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ON THE FREE ENERGY OF A GAUSSIAN DISORDERED MEAN FIELD MODEL

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Abstract. We study the limit of the quenched free energy of a disordered system derived from the Sherrington-Kirpatrick model where the usual ± 1 spins are replaced with Gaussian ones and influenced by an external field. Convergence rates are also given and their optimality discussed.

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1. Introduction

A disordered mean field model consits of a set of n sites whose spins interact randomly with each other regardless of their distance. It can be described via its Hamiltonian which represents the energy for a given configuration of the spins. We focus on Hamiltonians of the following form:

$$H_n(\varepsilon) = -\frac{1}{\sqrt{n}} \sum_{1 \le i < j \le n} g_{ij}\varepsilon_i\varepsilon_j - h\sum_{i=1}^{n}\varepsilon_i, \tag{1}$$

where $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$ is the configuration, a random vector with a given law P_ε, the pair interactions g_{ij} are i.i.d. $\mathcal{N}(0,1)$ random variables independent of ε and $h \in \mathbb{R}$ represents the action of an external field. The (quenched) free energy of such a system involves the inverse temperature β and is defined by

$$F_n(\beta) = \frac{1}{n}\ln \mathcal{E}\exp(-\beta H_n(\varepsilon)), \tag{2}$$

where $\mathcal{E}$ denotes the expectation with respect to P_ε. The physical definition of the free energy usually includes a factor $-1/\beta$ which is omitted here for simplicity. Note that $F_n(\beta)$ is random as a function of the interactions. As an example, the well-known Sherrington-Kirkpatrick model (SK model) has this frame with $\mathrm{P}_\varepsilon = \left(\frac{\delta_{-1}+\delta_1}{2}\right)^{\otimes n}$ and $h = 0$. One of the central questions in this area is to determine the almost sure behavior of $F_n(\beta)$ as the volume n goes to infinity. The main difficulty in the SK-model lies in the fact that the law of ε is not rotationally invariant and the limit of the free energy in this case is not completely established (see [1, 4, 5, 13]). Our aim is to give a rather complete description of $\lim_n F_n(\beta)$ with convergence rates in the simpler situation where $\mathrm{P}_\varepsilon = \mathcal{N}(0,1)^{\otimes n}$.

The paper is organized as follows. In the second section, we recall some known results from random matrix theory that are needed. In the third one, we state and prove the main result. In the last one, we establish the rate of convergence and discuss its optimality. Details are given in Appendix A and B.

2. Some known results on random matrices

Consider the symmetric random matrix $G = (g_{ij})_{1\le i,j\le n}$ obtained from the pair interactions by setting $g_{ii} = 0$ and $g_{ji} = g_{ij}$. The eigenvalues of $G/\sqrt{n}$ are denoted by $\lambda_1 \le \ldots \le \lambda_n$ (the dependence in the integer n is omitted) and we have then $G/\sqrt{n} = U^T \Lambda U$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and U orthogonal (U^T is the transpose matrix of U). From the invariance by an orthonormal change of basis of the law of G, it can be shown that Λ and U are independent and that U is Haar distributed on the orthogonal group (see Appendix A for details). The interested reader will find some other important facts in the books of Mehta [12] and Deift [6].

In this setting (also in a much more general one [3]), the Wigner theorem states that, with probability one, the empirical measure of the eigenvalues

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k} \tag{3}$$

converges weakly to the semi-circle distribution

$$\sigma(dx) = \frac{1}{2\pi} \sqrt{(4-x^2)}\, \mathbb{1}_{|x|\le 2}\, \mathrm{d}x.$$

Recently, Götze and Tikhomirov established in [7] that the convergence rate of the uniform distance between the expectation of the empirical distribution of the eigenvalues and the semi-circle's distribution is at most of order $n^{-1/2}$.

Note that Wigner's theorem immediately implies that $\liminf_n \lambda_n \ge 2$ almost surely. Bai proved in [2] that, under optimal moment conditions on the entries of the matrix that are satisfied here, $\lim_n \lambda_n = 2$ with probability one. A simple symmetry argument gives in addition $\lim_n \lambda_1 = -2$ almost surely. The fluctuation of λ_n is also known. Indeed, the distribution function of $n^{2/3}(\lambda_n - 2)$ converges to the Tracy-Widom one ([14, 15]) defined by

$$F_{\mathrm{TW}}(x) = \exp\left(-\frac{1}{2} \int_x^{\infty} q(t) + (t-x) q^2(t)\, \mathrm{d}t \right) \tag{4}$$

where q is the solution of $y'' = ty + 2y^3$ that is the Painlevé II differential equation, with the Airy function as an asymptotic. The existence and properties of q can be found in [9] where it is shown in particular that q is positive.

We will keep these notations and results in mind for the proofs.

3. Convergence of the free energy

We turn to the first result of the paper. Set for $\beta < 1/2$ and $h \in \mathbb{R}$,

$$F(\beta) = \frac{1}{2} \int \left[\ln\left(\frac{1}{1-\beta x}\right) + \frac{\beta^2 h^2}{1-\beta x} \right] \sigma(\mathrm{d}x).$$

Theorem 1. *With probability one, as n goes to infinity, the free energy $F_n(\beta)$ behaves as follows:*

(i) *If* $\beta < 1/2$, *then* $\lim_{n\to\infty} F_n(\beta) = F(\beta)$.
(ii) *If* $\beta = 1/2$, *then* $\limsup_{n\to\infty} F_n(\beta) = \infty$.
(iii) *If* $\beta > 1/2$, *then* $\lim_{n\to\infty} F_n(\beta) = \infty$.

For $\beta < 1/2$, the limit $F(\beta)$ of the free energy has an explicit form. Indeed, it can be shown (see Appendix B.1) that

$$F(\beta) = \frac{1}{4} - \frac{1}{2}\ln\left(\frac{1+\sqrt{1-4\beta^2}}{2}\right) + \frac{2\beta^2h^2-1}{8\beta^2}\left(1-\sqrt{1-4\beta^2}\right).$$

To have an idea of the shape of $F(\beta)$, note that F is increasing and convex with

$$F(\beta) \sim_0 \beta^2(1+h^2)/2, \qquad \lim_{\beta\to 1/2^-} F(\beta) = \frac{(h^2-1)+2\ln 2}{4},$$

and

$$\lim_{\beta\to 1/2^-} F'(\beta) = \begin{cases} 1 & \text{if } h = 0, \\ \infty & \text{if } h \neq 0. \end{cases}$$

Proof. We start with the proof of (i). Let $\beta < 1/2$ and $c \in]2, 1/\beta[$. It is convenient to introduce the function f defined by

$$f(x) = f_{\beta,c}(x) = \begin{cases} (1-\beta x)^{-1} & \text{if } x \leq c, \\ (1-\beta c)^{-1} & \text{if } x > c, \\ (1+\beta c)^{-1} & \text{if } x < -c. \end{cases} \tag{5}$$

We first put the Hamiltonian (1) into a diagonal form:

$$H_n(\varepsilon) = -\frac{1}{2}(U\varepsilon)^T\Lambda(U\varepsilon) - h\langle \mathbf{1}, \varepsilon\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$ and $\mathbf{1}$ the vector $(1,\ldots,1)$. It follows from the rotational invariance of P_ε that $H_n(\beta)$ is distributed as $-\frac{1}{2}\varepsilon^T\Lambda\varepsilon - h\langle\mathbf{1}, U^T\varepsilon\rangle$ and the free energy (2) becomes

$$F_n(\beta) = \frac{1}{n}\ln \mathcal{E}\exp\left\{\frac{\beta}{2}\,\varepsilon^T\Lambda\varepsilon + \beta h\langle\mathbf{1}, U^T\varepsilon\rangle\right\}. \tag{6}$$

For each ε_k, we have $\mathcal{E}\exp\{a\varepsilon_k^2 + b\varepsilon_k\} = (1-2a)^{-1/2}\exp\{b^2/(2-4a)\}$ if $a < 1/2$ and $b \in \mathbb{R}$. Thus, by independence of the ε_k's, taking suitable a and b, we get from (6) and Bai's theorem that almost surely, for n large enough, $F_n(\beta) = \widetilde{F_n}(\beta)$ where

$$\widetilde{F_n}(\beta) = \frac{1}{2n}\sum_{k=1}^{n}\ln f(\lambda_k) + \frac{\beta^2h^2}{2n}\sum_{k=1}^{n} f(\lambda_k)\left(\sum_{l=1}^{n} u_{kl}\right)^2, \tag{7}$$

and the u_{kl}'s are the entries of U. So, it suffices to work with $\widetilde{F_n}(\beta)$ which is always well defined. We rewrite (7) as

$$\widetilde{F_n}(\beta) = \frac{1}{2n}\sum_{k=1}^{n}\big(\ln f(\lambda_k) + \beta^2 h^2 f(\lambda_k)\big) + \frac{\beta^2 h^2}{2} R_n, \tag{8}$$

where we set

$$R_n = \frac{1}{n}\sum_{k=1}^{n} f(\lambda_k)\Big[\Big(\sum_{l=1}^{n} u_{kl}\Big)^2 - 1\Big]. \tag{9}$$

The first summand in (8) is an integral of a continuous bounded function with respect to the empirical measure σ_n. By Wigner's theorem, this integral converges a.s. to

$$\frac{1}{2}\int \big(\ln f(x) + \beta^2 h^2 f(x)\big)\,\sigma\,(\mathrm{d}x) = F(\beta),$$

since $[-c, c]$ contains the support $[-2, 2]$ of the measure σ.

It remains to prove that R_n converges a.s. to 0. This requires more attention. Let V an orthogonal matrix with $(n^{-1/2}, \dots, n^{-1/2})$ as first column. Since the random matrix UV is Haar distributed on the orthogonal group, its first column $v = (n^{-1/2}\sum_{l=1}^{n} u_{kl})_{1\le k\le n}$ is uniformly distributed on the unit sphere. Let $\xi = (\xi_1, \dots, \xi_n)$ be a standard Gaussian vector in $\mathbb{R}^n$ independent of Λ; it is a well-known property of Gaussian variables that the random vector v has the same distribution as $\xi/\|\xi\|$ where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and that $\|\xi\|$ and $\xi/\|\xi\|$ are independent. Set

$$\widetilde{R}_n = \frac{1}{n}\sum_{k=1}^{n} f(\lambda_k)\left[\frac{n\xi_k^2}{\|\xi\|^2} - 1\right], \quad S_n = \widetilde{R}_n\|\xi\|^2.$$

Note that $\widetilde{R}_n$ is distributed as R_n and for each given Λ, S_n is a two-order Gaussian chaos ([10] chap. 6) based on the variables ξ_k. A standard computation (see Appendix B.2) gives

$$\mathrm{E}R_n^2 = \mathrm{E}\widetilde{R}_n^2 = \frac{2\mathrm{E}\mathrm{Var}_{\sigma_n}(f)}{n+2}, \tag{10}$$

where Var_{σ_n} denotes the variance with respect to σ_n. The moments of a Gaussian chaos are equivalent (see [10] Theorem 5.10). In particular, we have for some universal constant $C_0 > 0$, $\mathrm{E}[S_n^4|\Lambda] \le C_0(\mathrm{E}[S_n^2|\Lambda])^2$, which yields, using again the independence of $\|\xi\|$, $\xi/\|\xi\|$ and Λ together with the equality of $\widetilde{R}_n$ and R_n in distribution,

$$\mathrm{E}\big[R_n^4 \mid \Lambda\big] \le C_0\Big(\mathrm{E}\big[R_n^2 \mid \Lambda\big]\Big)^2, \quad \mathrm{E}R_n^4 \le \frac{4C_0\,\mathrm{E}|\mathrm{Var}_{\sigma_n}(f)|^2}{(n+2)^2}. \tag{11}$$

By Wigner's theorem and dominated convergence, $\mathrm{E}|\mathrm{Var}_{\sigma_n}(f)|^2$ converges to $|\mathrm{Var}_\sigma(f)|^2$ and by Borel-Cantelli's lemma, we obtain for instance that

$$\mathrm{P}(|R_n| \leq n^{-1/8} \text{ eventually}) = 1.$$

This completes the proof of (i).

We turn to the proof of (ii) and show that there exists a tail event of positive probability (and therefore of probability one) contained in $\limsup_n \{\lambda_n \geq 2\}$. To this end, for integers $k < m$, denote by $\lambda_m^{(k)}$ the largest eigenvalue of the symmetric matrix $(m^{-1/2} g_{ij})_{k<i,j\leq m}$. Defined as a supremum, it's easy to see that $\lambda_m^{(k)}$ is non-increasing with k. Then the set

$$\Omega_0 = \bigcap_k \limsup_n \{\lambda_n^{(k)} \geq 2\}$$

is measurable with respect to the tail σ-field $\bigcap_k \sigma(g_{ij}, k < i < j)$ and a subset of $\limsup_n \{\lambda_n \geq 2\}$, since $\lambda_n^{(0)} = \lambda_n$. To prove that $\mathrm{P}(\Omega_0) > 0$, note first that

$$\mathrm{P}(\Omega_0) = \inf_k \mathrm{P}\big(\limsup_n \{\lambda_n^{(k)} \geq 2\}\big) \geq \inf_k \limsup_n \mathrm{P}(\lambda_n^{(k)} \geq 2).$$

Moreover, for each fixed k, the random matrix $(g_{ij})_{k<i,j\leq n}$ is distributed as $(g_{ij})_{0<i,j\leq n-k}$ whose largest eigenvalue is $(n-k)^{1/2}\lambda_{n-k}^{(0)}$ according to the introduced notation. Then, $\lambda_n^{(k)}$ is distributed as $(1-k/n)^{1/2}\lambda_{n-k}^{(0)}$ and

$$\begin{aligned} n^{2/3}(\lambda_n^{(k)} - 2) &\overset{law}{=} n^{2/3}\left[\left(1 - \frac{k}{n}\right)^{1/2} \lambda_{n-k}^{(0)} - 2\right], \\ &= \left(1 - \frac{k}{n}\right)^{1/2} \left[n^{2/3}\big(\lambda_{n-k}^{(0)} - 2\big)\right] + \mathrm{O}(n^{-1/3}). \end{aligned}$$

We deduce that $n^{2/3}(\lambda_n^{(k)} - 2)$ and $n^{2/3}\big(\lambda_{n-k}^{(0)} - 2\big)$ have the same asymptotic distribution which is the Tracy-Widom one. By inspecting (4), we obtain

$$\lim_n \mathrm{P}(\lambda_n^{(k)} \geq 2) = \lim_n \mathrm{P}\big(n^{2/3}(\lambda_n^{(k)} - 2) \geq 0\big) = 1 - F_{\mathrm{TW}}(0) > 0.$$

It follows that $\mathrm{P}(\Omega_0) \geq 1 - F_{\mathrm{TW}}(0) > 0$ and $\mathrm{P}(\limsup_n \{\lambda_n \geq 2\}) = 1$. Besides, as long as λ_n is less than 2, one has from (6)

$$F_n(1/2) = \frac{1}{2n} \sum_{k=1}^n \ln \frac{1}{1 - \lambda_k/2} + \frac{h^2}{8n} \sum_{k=1}^n \frac{1}{1 - \lambda_k/2} \left(\sum_{l=1}^n u_{kl}\right)^2,$$

with an explosion when λ_n reaches 2. By continuity, it is then natural to set $F_n(1/2) = \infty$ if $\lambda_n \geq 2$. But we showed that this last condition occurs infinitely often almost surely. This proves (ii).

Claim (iii) is obvious. □

4. Rate of convergence

In the case where $\beta < 1/2$, we can provide a rate of convergence for $F_n(\beta)$.

Theorem 2. *Assume that $\beta < 1/2$. For all $h \neq 0$, the rate of the a.s. convergence of $F_n(\beta)$ to $F(\beta)$ is at most of order $n^{-1/2} \ln n$. In the case $h = 0$, the rate can be improved to $n^{-1/2}$.*

Proof. We keep the notations introduced in the proof of Theorem 1. Recall that it suffices to work with $\widetilde{F_n}(\beta)$ and note that (8) is also

$$\widetilde{F_n}(\beta) = \mathrm{E}\big[\widetilde{F_n}(\beta)|\Lambda\big] + \frac{\beta^2 h^2}{2} R_n,$$

where R_n is defined by (9) and $\mathrm{E}\big[\,\cdot\,|\Lambda\big]$ corresponds to the partial integration with respect to U since U and Λ are independent.

Step 1: convergence rate of $\mathrm{E}\big[\widetilde{F_n}(\beta)|\Lambda\big]$. Let g be a smooth bounded increasing function with bounded derivative such that on the interval $[-c, c]$,

$$g(x) = \frac{1}{2}\big(\ln f(x) + \beta^2 h^2 f(x)\big), \tag{12}$$

where f is defined by (5). Note that

$$\mathrm{E}\big[\widetilde{F_n}(\beta)|\Lambda\big] = \int g \,\mathrm{d}\sigma_n = \frac{1}{n}\sum_{k=1}^{n} g(\lambda_k) = \frac{\mathrm{Tr}(G/\sqrt{n})}{n}, \quad F(\beta) = \int g\,\mathrm{d}\sigma.$$

Each entry of G satisfy the logarithmic Sobolev inequality with constant one (see [11] chap. 5). Set $\delta_n = |\mathrm{E}\int g\,\mathrm{d}\sigma_n - \int g\,\mathrm{d}\sigma|$. Following the notations of Guionnet-Zeitouni [8], and applying their corollary 1.6.b, we get for each real $\delta \geq \delta_n$

$$\mathrm{P}\left(\Big|\int g\,\mathrm{d}\sigma_n - \int g\,\mathrm{d}\sigma\Big| > \delta\right) \leq 2\exp\left(-\frac{n^2(\delta-\delta_n)^2}{8\|g'\|_\infty}\right). \tag{13}$$

We now estimate δ_n. By integration by parts (cf Appendix B.3),

$$\delta_n = \left|\mathrm{E}\int_{-\infty}^{+\infty} g'(x)\big[\sigma_n\big(]x,\infty[\big) - \sigma\big(]x,+\infty[\big)\big]\,\mathrm{d}x\right|.$$

According to [7] there exist a positive constant C_1 such that for all $n \geq 1$,

$$\sup_x \big|\sigma_n\big(]x,\infty[\big) - \sigma\big(]-\infty,x]\big)\big| \leq \frac{C_1}{n^{1/2}}.$$

Since g is bounded and increasing, δ_n is at most $2C_1 n^{-1/2}\sup|g|$. Then, inequality (13) with $\delta = 2\delta_n$ and Borel-cantelli's lemma together imply

$$\mathrm{P}\left(\exists N,\, \forall n \geq N,\, \left|\mathrm{E}\big[\widetilde{F_n}(\beta)|\Lambda\big] - F(\beta)\right| \leq \frac{2C_1 \sup|g|}{n^{1/2}}\right) = 1,$$

which ends the first step.

Step 2: convergence rate of R_n. It suffices to prove that for some constant $C_2 \geq 1$ independent of n and all $n \geq 1$ and $t > 0$,

$$\mathrm{P}\big(|R_n| > t\big) \leq C_2 \exp\Big(-\frac{t\sqrt{n}}{C_2}\Big). \tag{14}$$

Indeed, inequality (14) shows via Borel-cantelli's lemma that the rate of convergence of R_n to zero is of order $n^{-1/2}\ln n$.

A Gaussian chaos of order two is exponentially integrable (see [10], Theorem 6.7). More precisely, there exists a universal constant C_2' such that

$$\mathrm{E}\left[\exp\left(\frac{|S_n|}{C_2'\sqrt{\mathrm{E}[S_n^2 \,|\, \Lambda]}}\right)\,\middle|\, \Lambda\right] \leq C_2', \tag{15}$$

with $\mathrm{E}[S_n^2 \,|\, \Lambda] = 2n\mathrm{Var}_{\sigma_n}(f)$ (see Equality 18). In addition, we have

$$\mathrm{E}\big[S_n \,\big|\, \Lambda, \xi/\|\xi\|\big] = \mathrm{E}\|\xi\|^2 \widetilde{R}_n = n\widetilde{R}_n$$

and together with Jensen's conditional inequality, (15) gives

$$\mathrm{E}\left[\exp\left(\frac{n|R_n|}{C_2'\sqrt{2n\mathrm{Var}_{\sigma_n}(f)}}\right)\,\middle|\, \Lambda\right] \leq C_2',\ \mathrm{E}\exp\left(\frac{\sqrt{n}|R_n|}{C_2'\sqrt{2\sup f^2}}\right) \leq C_2',$$

which leads to (14) with $C_2 = C_2'\sqrt{2\sup f^2}$ via Markov's inequality. $\square$

In the case $h = 0$ we can not say that the rate is sharp since it is not known whether the rate $n^{-1/2}$ in Götze-Tikhomirov's theorem ([7]) is optimal. On the other hand, in the case where $h \neq 0$, a simple argument shows that it can not be smaller than $\mathrm{O}(n^{-1/2})$. We first show that there exists $n_0 \geq 1$ such that for all $n \geq n_0$,

$$\mathrm{P}(\sqrt{n}\,|R_n| > C_3) > 1/C_3, \tag{16}$$

for some constant $C_3 \geq 1$ independent of n.

By Paley-Zygmund's inequality, together with (10) and (11), we get for all $n \geq 1$

$$\mathrm{P}\left\{R_n^2 > \frac{\mathrm{E}R_n^2}{2}\right\} \geq \frac{\big(\mathrm{E}R_n^2\big)^2}{4\mathrm{E}R_n^4},\ \mathrm{P}\left\{R_n^2 > \frac{\mathrm{E}\mathrm{Var}_{\sigma_n}(f)}{n+2}\right\} \geq \frac{\big(\mathrm{E}\mathrm{Var}_{\sigma_n}(f)\big)^2}{4C_0\mathrm{E}|\mathrm{Var}_{\sigma_n}(f)|^2}.$$

The convergence of $\mathrm{Var}_{\sigma_n}(f)$ to $\mathrm{Var}_\sigma(f)$ in L^1 and L^2 imply that for n large enough, say $n \geq n_1$,

$$\mathrm{P}\big(nR_n^2 > 2\mathrm{Var}_\sigma(f)\big) \geq \frac{1}{8C_0}$$

which yields (16) with $C_3 = \min\big(\sqrt{2\mathrm{Var}_\sigma(f)}, (8C_0)^{-1}\big)$.

Now, suppose that there exits $M > 0$ and a positive (non random) sequence (τ_n) tending to infinity such that

$$\mathrm{P}(\exists n_2 \geq 1, \, \forall n \geq n_2, \, \tau_n \sqrt{n} |R_n| \leq M) = 1.$$

This would imply that

$$\forall \epsilon > 0, \, \exists n_2 \geq 1, \, \forall n \geq n_2, \, \mathrm{P}\big(\tau_n \sqrt{n} |R_n| > M\big) \leq \epsilon. \tag{17}$$

Since τ_n tends to infinity, there exists $n_3 \geq n_0$ such that for all $n \geq n_3$, τ_n is more than M/C_3 and by inequality (16) we get that for all $n \geq n_3$

$$\mathrm{P}\big(\tau_n \sqrt{n} |R_n| > M\big) > \frac{1}{C_3},$$

which refutes assertion (17) for $\epsilon = 1/C_3$. Therefore the convergence rate of R_n can not be better than $\mathrm{O}(n^{-1/2})$.

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Appendix A. Distribution of (Λ, U)

Set $G_n := G/\sqrt{n} = U^T \Lambda U$ with $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and U orthogonal as in section 2. U is almost surely determined if one requires that the first non-vanishing coordinate of each eigenvector is positive. Since G_n can be realized as a difference $W - W^T$ where W is a random matrix with independent identically distributed Gaussian entries, one has that for any orthogonal V, the matrix $V^T G_n V$ is distributed as G_n. Then, for each continuous bounded function h,

$$\mathrm{E}\, h\big(G_n\big) = \int \Big[\mathrm{E}\, h\big(V^T G_n V\big)\Big] \,\mathrm{d}Haar(V),$$

where $Haar$ denotes of course the Haar measure on the (compact) orthogonal group. Given U orthogonal, the product UV is also Haar-distributed (the Haar measure is "translation"-invariant). It follows then from the Fubini theorem that,

$$\begin{aligned} \mathrm{E}\, h\big(G_n\big) &= \mathrm{E}\Big[\int h\big((UV)^T \Lambda UV\big) \,\mathrm{d}Haar(V)\Big] \\ &= \mathrm{E}\Big[\int h\big(V^T \Lambda V\big) \,\mathrm{d}Haar(V)\Big], \end{aligned}$$

which proves that the law of (Λ, U) is the product measure $\mathrm{P}_\Lambda \otimes Haar$ where P_Λ denotes the law of Λ.

Appendix B. Auxillary computations

B.1. Computation of $F(\beta)$.

We only give the main steps as a guide for the reader. Let $s \geq 2\beta$ and consider

$$I(s) = \int_{-2}^{2} \frac{1}{s - \beta x} \frac{\sqrt{4 - x^2}}{2\pi} \,\mathrm{d}x, \quad J(s) = \int_{-2}^{2} \ln\left(\frac{1}{s - \beta x}\right) \frac{\sqrt{4 - x^2}}{2\pi} \,\mathrm{d}x.$$

By two changes of variables $x = 2\sin\theta$ and then $\theta = \arctan u$, we get

$$I(s) = \frac{s}{\pi \beta^2} \int_0^\infty \left[\frac{1}{1 + u^2} - \frac{s^2 - 4\beta^2}{(s^2 - 4\beta^2)u^2 + s^2}\right] \mathrm{d}u = \frac{1}{2\beta^2}\left(s - \sqrt{s^2 - 4\beta^2}\right).$$

Next, we have for $s \geq 2\beta$,

$$J(s) - J(2\beta) = \int_{2\beta}^{s} I(t) \mathrm{d}t = \int_{2\beta}^{s} \frac{1}{2\beta^2}\left(t - \sqrt{t^2 - 4\beta^2}\right) \mathrm{d}t,$$

which, after the change of variable $t = 2\beta \cosh x$, leads to

$$J(s) - J(2\beta) = \int_0^{\operatorname{argcosh}(s/2\beta)} \big(1 - \mathrm{e}^{-2x}\big) \mathrm{d}x.$$

If we set $x = 2\sin\theta$ in $J(2\beta)$, we arrive at

$$J(2\beta) = \ln\beta + \ln 2 + \frac{4}{\pi}\int_0^{\pi/2} \ln(\cos\theta)\cos^2\theta\,\mathrm{d}\theta = \ln\beta + \frac{1}{2},$$

Using then $\mathrm{argcosh} x = \ln(x + \sqrt{x^2-1})$, we finally get

$$J(s) = \frac{s}{4\beta^2}\left(s - \sqrt{s^2 - 4\beta^2}\right) + \ln\left(\frac{s + \sqrt{s^2-4\beta^2}}{2}\right) - \frac{1}{2}.$$

B.2. **Computation of second moment of R_n.**

Note that S_n can be written as

$$S_n = \sqrt{2}\sum_{k=1}^n \Big(f(\lambda_k) - \frac{1}{n}\sum_{l=1}^n f(\lambda_l)\Big)\frac{\xi_k^2 - 1}{\sqrt{2}},$$

and since the $2^{-1/2}(\xi_k^2 - 1)$'s are orthonormal in L^2 and independent of Λ, it is straightforward that

$$\mathrm{E}\big[S_n^2 \mid \Lambda\big] = 2\sum_{k=1}^n \Big(f(\lambda_k) - \frac{1}{n}\sum_{l=1}^n f(\lambda_l)\Big)^2 = 2n\mathrm{Var}_{\sigma_n}(f). \tag{18}$$

Recall that $S_n = \widetilde{R}_n\|\xi\|^2$; because of the independence of $\|\xi\|$, $\xi/\|\xi\|$ and Λ we get from (18)

$$\mathrm{E}\big[\widetilde{R}_n^2 \mid \Lambda\big]\mathrm{E}\|\xi\|^4 = 2n\mathrm{Var}_{\sigma_n}(f),$$

which implies

$$\mathrm{E}\big[R_n^2 \mid \Lambda\big] = \mathrm{E}\big[\widetilde{R}_n^2 \mid \Lambda\big] = \frac{2\mathrm{Var}_{\sigma_n}(f)}{n+2}.$$

B.3. **Integration by parts for δ_n.**

Recall the definition (12) of the function g. Using the change of variables $y = g(x)$ and Fubini's theorem, we obtain

$$\begin{aligned}\int g\,\mathrm{d}\sigma &= -\int_{-\infty}^{+\infty}\int_{\inf g}^{\sup g} \mathbb{1}_{y\geq g(t)}\,\mathrm{d}y\;\sigma(\mathrm{d}t) + \sup g\\ &= -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbb{1}_{x\geq t}\;g'(x)\;\mathrm{d}x\;\sigma(\mathrm{d}t) + \sup g\\ &= -\int_{-\infty}^{+\infty}\sigma\big(]-\infty, x]\big)\;g'(x)\;\mathrm{d}x + \sup g,\end{aligned}$$

and the same result holds for the empirical distribution σ_n. The difference beetween both results gives

$$\delta_n = \left|\mathrm{E}\!\int g\,\mathrm{d}\sigma_n - \int g\,\mathrm{d}\sigma\right| = \left|\mathrm{E}\!\int_{-\infty}^{+\infty} g'(x)\Big[\sigma_n\big(]-\infty, x]\big) - \sigma\big(]-\infty, x]\big)\Big]\mathrm{d}x\right|.$$

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