M P E J

Mathematical Physics Electronic Journal

ISSN 1086-6655
Volume 6, 2000

Paper 6
Received: May 8, 2000, Revised: Nov 29, 2000, Accepted: Dec 6, 2000
Editor: R. de la Llave

CIRCLE PACKING IN THE HYPERBOLIC PLANE

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Abstract. We consider circle packings in the hyperbolic plane, by finitely many congruent circles, which maximize the number of touching pairs. We show that such a packing has all of its centers located on the vertices of a triangulation of the hyperbolic plane by congruent equilateral triangles, provided the diameter d of the circles is such that an equilateral triangle in the hyperbolic plane of side length d has each of its angles is equal to $2\pi/N$ for some $N > 6$.

Research supported in part by Texas ARP Grants 152 and 158.

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By a *circle packing* we will mean a collection of circles, all of some fixed diameter d, in either the hyperbolic or Euclidean plane such that no two circles overlap except possibly in their boundaries.

For the hyperbolic plane there are well known difficulties in making sense of *optimally dense* circle packings; see for instance [1-4]. In this paper we analyze a class of packings with similar properties using an alternative to density. We will call a *finite* packing *optimal* if the number of tangencies between its circles is not less than the number of tangencies for any other circle packing with the same number of circles and the same radius. Heitmann and Radin [5] proved that any optimal packing of the Euclidean plane is such that the set of its centers is contained in an equilateral triangular lattice. In this paper, we prove a similar result in the hyperbolic plane. If the diameter d of an optimal circle packing is such that an equilateral triangle of side length d has angle $\frac{2\pi}{N}$ for some integer $N \geq 7$, then the set of centers of the packing is contained in the vertices of a tessellation of the hyperbolic plane by equilateral triangles (of side length d).

In both the Euclidean and hyperbolic settings these results can be interpreted physically as the determination of the internal structure and Wulff shape for the energy ground state of a model of matter composed of hard disks with contact attraction. A notable difference between the settings is that in hyperbolic space surface tension is not in fact a surface effect but is comparable in magnitude to the bulk energy.

I. Notation and Statement of Results.

From now on, fix an integer $N \geq 7$. Let d be such that an equilateral triangle of side length d has an interior angle equal to $\frac{2\pi}{N}$. Let ρ be the distance function in the hyperbolic plane, $\mathbb{H}^2$. For any subsets X and Y in the plane, let $\rho(X,Y)$ be the infimum over $\rho(x,y)$ where x is an element of X and y is an element of Y.

An *admissible graph* G is a finite geodesic graph in the hyperbolic plane that satisfies these conditions: (a) every pair of distinct vertices of G are at least a distance d apart, and (b) an edge exists between vertices v and w if and only if the distance between v and w is d.

An *optimal graph* is an admissible graph that has at least as many edges as any admissible graph with the same number of vertices.

There is a natural bijection from the set of finite circle packings of diameter d to the set of admissible graphs which restricts to a bijection between optimal packings and optimal graphs. The bijection is given by considering the centers of a circle packing to be the vertices of an admissible graph. Notice that the tangencies of the packing then correspond to the edges of the graph.

For any admissible graph G, let $V(G)$ denote its vertex set, $E(G)$ its edge set, $F(G)$ its face set, and ∂G its boundary, i.e. the subgraph of G that is contained in the closure of the unbounded component of the complement of G in the plane. For any set S, let $|S|$ denote the cardinality of S. For any face f of G, let $A(f)$ be the area of f. For any vertex v in the boundary of G, let $a(v)$ be the angle subtended by v, i.e. the angle interior to the polygon whose boundary is the boundary of G.

We now define the *spiral* graph on n vertices, S_n, as follows. Let T be the graph formed from the tiling of the plane by equilateral triangles with interior angles equal to $2\pi/N$. Let e be an edge of T and let v_1 and v_2 be its endpoints. We now proceed inductively. Assuming v_j has been chosen for $1 < j \leq i$ we let v_{i+1} be the unique vertex of T defined as follows. Order *all* the previously defined vertices v_j adjacent to v_i in the form v_{b_k}, $1 \leq k \leq t$, with $v_{b_1} = v_{i-1}$, v_{b_j} adjacent to $v_{b_{j+1}}$, and the triangle $v_{b_j} v_{b_{j+1}} v_i$ positively oriented. Then let v_{i+1} be the vertex adjacent to v_i and v_{b_t} such that the triangle $v_{b_t} v_{i+1} v_i$ is positively oriented. We define the spiral graph of order n, S_n, to be the admissible graph whose vertex set is $\{v_1, \dots, v_n\}$. Note that this uniquely determines S_n up to congruence.

S_n is easily seen to have the following properties.

a) All the faces of S_n are triangles.

b) There is an m such that the vertex set of the boundary of S_n is $\{v_m, v_{m+1}, \dots, v_n\}$.

c) $4\pi/N \leq a(v_m) \leq 2\pi - 4\pi/N$.

d) If $m < i < n$, then $4\pi/N \leq a(v_i) \leq 6\pi/N$.

e) $2\pi/N \leq a(v_n) \leq 4\pi/N$ and $a(v_m) = 4\pi/N$ if and only $a(v_m) = 2\pi - 4\pi/N$.

The main result follows immediately from

Theorem 1. *Using circles whose radius r is such that an equilateral triangle with side length $2r$ has an interior angle of $2\pi/N$ for some integer $N \geq 7$, then for this radius all spiral graphs are optimal and every face of any optimal graph is triangular.*

II. Proofs.

Before proving Theorem 1 we require two technical lemmas.

Lemma 1. *Let G be an admissable graph. Let $\ell_G = \min\{\rho(e, v) \mid e \in E(G),\ v \in V(G)$ and $v \notin e\}$. Let $\ell = \inf\{\ell_G \mid G$ is an admissable graph$\}$. Then ℓ is the length of the arc depicted in Figure 1:*

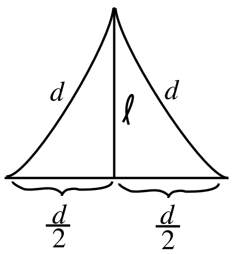


FIGURE 1

Proof. Let G be an admissible graph. Suppose that there exists $(v, e) \in V(G) \times E(G)$ such that $v \notin e$ and $\rho(v, e) < \ell$. Let v_1, v_2 be the endpoints of e. Without loss of generality, assume $\rho(v_1, v) \leq \rho(v_2, v)$. Let $p \in e$ such that $\rho(p, v) \leq \rho(x, v)$ for all $x \in e$. Then $\rho(p, v) < \ell < d < \rho(v, v_2)$. So there exists at point $v_2' \in e$ such that p is in the arc from v_1 to v_2' and $\rho(v, v_2') = \rho(v, v_1)$. See Figure 2. Let m be the midpoint of the arc from v_1 to v_2'.

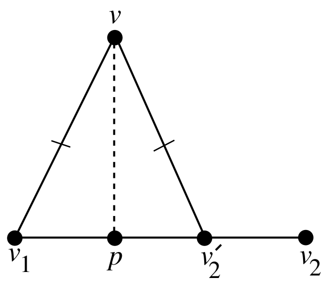


FIGURE 2

Without loss of generality assume p is in the arc from v_1 to m. By symmetry there exists a point p' on the arc from m to v_2' such that $\rho(v, p) = \rho(v, p')$. See Figure 3.

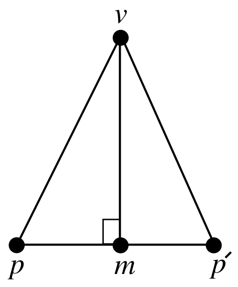


FIGURE 3

By the law of sines,

$$(*)\qquad \begin{aligned} \frac{\sinh\rho(m,v)}{\sin a(vpm)} &= \frac{\sinh\rho(v,p)}{1} \\ \Longrightarrow \quad \frac{\sinh\rho(m,v)}{\sinh\rho(v,p)} &= \sin a(vpm) \leq 1 \\ \Longrightarrow \quad \rho(m,v) &\leq \rho(v,p) \Longrightarrow m = p\ . \end{aligned}$$

By the law of cosines,

$$\begin{aligned} 0 = \cos\frac{\pi}{2} &= \frac{\cosh\rho(v,p)\cosh\rho(p,v_2') - \cosh\rho(v,v_2')}{\sinh\rho(v,p)\sinh\rho(p,v_2')} \\ \Rightarrow \cosh\rho(v,p) &= \frac{\cosh\rho(v,v_2')}{\cosh\rho(p,v_2')}\ . \end{aligned}$$

So $\rho(v,p)$ is minimized when $\rho(p,v_2')$ is maximized and $\rho(v,v_2')$ is minimized. This occurs precisely when $\rho(p,v_2') = d/2$ and $\rho(v,v_2') = d$, i.e., when $v_2 = v_2'$, vv_1v_2 form an equilateral triangle and $\rho(v,p) = \ell$. □

Lemma 2. *Let G be an admissible graph. Let $f \in F(G)$. If f has n sides then $A(f) \geq (\frac{n}{3})(\pi - 3\alpha)$.*

Proof. Let $e, f \in E(G)$ be different edges with endpoints v_1, v_2 and w_1, w_2 respectively. Let $v_3, w_3 \in \mathbb{H}^2$ such that the triangles $v_1v_2v_3$, $w_1w_2w_3$ have interior angles at v_1, v_2, w_1 and w_2 equal to $\alpha/2$.

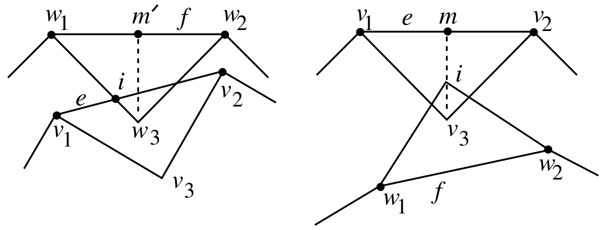


FIGURE 4

Suppose for a contradiction that the triangles $v_1v_2v_3$ and $w_1w_2w_3$ intersect in their interiors. See Figure 4. Since $\mathrm{Int}\, e \cap \mathrm{Int}\, f = \emptyset$ this implies that the triangle $v_1v_2v_3$ intersects $w_1w_2w_3$ in the arcs w_1w_3 and w_2w_3. Let m, m' be the midpoints of e and f respectively. Then $v_1v_2v_3$ intersects the arc $m'w_3$. If v_1v_2 intersects $m'w_3$ then without loss of generality $v_1v_2 \cap w_1w_3 \neq \emptyset$. Let $\{i\} = v_1v_2 \cap w_1w_3$. Then $\rho(w_1, e) \leq \rho(w_1, i) < \rho(w_1, w_3) < \ell$. This contradicts the previous lemma. So $v_1v_2 \cap m'w_3 = \emptyset$. So either v_1v_3 or v_2v_3 intersects $m'w_3$. Without loss of generality v_1v_3 intersects $m'w_3$. Let $\{i\} = v_1v_3 \cap m'w_3$. Then $\rho(v_1, f) \leq \rho(v_1, i) + \rho(i, m') < \rho(v_1, v_3) + \rho(w_3, m') = \ell$ contradicting the previous lemma. Thus $v_1v_2v_3$ and $w_1w_2w_3$ do not intersect in their interiors.

For each edge e of f form the triangle $v_1v_2v_3$ as above so that this triangle lies in the interior of f. Since none of these triangles overlap and each has area $= \frac{1}{3}(\pi - 3\alpha)$, the lemma is proved.

Proof of Theorem 1. If G is a graph of one vertex, then G is spiral and each face of G is a triangle. Let $n \geq 1$. Assume for induction that if G is an admissible graph such that $|V(G)| \leq n$ then if G is spiral then G is optimal and if G is optimal then each face of G is a triangle.

Let O be an optimal graph and let S be a spiral graph such that $|V(O)| = |V(S)| = n+1$. Define F_k as the number of k-gon faces of O. Then $\sum_{k-3}^{\infty} kF_k = 2(|E(O)| - |E(\partial O)|) + |E(\partial O)|$. Combined with the Euler characteristic formula $V - E + F = 1$, this gives:

$$
\begin{aligned}
(1) \qquad & 3|V(O)| - 3 - \left(|E(\partial O) + \sum_{k=4}^{\infty} (k-3) F_k \right) \\
& = |E(O)| \geq |E(S)| = 3|V(S)| - 3 - |E(\partial S)| \; .
\end{aligned}
$$

The last equality holds because all faces of S are triangles. Thus $|E(\partial O)| \leq |E(\partial S)|$ with equality if and only if S is optimal and every face of O is a triangle. Assume for a contradiction that $|E(\partial O)| < |E(\partial S)|$.

Let O' be the admissible graph with vertex set $V(O') = V(O) - V(\partial O)$. If O' is empty then by (1) the theorem is true. So assume O' is not empty. Order the vertices of S, $(v_1, \ldots, v_n)$ as in the definition of spiral. Let S' be the admissible graph with $V(S') = \{v_1, \ldots, v_s\}$ and $s = |V(O')|$. Then S' is spiral. By induction, S' is optimal so $|E(O')| \leq |E(S')|$.

By Gauss-Bonnet,

$$A(O) - A(S) = \sum_{v \in \partial O} \pi - a(v) - \sum_{w \in \partial S} \pi - a(w) \tag{2}$$

$$\sum_{v \in \partial O} \pi - a(v) \leq |E(\partial O)|\pi - (|E(O)| - |E(O')|)\alpha \tag{3}$$

because there are at least $|E(O)| - |E(O')|$ pairs of adjacent edges touching ∂O and each pair contributes at least α to the sum.

$$\sum_{w \in \partial S} \pi - a(w) = \sum_{w \in \partial S - \partial S'} \pi - a(w) + \sum_{w \in \partial S \cap \partial S'} \pi - a(w) \tag{4}$$

$$= \pi|E(\partial O)| - (|E(S)| - |E(S')| - 1)\alpha + \sum_{w \in \partial S \cap \partial S'} \pi - a(w) \tag{5}$$

because $|V(\partial S) - V(\partial S')| = |E(\partial O)|$, there are exactly $(|E(S)| - |E(S')| - 1)$ pairs of adjacent edges radiating from the vertices in $V(\partial S) - V(\partial S')$ (since if $a(v_1) = \alpha$ for $v_i \in \partial S$ then $i = n$ and the edge connecting v_{n-1} to v_m is not counted twice), and each pair contributes exactly α to the sum since every face of S is a triangle.

$$\sum_{w \in \partial S \cap \partial S'} \pi - a(w) \geq \pi - (2\pi - 2\alpha) + (\pi - 3\alpha)\Big[|E(\partial S)| - |E(\partial O)| - 1\Big] \tag{6}$$

since $a(v_m) \leq (2\pi - 2\alpha)$ and $a(v_i) \leq 3\alpha$ for $m < i < n$, and

$$|V(\partial S) \cap V(\partial S')| = |E(\partial S)| - |E(\partial O)|.$$

By lemma 2,

$$A(O) - A(S) = \sum_{f \in F(O)} A(f) - \sum_{f \in F(S)} A(f) \tag{7}$$
$$\geq (|F(O)| - |F(S)|)(\pi - 3\alpha)$$
$$= (|E(O)| - |E(S)|)(\pi - 3\alpha)$$

and equality holds if and only if all faces of O are triangles. The last equality holds from the Euler characteristic formula.

Equations (2)–(7) and $\alpha = 2\pi/N$ imply

$$|E(O)| - |E(S)| + |E(S')| - |E(O')| \tag{8}$$
$$\leq N - 6 - \left(\frac{N-6}{2}\right)(|E(\partial S)| - |E(\partial O)|) - \left(\frac{N-6}{2}\right)(|E(O)| - |E(S)|)$$

with equality holding only if all faces of O are triangles. Hence $|E(\partial S)| - |E(\partial O)| \leq 2$ with equality holding only if $|E(O)| = |E(S)|$ and all faces of O are triangles. This contradicts (1). So we may assume $|E(\partial S)| - |E(\partial O)| = 1$.

By (8),

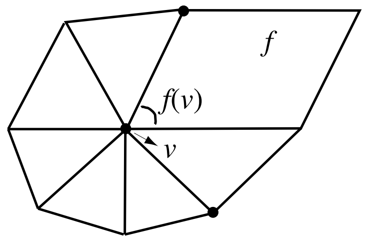
$$(|E(O)| - |E(S)|)\left(\frac{N-4}{2}\right) \leq \frac{N-6}{2} + |E(O')| - |E(S')| \; . \tag{9}$$

since $|E(O')| \leq |E(S')|$, $|E(O)| = |E(S)|$, i.e., S is optimal.

By (1), there exists exactly one nontriangular face, f of O and it is a 4-gon. Suppose, for a contradiction, that there exists a vertex v in f which is not in ∂O. Let k be the number of triangular faces of O containing v. Since v is in the interior of O,

$$f(v) = 2\pi - k\alpha = (N - k)\alpha \tag{10}$$

where $f(v) =$ the interior angle of f at v. But this implies



that two opposite vertices of f are distance less than or equal to d apart, contradicting that f is a face of O. Thus all vertices of f are in ∂O. Thus any vertex in the interior of O is contained in N edges of O. For any admissible graph G, let $\hat{E}(G) = \{e \in E(G) | e \cap V(\partial G) \neq \emptyset\}$. Then,

$$
\begin{aligned}
\text{(11)} \qquad N(|V(O) - V(\partial O)|) &= 2(|E(O) - \hat{E}(O)|) + |\hat{E}(O) - E(\partial O)| \\
&= 2|E(O)| - |\hat{E}(O)| - |E(\partial O)| \\
&= 2|E(S)| - |\hat{E}(O)| - (|E(\partial S)| - 1) \\
\text{(11a)} \qquad N(|V(O)| - |V(\partial O)|) &= N(|V(S)| - (|V(\partial S)| - 1)) \\
&= 2|E(S)| - |\hat{E}(S)| - |E(\partial S)| + N \ .
\end{aligned}
$$

So,

$$|\hat{E}(O)| = |\hat{E}(S)| - N + 1 \ . \tag{12}$$

Every edge of $\hat{E}(S)$ which is in $E(S')$ is connected to the unique vertex $v \in V(\partial S) \cap V(\partial S')$. Since $v \in \partial S$, at least 2 edges radiating from v touch other vertices on ∂S and are therefore not in S'. Hence,

$$|\hat{E}(S)| \leq |E(S)| - |E(S')| + N - 3 \ . \tag{13}$$

(12) and (13) imply

$$|E(O')| = |E(O)| - |\hat{E}(O)| = |E(S)| - |\hat{E}(S)| + N - 1 \geq |E(S')| + 2 \ .$$

This contradicts that S' is optimal. Thus $|E(\partial O)| = |E(\partial S)|$ and by (1), the theorem is proved. $\square$

III. Conclusion.

For some specific diameters (other than those we have just considered) it is easy to guess what the optimal packings look like. For example, let P be a 6-gon such that one angle of P is equal to $\pi/2$ and all other angles are equal to $2\pi/3$. Suppose also that the sides of P that make the $\pi/2$ angle have equal length and the other four sides all have equal length as well. Then P is determined up to congruence and P admits a unique tessellation of the plane for which any copy of P is a fundamental domain. With respect to the diameter of the incircle of P, we believe that any optimal packing will have its center set contained in the center set of a tessellation by P.

For most diameters d, we conjecture that any limit of optimal packings (for d) does not fill space well; in fact, it appears that they may be very "narrow". Imagine placing one circle in the plane after another in such a way as to maximize the number of tangencies at each step. It is easy to increase the number of tangencies by two in a single step but the opportunity to increase the tangencies by three or more is rare. By construction, if d is smaller than the diameter of the incircle of P then the maximum number of tangencies in a circle packing of diameter d with n circles is at least $2n-3+[\frac{n}{12}]+[\frac{n+3}{12}]$. This is the best bound we have so far.

Acknowledgements

I would like to thank Charles Radin for suggesting this problem and for many helpful conversations.

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