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Hilbert's Space Filling Curves and Geodesic Laminations

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Abstract

We present a modified version of the classical Hilbert's space filling curve and we associate to this curve a geodesic lamination on the disk together with a transversal measure. The lamination helps us to understand how the points of the interval are mapped to the square. We generalize this construction to space filling curves from the interval to the regular n-gon.

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1 Introduction

The first space filling curve was introduced by Peano [6] in 1890. The drawing of this curve was shown in [7]. During the following year Hilbert [3] published another example of a space filling curve. Later other curves were introduced, among others by Lebesgue [5], Sierpiński [12], Schoenberg [11]. All these examples are based on the representation of the numbers on the interval in some integer base and an iterated function system on the plane. Hilbert's curve was done using base 4 and Peano's in base 9. The digits of the representation of the numbers are used to know the order in which to apply the maps of the iterated function system. More recently, space filling curves have been studied among others [8, 9, 10], and references within. In [10] is described the history of the space filling curves. In a different context, in [1] another type of space filling curve was introduced. This curve has a rich dynamical behaviour. And it has been studied later in [13, 14]. In [13] the author constructed a geodesic lamination on the disc associated to the space filling curve defined in [1]. This lamination helps us to understand the geometry and the dynamics involved.

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We would like to associate a geodesic lamination to a space filling curve such that points joined by geodesics are mapped to the same point in the plane by the space filling curve. In the classical examples it is not possible to associate a geodesic lamination to the space filling curve with these properties as we shall explain. Therefore we modified Hilbert's classical space filling curve. We associate to this new curve a geodesic lamination on the disk with the desired property. This lamination comes with a transverse measure, which helps us to understand the geometry of the space filling curve. The symmetries of the lamination are the symmetries of the square, i.e. are given by the dihedral group D_4. The results are summarized in Theorem 1.

Geodesic laminations on the disk have been studied previously in different contexts. In [4] and references within, geodesic laminations are used in the study of quadratic Julia sets. In [13, 15] geodesic laminations turn up as geometric models of some type of symbolic dynamics.

In section 4 the iterated function system on the square is changed, so that the squares are visited in a different order from the clock-wise or anti clock-wise in Hilbert's modified curve. We obtain a different curve, from which we get a different lamination, sharing the same properties as the previous one. However it has fewer symmetries. We also explain that essentially these two laminations are the only laminations associated to this type of curve on the square.

The constructions shown in this paper can be generalized to higher dimensions.

In the last section we define, in a similar way, space filling curves from the interval to the regular n-gon. In this case we can also construct geodesic laminations with similar properties. However there are some differences between the case n even and odd.

2 Revised version of the Hilbert's curve

We shall describe the construction of Hilbert's classical space filling curve, via iterated function systems (IFS), as is done in [10].

Definition 2.1 ([2], page 80) *An* iterated function system or IFS *consists of a complete metric space X together with a finite set of contraction mappings.*

The contraction mappings induce a map on the space of all compact subsets of X. We consider this space provided with the Hausdorff metric, here the induced map is a contraction. Its fixed point is called *the attractor of the IFS.*

Let $\{H_0, H_1, H_2, H_3\}$ be an IFS on $\mathcal{R} = \{x + iy \in \mathbb{C} \,|\, 0 \le x, y \le 1\}$, where

$$H_0(z) = \tfrac{\overline{z}i}{2}, \quad H_1(z) = \tfrac{z}{2} + \tfrac{i}{2}, \quad H_2(z) = \tfrac{z}{2} + \tfrac{1+i}{2}, \quad H_3(z) = -\tfrac{\overline{z}i}{2} + \tfrac{i}{2} + 1.$$

Let $\{h_0, h_1, h_2, h_3\}$ be an IFS on $I = [0, 1]$, where $h_k(t) = t/4 + k/4$, for $0 \le k \le 3$. The attractor of the former IFS is the square $\mathcal{R}$ and of the latter is the interval I. This IFS on the interval defines the numeration system base 4, i.e. $t = \sum_{n=1}^{\infty} a_n/4^n$, with $0 \le a_n \le 3$ if and only if $t = \lim_{n\to\infty} h_{a_1} \cdots h_{a_n}(I)$.

The space filling curve $\xi_H : I \to \mathcal{R}$ is defined as

$$\xi_H(t) = \lim_{n\to\infty} H_{a_1} H_{a_2} \cdots H_{a_n}(\mathcal{R}),$$

where $t = \sum_{n=1}^{\infty} a_n/4^n$. This map is continuous and surjective.

We will consider a modified version of this curve: Let $\{H_0, H_1, H_2, H_3\}$ be an IFS on $\mathcal{R}$ and $\{h_0, h_1, h_2, h_3\}$ be an IFS on $I = [0, 1]$, where

$$H_0(z) = -\tfrac{z}{2} + \tfrac{1+i}{2}, \quad H_1(z) = \tfrac{iz}{2} + \tfrac{1+i}{2}, \quad H_2(z) = \tfrac{z}{2} + \tfrac{1+i}{2}, \quad H_3(z) = \tfrac{-zi}{2} + \tfrac{1+i}{2}.$$

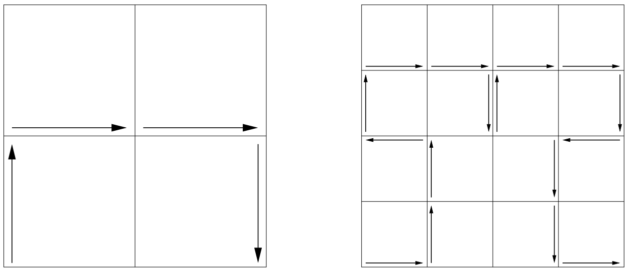

Figure 1: The classical Hilbert's curve.

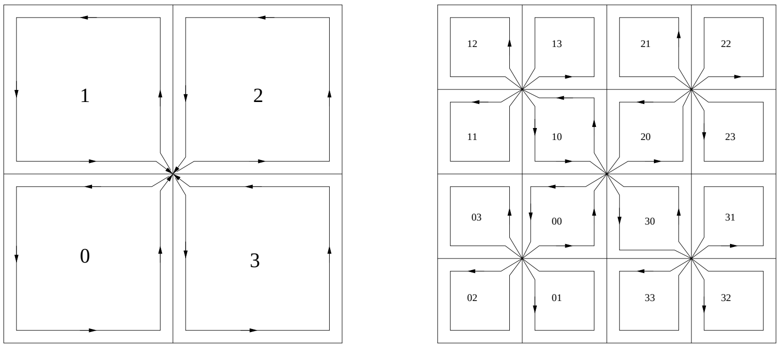

Figure 2: The modified Hilbert's curve.

And

$$h_0(t) = \begin{cases} \frac{t}{4} + \frac{7}{32} & \text{if} \quad 0 \le t < \frac{1}{8} \\ \frac{t}{4} - \frac{1}{32} & \text{if} \quad \frac{1}{8} \le t < 1, \end{cases} \qquad h_1(t) = \begin{cases} \frac{t}{4} + \frac{15}{32} & \text{if} \quad 0 \le t < \frac{1}{8} \\ \frac{t}{4} + \frac{7}{32} & \text{if} \quad \frac{1}{8} \le t < 1, \end{cases}$$

$$h_2(t) = \begin{cases} \frac{t}{4} + \frac{23}{32} & \text{if} \quad 0 \le t < \frac{1}{8} \\ \frac{t}{4} + \frac{15}{32} & \text{if} \quad \frac{1}{8} \le t < 1, \end{cases} \qquad h_3(t) = \begin{cases} \frac{t}{4} + \frac{31}{32} & \text{if} \quad 0 \le t < \frac{1}{8} \\ \frac{t}{4} + \frac{23}{32} & \text{if} \quad \frac{1}{8} \le t < 1. \end{cases}$$

Note that $h_k(t) = h_0(t) + k/4$ for $k = 1, 2, 3$. We will denote by $I_k = [k/4, (k+1)/4) = h_k(I)$, for $k = 0, 1, 2, 3$, these are the intervals defined by the IFS.

And the space filling curve $\xi_M : I \to \mathcal{R}$ is defined in a similar way, as the classical curve: $\xi_M(t) = \lim_{n\to\infty} H_{a_1} H_{a_2} \cdots H_{a_n}(\mathcal{R})$, where $t = \lim_{n\to\infty} h_{a_1} h_{a_2} \cdots h_{a_n}(I)$. Since each h_k and H_k is a contraction, t and $\xi_M(t)$ are well defined. In a similar way to the classical case we can prove that ξ_M is continuous and surjective. Furthermore, it can be proved that it is Hölder continuous, with exponent $1/2$.

One of the main differences in these two curves is that in the classical version $\xi_H(0) = 0$ and $\xi_H(1) = 1$. And in the revised version the images of the points $0, 1/4, 1/2, 3/4, 1$, i.e. the extremities of the intervals that defined the IFS on the interval, are the same point: $(1+i)/2$, the geometrical centre of the square $\mathcal{R}$.

3 Geodesic lamination on the disk

Let $\mathbb{D}^2$ be the closed unit disk in the plane, and $\mathbb{S}^1$ its boundary. We identify $\mathbb{S}^1$ with $I = [0, 1)$. Since the image of 0 and 1 are the same under the maps of the IFS on the interval. We think of the IFS: $\{h_0, h_1, h_2, h_3\}$ as acting on the boundary of the disk.

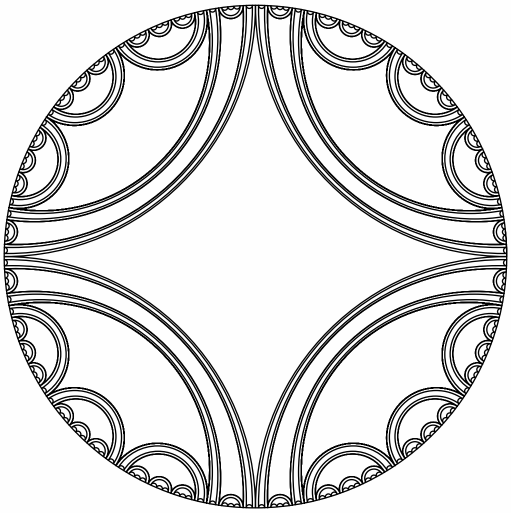

Figure 3: The geodesic lamination Λ.

The construction of the geodesic lamination Λ is as follows: We consider the extremities of the intervals defined by the IFS, i.e. $t_k = k/4$ with $k = 0, 1, 2, 3$. And we join pairwise consecutive extremities, i.e. we join t_k with t_j, where $j = k + 1 \pmod 4$ for $0 \le k \le 3$, by arcs of circles that meet the boundary of $\mathbb{S}^1$ perpenticularly. If we think in the hyperbolic disk, these arcs are geodesics there. Therefore we will call these arcs geodesics.

Let $a_1 \ldots a_n$ be a word in the alphabet $\{0, 1, 2, 3\}$. We join by a geodesic the point $h_{a_1} \cdots h_{a_n}(t_k)$ with $h_{a_1} \cdots h_{a_n}(t_j)$, where $j = k + 1 \pmod 4$ for $k = 0, 1, 2, 3$. We do this for all possible words in this alphabet and later we take the closure in the Hausdorff topology of $\mathbb{D}^2$. The elements of $\mathbb{D}^2$ are either geodesics or points in $\mathbb{S}^1$. In the latter case the points are called *degenerate geodesics.*

Definition 3.1 *A* geodesic lamination *on $\mathbb{D}^2$ is a non-empty closed set of geodesics of the disk and that any two of these geodesics do not intersect except at their end points.*

Proposition 3.1 Λ *is a geodesic lamination on $\mathbb{D}^2$.*

Proof: Let $a_1 \cdots a_m$ and $b_1 \cdots b_l$ be two words in the alphabet $\{0, 1, 2, 3\}$. Suppose that there is an intersection between the geodesics that join the images of the t_k's under $h_{a_1} \cdots h_{a_m}$ and $h_{b_1} \cdots h_{b_l}$. Therefore the interiors of $h_{a_1} \cdots h_{a_m} h_{a_{m+1}}(I)$ and $h_{b_1} \cdots h_{b_l} h_{b_{l+1}}(I)$ have non-empty intersection, for some $0 \le a_{m+1}, b_{l+1} \le 3$. This is not possible, unless one of the words: $a_1 \ldots a_m a_{m+1}$, $b_1 \ldots b_l b_{l+1}$ is a sub-word of the other, since the interiors of $h_k(I)$ and $h_j(I)$ are disjoint for $k \ne j$. ∎

Proposition 3.2 *The lamination Λ is invariant under the group of symmetries of the square D_4.*

Proof: Let $R_{1/4} : I \to I$ be the rotation by $1/4$. Since $h_k(t) = R^k_{1/4}(h_0(t))$, for $0 \le k \le 3$. We have that for any t_j and any word $a_1 \cdots a_m$: $R_{1/4}(h_{a_1} \cdots h_{a_m}(t_j)) = h_{b_1} h_{a_2} \cdots h_{a_m}(t_j)$ where $b_1 = a_1 + 1 \pmod 4$. So by the construction of the lamination, it follows that Λ is invariant under the rotation $R_{1/4}$.

The symmetry related to the invariance under the reflection along the horizontal edge is equivalent to the fact if t, t' are joined then $1 - t$, $1 - t'$ are also joined. It follows from $h_k(t_j) = 1 - h_{3-k}(t_{r(j)})$ for $0 \le j, k \le 3$, where $r(0) = 1$, $r(1) = 0$, $r(2) = 3$ and $r(3) = 2$. The invariance under the reflection in the vertical axis comes from the fact: $h_k(t_j) = (2 + k)/4 - h_{k+1}(t_{r(j)})$ for $k = 0, 2$ and $0 \le j \le 3$. The invariance under the reflection in the diagonals is obtained in a similar way. ∎

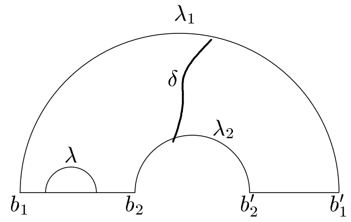

Figure 4: The construction of C_δ.

Proposition 3.3 *Let λ be an element of Λ with end points b and b'. Then $\xi_M(b) = \xi_M(b')$.*

Proof: By the construction of Λ there are geodesics λ_k such that they join the images under the IFS of the t_k's. And they converge to λ.

If the end points of λ_k are b_k and b_k' then $\xi_M(b_k) = \xi_M(b_k')$. So by the continuity of the space filling curve: $\xi_M(b) = \xi_M(b')$. ■

However the converse of Proposition 3.3 is not true. If we take a point in the boundary between $H_k(\mathcal{R})$ and $H_{k+1}(\mathcal{R})$, different from the centre of $\mathcal{R}$: $(1+i)/2$, then its preimages lie in I_k and I_{k+1} so they cannot be joined.

These properties allow us to define a map $\Xi : \Lambda \to \mathcal{R}$ as follows: Let λ be a geodesic of Λ with end points b, b'. So $\Xi(\lambda) := \xi_M(b)$. By Proposition 3.3 this map is well defined. The map Ξ is continuous and surjective since ξ_M has these properties.

Remark: In the classical case we do not end up with a lamination because the four points that are mapped to the central point of the squares are not boundary points of the intervals that define the IFS, i.e. I_j with $j = 0, 1, 2, 3$. So if these points are joined and we iterate the process, we end up with crossings in the geodesics. This situation is also found in the other classical space filling curves: Peano's, Lebesgue's, etc.

3.1 The transverse measure to the lamination

Let δ be any arc in $\mathbb{D}^2$ joining two distinct geodesics of the lamination. It can be slid along the geodesics towards the boundary of the disk according the two possible directions in which the geodesics can be oriented. This procedure gives rise to a Cantor set in the boundary of the disk, say C_δ. More precisely: Let λ_1 and λ_2 be the geodesics in Λ that are joined by an arc δ. This procedure defines two disjoint intervals on the circle $J = [b_1, b_2]$ and $J' = [b_2', b_1']$ where b_k, b_k' are the end points of λ_k for $k = 1, 2$. Let λ be a geodesic in the lamination such that its end points lie on the same interval J or J', we remove from $J \cup J'$ the open interval whose extremities are the end points of λ. See figure 4. The set C_δ is obtained in this way when all the geodesics in Λ with end points in J or J' are considered.

Let δ be any transverse arc to Λ. We define $\mu(\delta) = \mathcal{M}_{s_0}(C_\delta)$ where $\mathcal{M}_{s_0}$ is the s_0-Hausdorff measure and s_0 is the Hausdorff dimension of C_δ.

Proposition 3.4 *For every transverse curve δ to Λ, the Hausdorff dimension of C_δ is $s_0 = 1/2$.*

Before proving this proposition we need to describe the structure of the sets of the form $h_{a_1} \cdots h_{a_n}(I)$, and how $h_{a_1} \cdots h_{a_n} h_{a_{n+1}}(I)$ fits in the previous set. We call *cylinders* all the set of the form $h_{a_1} \cdots h_{a_n}(I)$ for $a_1 \dots a_n$ a word in the alphabet $\{0, 1, 2, 3\}$. In order to understand the structure of the cylinders, we give the following model for them:

Let

$$0 \le x_1 < y_1 < y_2 < x_2 \le x_3 < y_3 < y_4 < x_4 \le 1$$

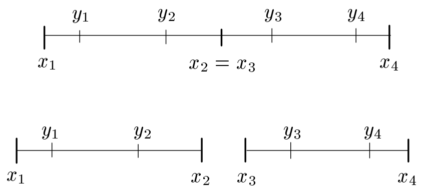

Figure 5: Different type of cylinders.

with the relations

$$\frac{|y_2 - y_1|}{|x_2 - x_1| + |x_4 - x_3|} = \frac{1}{4}, \quad \frac{|x_2 - y_2| + |y_3 - x_3|}{|x_2 - x_1| + |x_4 - x_3|} = \frac{1}{4},$$

$$\frac{|y_4 - y_3|}{|x_2 - x_1| + |x_4 - x_3|} = \frac{1}{4}, \quad \frac{|y_1 - x_1| + |y_4 - x_4|}{|x_2 - x_1| + |x_4 - x_3|} = \frac{1}{4}.$$

The extremities of the cylinder associated to any word, say $a_1 \dots a_n$, will be the points x_j's. And the y_j's will be the extremities of its subcylinders, i.e. the cylinders associated to the words $a_1 \dots a_n a_{n+1}$. Let us consider $[x_1, x_2) \cup [x_3, x_4)$, we will see all the cylinders are of this form. See figure 5. There are two possibilities:

- If $x_2 = x_3$ we say that the cylinder is of type 1.
- If $x_2 < x_3$, we say that the cylinder is of type 2.

Starting with the intervals I_1, I_2, I_3, I_4 and using induction one can prove the following fact about the cylinders:

If the cylinder corresponding to the word $a_1 \dots a_n$ is of type 1 then the cylinder corresponding to $a_1 \dots a_n j$ is of type 1 if $j = 1, 2, 3$ and is of type 2 if $j = 0$. And if the cylinder corresponding to the word $a_1 \dots a_n$ is of type 2 then the cylinder corresponding to $a_1 \dots a_n j$ is of type 1 if $j = 1, 3$ and is of type 2 if $j = 0, 2$.

Proposition 3.5 *The lamination Λ is obtained by joining by geodesics the neighbouring extremities of the cylinders, for all the cylinders and then taking the closure, in the Hausdorff topology, of the union of all these geodesics.*

Proof: By joining neighbouring extremities of $[x_1, x_2) \cup [x_3, x_4)$ we mean joining x_2 with x_3 and x_1 with x_4.

Let $h_{a_1} \cdots h_{a_n}(I)$ be a cylinder. Its extremities are of the form: either $h_{a_1} \cdots h_{a_{n-1}}(t_k)$ for some $0 \le k \le 3$, or $h_{a_1} \cdots h_{a_l}(\tilde{t})$, for some $1 \le l \le n$, where $\tilde{t} = 1/8$, the discontinuity point of the maps $h_k : I \to I$; note $h_k(\tilde{t}) = t_k$. Since the neighbouring extremities correspond to images of consecutive t_k's. The neighbouring extremities of a cylinder are joined by geodesics, according to the definition of the lamination Λ.

On the other hand. Given $h_{b_1} \cdots h_{b_n}(t_k)$ and $h_{b_1} \cdots h_{b_n}(t_j)$, with $j = k + 1 \pmod 4$. These points are neighbouring extremities of the cylinder $h_{b_1} \cdots h_{b_n} h_k(I)$. ■

Proof of Proposition 3.4: Without loss of generality we can suppose that the end points of δ are on geodesics that join extreme points of the same cylinder, and we can suppose that this cylinder is of type 2. So the cylinder is of the form $[x_1, x_2) \cup [x_3, x_4)$, which is subdivided into the following cylinders: $[x_1, y_1) \cup [y_4, x_4)$, $[y_2, x_2) \cup [x_3, y_3)$, $[y_1, y_2)$ and $[y_3, y_4)$. In the first step of the construction of the limit set C_δ the intervals $[y_1, y_2)$, $[y_3, y_4)$ are removed (see figure 6).

So C_δ is obtained as intersection of cylinders: $C_\delta = \cap_{j \ge 0} K_j$, where in this case $K_0 = [x_1, x_2) \cup [x_3, x_4)$, $K_1 = K_1^1 \cup K_1^2$, $K_1^1 = [x_1, y1) \cup [y_4, x_4)$ and $K_1^2 = [y_2, x_2) \cup [x_3, y_3)$. Note that K_1^1 and K_1^2

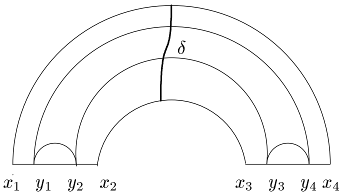

Figure 6: The construction of C_δ.

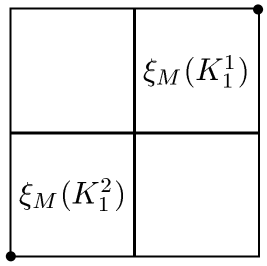

Figure 7: The image of a cylinder under ξ_M and its sub-division.

are cylinders of type 2. We continue the subdivision of these cylinders in the same way as for the parent cylinder. This process can be described in term of an IFS, $\{\varphi_1, \varphi_2\}$ such that $K_1^j = \varphi_j(K_0)$ for $j = 1, 2$. This IFS satisfies the open set condition. Since each φ_j has a contraction factor of $1/4$, the resulting Cantor set has Hausdorff dimension $1/2$. Moreover it can be proved using standard techniques that $0 < \mathcal{M}_{1/2}(C_\delta) < \infty$. ■

Proposition 3.6 *Let δ be any arc transversal to Λ whose end points are in the geodesics λ_1 and λ_2. The image of the set C_δ under ξ is the line segment that joins $\Xi(\lambda_1)$ and $\Xi(\lambda_2)$.*

Proof: Without loss of generality we can suppose that the end points of the arc δ are in the geodesics that join extreme points of a cylinder of type 2. Say $[x_1, x_2) \cup [x_3, x_4)$. See figure 6. As we showed in the proof of Proposition 3.4, $C_\delta = \cap_{j \geq 0} K_j$ where $K_0 = [x_1, x_2) \cup [x_3, x_4)$, $K_1 = K_1^1 \cup K_1^2$, $K_1^1 = [x_1, y1) \cup [y_4, x_4]$ and $K_1^2 = [y_2, x_2) \cup [x_3, y_3)$. The cylinder K_0 is mapped by ξ_M into a sub-square of $\mathcal{R}$. By the definition of the space filling curve $\xi_M(x_1)$ and $\xi_M(x_2)$ are opposite corners of this sub-square.

The cylinder $[x_1, x_2) \cup [x_3, x_4)$ is subdivided into the sub-cylinders: $[x_1, y_1) \cup [y_4, x_4)$, $[y_2, x_2) \cup [x_3, y_3)$, $[y_1, y_2)$ and $[y_3, y_4)$. And the image under ξ_M of each of these cylinders corresponds to a sub-square of $\xi_M(K_0)$. All these squares have the same area and they intersect in only one point, the centre of $\xi_M(K_0)$. So $\xi_M(K_1)$ is the collection of two of these four sub-squares. Then $\xi_M(\cap_{j=0}^n K_j)$ is a collection of small squares whose diagonal are in the line segment that joins $\xi_M(x_1)$ and $\xi_M(x_2)$. So in the limit we get $\xi_M(\cap_{j \geq 0} K_j)$ is the line segment that joins $\xi_M(x_1)$ and $\xi_M(x_2)$. ■

Let $F : \Lambda \to \Lambda$ be a map on the lamination, defined as follows. Let λ be a geodesic on the lamination with end points b and b' (if λ is degenerate $b = b'$). So the image of λ under F is the geodesic that joins $f(b)$ and $f(b')$ where f is the expanding map on the interval defined by the inverses of the maps in the IFS, i.e. $f(t) = h_k^{-1}(t)$ if $t \in I_k$. The continuity of F follows from the continuity of f. On the other hand if $\lambda \in \Lambda$ is such that it joins $h_{a_1} \cdots h_{a_n}(t_k)$ and $h_{a_1} \cdots h_{a_n}(t_j)$, for $n \geq 2$ and $j = k + 1 \pmod 4$, then $F(\lambda)$ joins $h_{a_2} \cdots h_{a_n}(t_k)$ with $h_{a_2} \cdots h_{a_n}(t_j)$. If $n = 1$ then $F(\lambda) = 1/8$, and it can be easily checked that this point is a degenerate geodesic. Therefore $F(\Lambda) \subset \Lambda$.

The domain of F can be extended to the set of equivalence classes of transverse curves to the lamination Λ. Given δ and δ' two transverse curves to Λ, we say that $\delta \sim \delta'$ if the end points of each curve lie in the same pair of distinct geodesics and $C_\delta = C_{\delta'}$. Therefore $\mu(\delta) = \mu(\delta')$. We extend the definition of the map F to the transversal curves to Λ and their equivalence classes. The curve $F(\delta)$ is defined as a curve transversal only to all $F(\lambda)$ where λ are the geodesics transversal to δ. It is clear that this definition is extended to the equivalence classes of transversal curves.

Proposition 3.7 *The map F has the property $F_*\mu = 2\mu$.*

Proof: By definition $F_*(\mu(\delta)) = \mu(F^{-1}(\delta)) = \mathcal{M}_{s_0}(C_{F^{-1}(\delta)})$. On the other hand the Cantor set C_δ is of the form $\cap_{j\geq 0} K_j$ where K_j is a finite union of closed intervals, as shown in the proof of Proposition 3.4. So $f^{-1}(C_\delta) = \cap_{j\geq 0} f^{-1}(K_j)$ and $C_{F^{-1}(\delta)} = f^{-1}(C_\delta)$. Since f is 4 to 1 and each of the branches of f^{-1}, i.e. the maps h_k's, is a contraction with a factor of $1/4$, we have

$$\begin{aligned}\mathcal{M}_{s_0}(C_{F^{-1}(\delta)}) = \mathcal{M}_{s_0}(f^{-1}(C_\delta)) &= \mathcal{M}_{s_0}(\bigcup_{k=0}^{3} h_k(C_\delta)) = \sum_{k=0}^{3} \mathcal{M}_{s_0}(h_k(C_\delta)) = \\ = 4^{1-s_0}\mathcal{M}_{s_0}(C_\delta) = 2\mathcal{M}_{s_0}(C_\delta) &= 2\mu(\delta).\end{aligned}$$

Hence $F_*\mu = 2\mu$. ■

We can summarize the previous results in the following Theorem:

Theorem 1 *There exists a geodesic lamination Λ on the disk, associated to Hilbert's modified space filling curve ξ_M. This lamination has the following properties:*

1. *The end points of each element of Λ are mapped to the same point in the square by the space filling curve ξ_M.*
2. *Λ is invariant under the group of symmetries of the square, the dihedral group D_4.*
3. *The lamination has a transverse measure μ and there is a continuous map $F : \Lambda \to \Lambda$, so that $F_*\mu = 2\mu$.*
4. *For any transverse arc to Λ, there is a limit set on the boundary of the disk, whose Hausdorff dimension is $1/2$. And, the image of this limit set under ξ_M is a straight line between the points, which are the images under ξ_M of the end points of the geodesics, joined by the transverse arc.*

4 Another version of the Hilbert's curve and the corresponding lamination

In this section, we will show a variation of Hilbert's modified space filling curve studied previously in the paper. In a similar way as before, we construct a geodesic lamination associated to this curve, which has the same properties as the previous one. However it has fewer symmetries. It is not invariant under the dihedral group D_4. But it is invariant under rotation by $1/4$.

We consider the IFS, $\{\hat{h}_0, \hat{h}_1, \hat{h}_2, \hat{h}_3\}$ on the interval:

$$\hat{h}_0(t) = \begin{cases} \frac{t}{4} + \frac{1}{64} & \text{if} \quad 0 \leq t < \frac{15}{16} \\ \\ \frac{t}{4} - \frac{15}{64} & \text{if} \quad \frac{15}{16} \leq t < 1, \end{cases}$$

and $\hat{h}_k(t) = \hat{h}_0(t) + k/4$ for $k = 1, 2$, or 3. Let $\{\hat{H}_0, \hat{H}_1, \hat{H}_2, \hat{H}_3\}$ be the IFS on $\mathcal{R}$, defined by: $\hat{H}_0 = H_3$, $\hat{H}_1 = H_1$, $\hat{H}_2 = H_2$ and $\hat{H}_3 = H_0$. Here H_k is the map of the IFS on the square studied previously. But we modified the order in which the squares are visited.

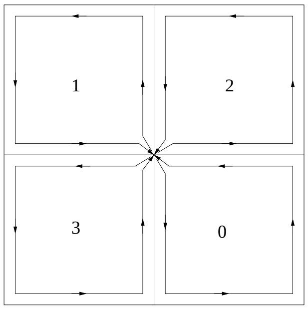

Figure 8: Other version of the Hilbert's modified curve.

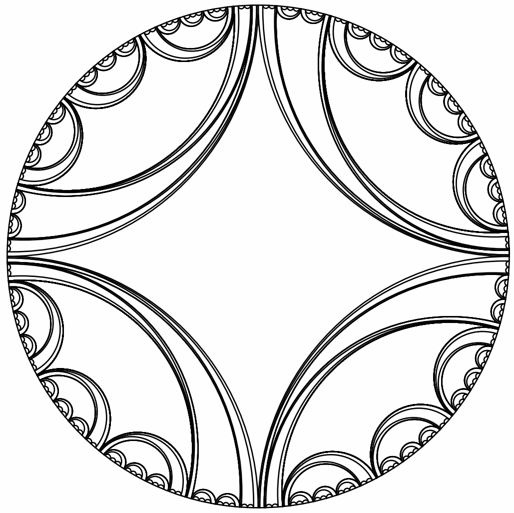

Figure 9: The geodesic lamination associated to the other version of Hilbert's curve.

The space filling curve $\hat{\xi}_M : I \to \mathcal{R}$ is defined in the same way as before:

$$\hat{\xi}_M(t) = \lim_{n\to\infty} \hat{H}_{a_1}\hat{H}_{a_2}\cdots\hat{H}_{a_n}(\mathcal{R}),$$

where $t = \lim_{n\to\infty} \hat{h}_{a_1}\hat{h}_{a_2}\cdots\hat{h}_{a_n}(I)$. This map is continuous surjective and $\hat{\xi}_M(t_j) = (1+i)/2$.

Up to re-labeling of the indices of the maps of the IFS on the square, these two laminations are the only two which support space filling curves on $\mathcal{R}$. This is mainly because the space filling curve should start and finish at the centre of the square, and the curve visits this point four times in total. The square is subdivided into four equal sub-squares. The lamination studied in the previous sections is obtained when the sub-squares are visited in clock-wise (or anti clock-wise) order, as shown in figure 2. We obtain the second lamination when we re-label the indices of the IFS on the plane such that there is a sub-square which is followed by its diagonally opposite subsquare, as happens in the order shown in Figure 8.

In a similar way one can obtain a space filling curve onto the cube in $\mathbb{R}^3$. Here the IFS consists of 8 maps and the corresponding laminations have 8-sided hyperbolic polygons.

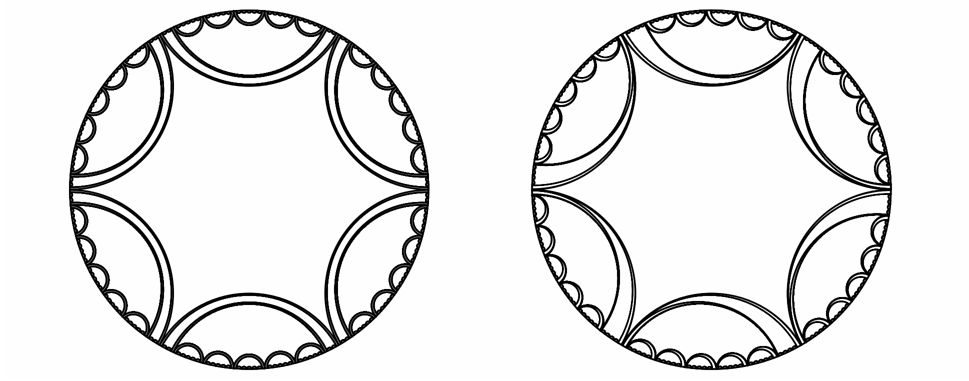

Figure 10: Geodesic laminations associated to space filling curves on the Hexagon.

5 Space filling curves on the n-gon and geodesic laminations

In this section we will generalize the previous constructions of space filling curve and its associated geodesic lamination to space filling curves that map the interval to a regular n-gon. We omit the proofs and computations since they are similar to those given in the previous sections. Let $\mathcal{G}$ be the regular n-gon inscribed in the unit disk and having $z = 1$ as one of its vertices. There are differences between the cases n even and odd.

Let us suppose that n is even. We will consider the following IFS on the regular n-gon: $\{H_k\}_{k=0,\dots,n-1}$, where $H_k(z) = e^{i2k\pi/n}(z+1)/2$. The attractor of this IFS is the n-gon, $\mathcal{G}$. Unlike the IFS studied in the previous sections this IFS, if $n \neq 4$, does not satisfy the open set condition, i.e. $H_j(\mathcal{G})$ and $H_k(\mathcal{G})$ intersect in a set of positive Lebesgue measure, for some $j \neq k$. But it has the important property: the intersection of all $H_k(\mathcal{G})$ is the origin, the geometrical centre of the n-gon. Let $\{h_k\}_{k=0,\dots,n-1}$ be an IFS on the interval I, where

$$h_0(t) = \begin{cases} \frac{t}{n} + \frac{n-1}{2n^2} & \text{if} \quad 0 \le t < \frac{n+1}{2n} \\[2ex] \frac{t}{n} - \frac{n-1}{2n^2} - \frac{1}{n} & \text{if} \quad \frac{n+1}{2n} \le t < 1, \end{cases}$$

and $h_k(t) = h_0(t) + k/n$ for $k = 1, \dots, n-1$. In a similar way, as before, we define the space filling curve as: $\xi_{\mathcal{G}(n)}(t) = \lim_{m\to\infty} H_{a_1} H_{a_2} \cdots H_{a_m}(\mathcal{G})$, where $t = \lim_{m\to\infty} h_{a_1} h_{a_2} \cdots h_{a_m}(I)$. This map is well defined, continuous and surjective. Furthermore the images of the points k/n, i.e. the extremities of $h_k(I)$, is the origin $z = 0$, the center of the n-gon.

As in the previous sections, we can associate a geodesic lamination to this space filling curve. This lamination is invariant under the dihedral group D_n, i.e. it has the same symmetries as the n-gon. We can prove in a similar way as in the previous sections that this lamination has all the properties expounded in Theorem 1. Although the IFS on the n-gon has overlaps, Proposition 3.6 is still valid. This is due to the fact that we eliminate the cylinders that cause overlaps on the n-gon, when the limit set C_δ is constructed. So we end up with sub-gons that do not have overlaps on their interiors. We summarize the properties of this lamination in Theorem 2.

If we re-label the indices of the IFS on the plane so that $H_k(\mathcal{G})$ are arranged anti clock-wise or clock-wise we get the same lamination. However if we arrange the sub-gons $H_k(\mathcal{G})$ such that there is at least one sub-gon which is followed by its diagonally opposite, we obtain a different lamination. In this case we have to define a new IFS on the interval. For instance we use the following IFS on the n-gon, which has the desired property: $\{\hat{H}_k\}_{k=0,\dots,n-1}$, where $\hat{H}_k = H_{r(j)}$ and r is the re-label function on the indices:

$$r(j) = \begin{cases} 0 & \text{if } j = 0 \\ j + 1 - \frac{n}{2} & \text{if } n/2 \le j \le n-1 \\ n - j & \text{if } 1 \le j < n/2. \end{cases}$$

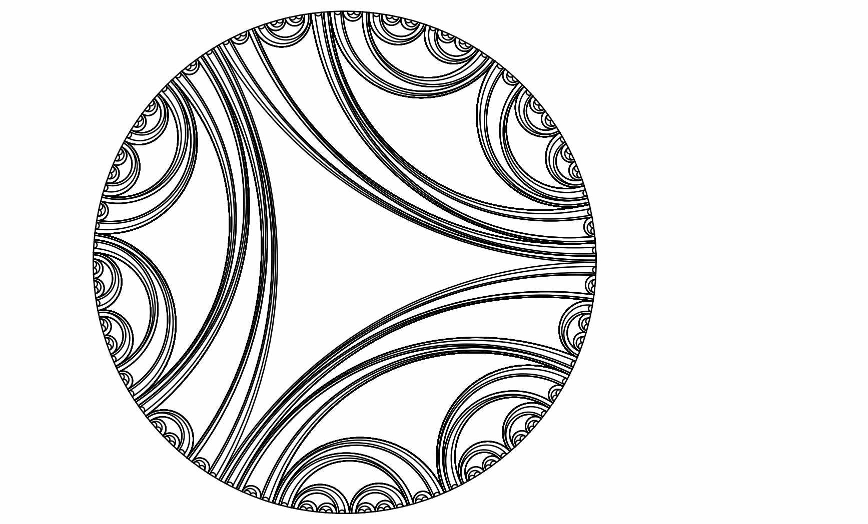

Figure 11: Geodesic lamination associated to a space filling curve on the triangle.

We consider the IFS, $\{\hat{h}_k\}_{k=0,\ldots,n-1}$ on the interval, where

$$\hat{h}_0(t) = \begin{cases} \frac{t}{n} + \frac{n-2}{n^2} + \frac{1}{2n^2} & \text{if} \quad 0 \leq t < \frac{n+1}{2n} \\ \frac{t}{n} - \frac{3}{2n^2} & \text{if} \quad \frac{n+1}{2n} \leq t < 1, \end{cases}$$

and $\hat{h}_k(t) = \hat{h}_0(t) + k/n$ for $k = 1, \ldots, n-1$. In a similar way we obtain a space filling curve $\hat{\xi}_{\mathcal{G}(n)}$and a geodesic lamination $\hat{\Lambda}(n)$. This geodesic lamination has the same properties as the lamination associated to $\xi_{\mathcal{G}(n)}$. However the symmetries are different. This lamination is not invariant under the dihedral group D_n. It is invariant under rotations by $1/n$. In summary we get the following Theorem:

Theorem 2 *Let $\mathcal{G}(n)$ be the regular n-gon, for n even. Let $\xi_{\mathcal{G}(n)}$, $\hat{\xi}_{\mathcal{G}(n)}$ be the space filling curves from the interval to the n-gon, defined above. Then, there are geodesic laminations $\Lambda(n)$ and $\hat{\Lambda}(n)$ on the disk, associated to $\xi_{\mathcal{G}(n)}$ and $\hat{\xi}_{\mathcal{G}(n)}$ respectively. These laminations have the following properties:*

1. *The end points of the elements of $\Lambda(n)$ ($\hat{\Lambda}(n)$) are mapped to the same point in the n-gon by the space filling curve $\xi_{\mathcal{G}(n)}$ ($\hat{\xi}_{\mathcal{G}(n)}$).*
2. *$\Lambda(n)$ is invariant under the group of symmetries of the n-gon, the dihedral group D_n. And $\hat{\Lambda}(n)$ is invariant under the rotation by $1/n$.*
3. *For any transverse arc to $\Lambda(n)$, there is a limit set on the boundary of the disk, whose Hausdorff dimension is $s_0 = \log 2/\log n$. And, the image of this limit set under $\xi_{\mathcal{G}(n)}$ is a straight line between the points, which are the images under $\xi_{\mathcal{G}(n)}$ of the end points of the geodesics, joined by the transverse arc. Similarly for $\hat{\Lambda}(n)$.*
4. *Each lamination has a transverse measure μ and there is a continuous map $F : \Lambda(n) \to \Lambda(n)$ (or $F : \hat{\Lambda}(n) \to \hat{\Lambda}(n)$), so that $F_*\mu = n^{1-s_0}\mu$.*

If n is odd, we consider the following IFS on the plane: $\{H_k\}_{k=0,\ldots,n-1}$, where $H_k(z) = e^{i2k\pi/n}(Lz + 1 - L)$ and $L = 1/(1 + \cos(\pi/n))$. The attractor of this IFS is the n-gon. Here some images of $\mathcal{G}$ under the maps of the IFS, overlap in a set of positive Lebesgue measure. The intersection of all $H_k(\mathcal{G})$ is the origin, the centre of the n-gon $\mathcal{G}$. The IFS on the interval is: $\{h_k\}_{k=0,\ldots,n-1}$, where

$$h_0(t) = \begin{cases} \frac{t}{n} + \frac{1}{n^2} + \frac{1}{n^3} & \text{if} \quad 0 \leq t < \frac{n^2-n-1}{n^2} \\ \frac{t}{n} + \frac{1}{n^2} + \frac{1}{n^3} - \frac{1}{n} & \text{if} \quad \frac{n^2-n-1}{n^2} \leq t < 1, \end{cases}$$

and $h_k(t) = h_0(t) + k/n$ for $n = 0, \ldots, n-1$. We define, as before, a space filling curve with the same properties. And we obtain a geodesic lamination on the disk, whose symmetries are given by rotation by $1/n$. Summarizing we get the following Theorem:

Theorem 3 *Let $\mathcal{G}(n)$ be the regular n-gon, for n odd. Let $\xi_{\mathcal{G}(n)}$ be the space filling curve from the interval to the n-gon, defined above. Then, there is a geodesic laminations $\Lambda(n)$ on the disk, associated to $\xi_{\mathcal{G}(n)}$. This lamination has the following properties:*

1. *The end points of each element of $\Lambda(n)$ are mapped to the same point in the n-gon by the space filling curve $\xi_{\mathcal{G}(n)}$.*
2. *$\Lambda(n)$ is invariant under the rotation by $1/n$.*
3. *For any arc transverse to $\Lambda(n)$, there is a limit set on the boundary of the disk, whose Hausdorff dimension is $s_0 = \log 2/\log n$.*
4. *Each lamination has a transverse measure μ and there is a continuous map $F : \Lambda(n) \to \Lambda(n)$, so that $F_*\mu = n^{1-s_0}\mu$.*

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