_0, (I - E_\Delta)x_0\rangle$, because $E_\Delta x_0 \in X_\Delta$. So, $\|x_0\|^2 = \langle x_0, (I - E_\Delta)x_0\rangle$. From this, since $\langle x_0, (I-E_\Delta)x_0\rangle \leq \|x_0\|\,\|(I-E_\Delta)x_0\|$, it follows that $\|x_0\| \leq \|(I-E_\Delta)x_0\|$, which together with $\|(I - E_\Delta)x_0\| \leq \|x_0\|$ proves $\|(I - E_\Delta)x_0\| = \|x_0\|$. Hence, $\langle x_0, (I-E_\Delta)x_0\rangle = \|x_0\|\,\|(I-E_\Delta)x_0\|$. From this, since X is strictly convex, it follows that $x_0 = (I - E_\Delta)x_0$. So, $x_0 \in X'_\Delta$.

Now let Δ be an arbitrary open set in $(-\infty, 0]$ (referring to the relative topology of $(-\infty, 0]$ induced by the topology of $\mathbf{R}$), and let $\Delta = \bigcup_{i\geq 1} \Delta_i$, Δ_i – mutually disjoint intervals in $(-\infty, 0]$. Set

$$X_\Delta^+ := \bigvee_{i\geq 1} X_{\Delta_i} \quad \text{and} \quad X'_\Delta := \bigcap_{i\geq 1} X'_{\Delta_i}.$$

Here, the linear hull of subspaces $X_{\Delta_i} = E_{\Delta_i}(X)$ is denoted by $\bigvee_{i\geq 1} X_{\Delta_i}$. So, $x_0 \in X_\Delta^+$ if and only if x can be written in the form $x = \sum_{i=1}^n x_i$, $x_i \in X_{\Delta_i}$, for some $n \in \mathbf{N}$. If $x \in X_\Delta^+$ then $x = \sum_{i=1}^n x_i = \sum_{i=1}^n E_{\Delta_i} x_i$ (by (3.18)). From this, and $E_{\Delta_i} E_{\Delta_j} = 0$, $E_{\Delta_i}(I - E_{\Delta_i}) = 0$ $(i, j = 1, 2, \ldots, n,\ i \neq j)$, we obtain $E_{\Delta_i} x = E_{\Delta_i} x_i$, thus $x = \sum_{i=1}^n E_{\Delta_i} x$. So, for $x \in X_\Delta^+$,

$$x = \sum_{i=1}^n E_{\Delta_i} x \quad \text{for some } n \in \mathbf{N}. \tag{3.21}$$

Clearly, the linear manifold X_Δ^+, and the closed subspace X'_Δ are invariant relative to the operator A. It is easy to see that

$$x_0 \perp X_\Delta^+ \implies x_0 \in X'_\Delta. \tag{3.22}$$

The proof is similar to the one of the statement (3.20).

From (3.22) it follows that $x_0 \perp X_\Delta^+$ and $x_0 \perp X'_\Delta$ imply $x_0 = 0$, thus $X_\Delta^+ + X'_\Delta$ is dense in X. Since $X_\Delta^+ \cap X'_\Delta = \{0\}$, the sum $X_\Delta^+ + X'_\Delta$ is direct, denoted by $X_\Delta^+ \dot{+} X'_\Delta$. Let us define the operator E_Δ on the linear manifold $X_\Delta^+ \dot{+} X'_\Delta$ in the following way: Every $x \in X_\Delta^+ \dot{+} X'_\Delta$ can be written in a unique way in the form

$$x = x_\Delta + x'_\Delta, \qquad x_\Delta \in X_\Delta^+, \quad x'_\Delta \in X'_\Delta,$$

so, we can define $E_\Delta x := x_\Delta$. By (3.21), $E_\Delta x = \big(\sum_{i=1}^n E_{\Delta_i}\big)x$. The operator $\sum_{i=1}^n E_{\Delta_i}$ is a hermitian projector, because operators E_{Δ_i} are hermitian and $E_{\Delta_i} E_{\Delta_j} = 0$, $i \neq j$. Hence $E_\Delta^2 x = E_\Delta x$ and $\|E_\Delta x\| \le \|x\|$ $(x \in X_\Delta^+ \dot{+} X'_\Delta)$.

Now, we see that E_Δ can be extended to the hermitian projector E_Δ defined on the whole X. Clearly, $E_\Delta x = x$ for $x \in X_\Delta^+$ and $E_\Delta x = 0$ for $x \in X'_\Delta$. Hence $E_\Delta x = x$ for $x \in X_\Delta := \overline{X}_\Delta^+$. Since $X_\Delta^+ \dot{+} X'_\Delta$ is dense in X, and X'_Δ is closed, it follows that $X = X_\Delta + X'_\Delta$. From $E_\Delta x = x$ for $x \in X_\Delta$ and $E_\Delta x = 0$ for $x \in X'_\Delta$, it follows that $X = X_\Delta \dot{+} X'_\Delta$.

It is easy to see that $E_\Delta = \sum_i E_{\Delta_i}$ in the strong operator topology on X. This is obvious if the number of intervals Δ_i is finite. Let the number of intervals Δ_i be infinite. Any vector $x \in X$ can be written in the form

$$x = \lim_{n\to\infty} (\tilde{x}_n + x'_n), \quad \tilde{x}_n \in X_\Delta, \ x'_n \in X'_\Delta, \quad \text{where } \tilde{x}_n = \sum_{i=1}^n x_i^{(n)}, \ x_i^{(n)} \in X_{\Delta_i}.$$

Then,

$$\begin{aligned} E_\Delta x &= \lim_{n\to\infty} \tilde{x}_n = \lim_{n\to\infty} \sum_{i=1}^n x_i^{(n)} = \lim_{n\to\infty} \sum_{i=1}^n E_{\Delta_i} x_i^{(n)} \\ &= \lim_{n\to\infty} \sum_{i=1}^n E_{\Delta_i} \tilde{x}_n = \lim_{n\to\infty} \Big(\sum_{i=1}^n E_{\Delta_i}\Big) x = \sum_{i\geqslant 1} E_{\Delta_i} x. \end{aligned}$$

Let now Δ and $\tilde{\Delta}$ be open sets in $(-\infty, 0]$. If $\Delta \subseteq \tilde{\Delta}$, and if $X = X_\Delta \dot{+} X'_\Delta$, $X = X_{\tilde{\Delta}} \dot{+} X'_{\tilde{\Delta}}$ are corresponding decompositions of the space X, then $X_\Delta \subseteq X_{\tilde{\Delta}}$ and $X'_\Delta \supseteq X'_{\tilde{\Delta}}$. This easily follows from the fact that the claim holds in the case when Δ and $\tilde{\Delta}$ are intervals. (In that case, thew claim follows from the fact that $E_\Delta E_{\tilde{\Delta}} = E_\Delta$ for $\Delta \subseteq \tilde{\Delta}$.) Furthermore, for open sets Δ and $\tilde{\Delta}$ (in $(-\infty, 0]$), it follows $E_\Delta E_{\tilde{\Delta}} = E_{\tilde{\Delta}} E_\Delta = E_\Delta$ if $\Delta \subseteq \tilde{\Delta}$. This can be written in the form $E_\Delta \le E_{\tilde{\Delta}}$. We saw that $X_\Delta = \overline{\bigvee_i X_{\Delta_i}}$ (by definition) in the case when Δ is an open set such that $\Delta = \bigcup_i \Delta_i$, Δ_i – mutually disjoint intervals. It is easy to see that the same is valid in the case when Δ_i are open sets. If $\Delta' \subseteq (-\infty, 0]$ is a closed set, then we can define

$E_{\Delta'} := I - E_\Delta$, where $\Delta = (-\infty, 0] \setminus \Delta'$ is an open set in $(-\infty, 0]$.

Hence, if $X_\Delta \dot{+} X'_\Delta$ is the decomposition of X corresponding to the open set Δ (in $(-\infty, 0]$), then

$E_{\Delta'} x = x$ for $x \in X_{\Delta'}$ and $E_{\Delta'} x = 0$ for $x \in X_\Delta$.

So, for each open or closed set δ in $(-\infty, 0]$ there is the hermitian projector E_δ.

Let Δ_i be arbitrary mutually disjoint open sets (in $(-\infty, 0]$) and $\Delta = \bigcup_i \Delta_i$. We have already proved that $E_\Delta = \sum_i E_{\Delta_i}$ in the strong operator topology on X, in the case when Δ_i are intervals. But, that fact has played no role in the mentioned proof. Hence, $E_\Delta = \sum_i E_{\Delta_i}$ holds in the case when Δ_i are arbitrary open sets. If Δ, $\tilde{\Delta}$ are open sets in $(-\infty, 0]$, then it is easy to see that $X_{\Delta\cap\tilde{\Delta}} = X_\Delta \cap X_{\tilde{\Delta}}$. Thus

$$E_{\Delta\cap\tilde{\Delta}} = E_\Delta E_{\tilde{\Delta}}. \tag{3.23}$$

It is easy to prove that

$$E_{\Delta\cup\tilde{\Delta}} = E_\Delta + E_{\tilde{\Delta}} - E_{\Delta\cap\tilde{\Delta}} \tag{3.24}$$

for all open sets Δ and $\tilde{\Delta}$. Namely we have

$$X_{\Delta\cup\tilde{\Delta}} = E_{\Delta\cup\tilde{\Delta}}(X) = (E_\Delta - E_{\Delta\cap\tilde{\Delta}})(X)\dot{+}(E_{\tilde{\Delta}} - E_{\Delta\cap\tilde{\Delta}})(X)\dot{+}E_{\Delta\cap\tilde{\Delta}}(X),$$

which implies (3.24).

If Δ, $\tilde{\Delta}$ are closed sets in $(-\infty, 0]$, then (3.23) and (3.24) hold, too. Indeed, if $\Delta = (-\infty, 0] \setminus G$, $\tilde{\Delta} = (-\infty, 0] \setminus \tilde{G}$, where G and $\tilde{G}$ are open sets in $(-\infty, 0]$, then

$$E_{\Delta\cap\tilde{\Delta}} = I - E_{G\cup\tilde{G}} = I - E_G - E_{\tilde{G}} + E_{G\cap\tilde{G}} = (I - E_G)(I - E_{\tilde{G}}) = E_\Delta E_{\tilde{\Delta}}.$$

The relation (3.24) can be proved in a similar way.

Furthermore, by the definition of the operator E_a, $a \geq 0$, we easily get $AE_a x = E_a Ax$, $x \in D(A)$, $a \geq 0$, thus $AE_{\Delta_i} x = E_{\Delta_i} Ax$, $x \in D(A)$, Δ_i – an open interval. From this, and from $E_\Delta = \sum_i E_{\Delta_i}$, $\Delta = \bigcup_i \Delta_i$, Δ_i – mutually disjoint open intervals, and since the operator A is closed, it follows that $AE_\Delta x = E_\Delta Ax$, $x \in D(A)$, Δ – an open set. It is obvious that the last equality holds in the case when Δ is a closed set in $(-\infty, 0]$, too.

Further, the resolvent $R(\lambda^2, A)$ of A can now be written in the form

$$R(\lambda^2, A)x = \int_0^\infty \frac{1}{\lambda^2 + u^2}\, dE_u x$$

(obtained by (2.8), because $F_a x = \int_0^a E_u x\, du$, thus $dF_a x = E_a x\, da$). From this, we obtain

$$R(\lambda^2, A)(E_b - E_a)x = \int_a^b \frac{1}{\lambda^2 + u^2}\, dE_u x, \quad \text{for every } 0 < a < b.$$

According to the previous notices: $\Delta = (-b^2, -a^2)$, $E_\Delta = E_b - E_a$, $I = (a, b)$,

$$R(\lambda^2, A)E_\Delta x = \int_I \frac{1}{\lambda^2 + u^2}\, dE_u x.$$

The same holds for any open set $\Delta = \bigcup_i \Delta_i$, Δ_i – open intervals in $(-\infty, 0]$, where the corresponding open set in $[0, +\infty)$ is denoted by I. This also holds when Δ is a closed set in $(-\infty, 0]$, because it is the complement of the open set in $(-\infty, 0]$ with respect to $(-\infty, 0]$. From this relation, it readily follows that the spectrum of the operator A_Δ is a subset of $\overline{\Delta}$, where A_Δ denotes the restriction of the operator A on the subspace $E_\Delta(X)$.

In the definition of the projector E_Δ, $\Delta = \bigcup_i \Delta_i$, and in the proofs of its properties, the fact that Δ_i are open intervals has played no essential role. Starting from open or closed sets in $(-\infty, 0]$ instead from open intervals, we define the projectors E_Δ for sets Δ in $(-\infty, 0]$ which are unions of a finite or infinite countable number of open or closed sets in $(-\infty, 0]$, and for their complements, too. All properties of the operator E_Δ hold for these sets, as well. By repeating this procedure we can verify that for each Borel set Δ in $(-\infty, 0]$ there is the projector E_Δ with earlier described properties.

Let us assume that A is a spectral scalar type operator with the spectral family E_u, $u \geq 0$ consisting of hermitian bounded linear projectors. Then, for all $x \in X$, and for $\lambda \in \mathbf{C} \setminus (-\infty, 0]$,

$$R(\lambda^2, A)x = \int_0^\infty \frac{1}{\lambda^2 + u^2}\, dE_u x = -\int_0^\infty \left(\frac{1}{\lambda^2 + u^2}\right)' E_u x\, du.$$

By the definition of operators F_a, $a \geq 0$ we get

$$F_a x = \int_0^a E_u x\, du, \qquad x \in X, \quad a \geq 0,$$

so, we conclude that the operators F_a, $a \geq 0$ are hermitian, and thus by (2.7), all operators $C(t)$, $t \in \mathbf{R}$ are hermitian. The Theorem is proved. ■

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