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LOCAL PERTURBATIONS OF ENERGY AND KAC'S RETURN TIME THEOREM

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To Gérard Rauzy on the occasion of his sixtieth birthday.

Abstract. We introduce the notion of local perturbations for normalized energies and study their effect on the level of equilibrium measures. Using coupling technics and Kac's return time theorem, we obtain some $\bar{d}$-estimates for the equilibrium measures. These reveal stability of certain energies under local perturbations. They also show how some weak-$\star$ convergence of equilibrium may be obtained in absence of $\| \|_\infty$-accuracy of the energies.

1. Introduction

This paper concerns Statistical Mechanics - Thermodynamic Formalism (see [9] for basics). However it entirely translates to Probability Theory, where it concerns chains with complete connections [2,3]. There the log of the local transitions for the chain is the normalized energy for Thermodynamics; they describe microscopical interactions for a system with many particles.

Given a transition function g, the pre-cited theories associate to it equilibrium measures. These are stationary and describe the macroscopical aspect of the system after a long time.

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We shall restrict to the case of uniqueness of the equilibrium measure. For such a case to hold there is a relatively important literature - also studying the Ergodic Theoretic properties of the single measure - see e.g. [4]. For a given g we denote by μ_g the uniquely associated equilibrium measure (by hypothesis).

A natural question is the study of the sensitivity, or stability, of the system, for a given interaction g, under perturbation of g. Let $\tilde{g}$ be another local transition function. It always is a perturbation of g. Then the perturbation theory asks about the behavior of $\mu_g - \mu_{\tilde{g}}$ when $\tilde{g}$ is close to g. That is *what is the effect of microscopical perturbations on the macroscopical aspect of the system at equilibrium* ?

Now the meaning of "$\tilde{g}$ *is close to* g" has to be specified. Usually $\tilde{g}$ is thought of as being close to g when $\| g - \tilde{g} \|_\infty$ is small - which is relevant to the Perturbation Theory of Markov chains also : this is $\| \; \|_\infty$ **perturbation theory**.

In this note we introduce and study **local perturbation theory** : we think of $\tilde{g}$ *being close to* g *if* $\mu_{\tilde{g}}(\{g \neq \tilde{g}\})$ *is small* (cf. the first statement of Theorem 1).

For classical $\| \; \|_\infty$-perturbation the reader will find in [9] -using [11]- the corresponding basic stability results, and in [2,3] more quantitative aspects. Essentially under the hypothesis of uniqueness of equilibrium it follows that $\mu_{\tilde{g}} \underset{\text{weak-}\star}{\longrightarrow} \mu_g$ as $\| \tilde{g} - g \|_\infty \to 0$. Convergence in the $\bar{d}$-metric under additional regularity assumptions on g are detailed in [2,3].

However the technics developed to prove this stability under classical perturbation do not apply to the case of local perturbations. The reader will find in Example 3 of the following section a very simple example where this is illustrated.

We have proved in Theorem 1 the stability under local perturbations for certain energies. The proof required the introduction of new ingredients. The main novelty is the use of Kac's return time theorem [5,10], to overcome the absence of $\| \; \|_\infty$-accuracy for $\tilde{g} - g$. Otherwise we use now standard -still powerful- coupling technics [1,2,3,4], [12]. Let us mention that in [7] another technic -of algebraic nature- is developed to prove stability.

We stress that our result also proves that there is no hope to detect empirically strong local variations of the law of a process, producing a time series - whence to determine its law.

The paper is organized as follows. Section 2 introduces the basic notations, presents in Theorem 1 the main result. In Example 3 we choose the simplest case to show how our stability result applies while the classical perturbation results do not.

In Section 3 we briefly sketch our proof. The final Section contains the details of the different steps used to prove Theorem 1.

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2. Notations and statement of result

Our results are presented within the simplest framework as generalizations to larger alphabets and/or sub-shifts of finite type are somewhat easy to guess, but would induce loss of clarity in the exposition.

2.1 : the shift, $\mathcal{G}$-functions, g-measures, and ergodicity.

Let $X = \{0,1\}^{\mathbb{N}}$, and $S : X \to X$ its usual one-sided shift map. This is a covering transformation in the sense of Keane [6]. He proves for such that invariant Borel probability measures and so-called $\mathcal{G}$-functions are intimately related. Now from the Thermodynamical view-point the log of $\mathcal{G}$-functions are normalized energies.

To get something working usual is to restrict generality and define the set of $\mathcal{G}$-functions as follows : we let X have product topology and

$$\mathcal{G} = \{g : X \to]0,1[: g \text{ continuous and for any } x \in X, \sum_{y \in S^{-1}x} g(y) = 1\}.$$

It is standard that such gs are local transitions for chains with complete connections - see e.g. [2].

Let $M(X)$ denote the set of Borel probability measures on X. Let $\mathcal{C}(X)$ denote the set of real valued continuous maps on X, endowed with $\| \|_\infty$. A $g \in \mathcal{G}$ defines a transfer operator $\mathcal{L}_g$ acting continuously, linearly, positively, and contracting $\| \|_\infty$ on $\mathcal{C}(X)$:

$$\mathcal{L}_g f(x) = \sum_{y \in S^{-1}x} g(y) f(y), \quad x \in X.$$

Since moreover $\mathcal{L}_g 1 = 1$, its dual acts on $M(X)$: now $M(X)$ is compact convex whence the Schauder - Tychonov fixed point theorem yields that it has a fixed point.

Such a μ is an *g–measure or equilibrium measure* : since $\mathcal{L}_g(f \circ S) = f$, it must be a S-invariant one. From the Statistical Mechanical view-point, the μ describes macroscopical evolution while the g does so for the microscopical one.

When several such μ exist we speak of phase transition while otherwise we shall agree to say that g is ergodic. If it is, we denote the only μ by μ_g. This will be the case by hypothesis.

2.2 : Variations, and (pseudo)-distances for measures.

For $g, \tilde{g}$ ergodic, if $g \neq \tilde{g}$, then μ_g and $\mu_{\tilde{g}}$ are singular, hence the variation norm is bad to measure $\mu_g - \mu_{\tilde{g}}$. From the macroscopical point of view, measures of cylinders make sense : let

$$w = (w_0, \ldots, w_{|w|-1}) \in \{0,1\}^{|w|}$$

be a word of length $|w|$. Let $[w]$ be the set of $x \in X$ that have their first $|w|$ coordinates equal to those of w.

Then for given $m \geq 1$, we have two probability vectors :

$$\pi_m(g) = (\mu_g([w]))_{|w|=m},$$

and $\pi_m(\tilde{g})$ is defined similarly. Then we can measure

$$\bar{d}_m(\mu_g, \mu_{\tilde{g}}) = \| \pi_m(g) - \pi_m(\tilde{g}) \|_1 = \sum_{|w|=m} |\mu_g([w]) - \mu_{\tilde{g}}([w])|.$$

Ornstein [8] proved fruitful the $\bar{d}$ distance :

$$\bar{d}(\mu_g, \mu_{\tilde{g}}) = \inf\{\nu([(0,1)] \cup [(1,0)]) : \nu \in J(\mu_g, \mu_{\tilde{g}})\},$$

where $J(\mu_g, \mu_{\tilde{g}})$ is the set of joinings between μ_g and $\mu_{\tilde{g}}$, that is the $S \times S$-invariant probability measures on $X \times X$ that go for the first natural projection to μ_g and for the second to $\mu_{\tilde{g}}$.

We shall evaluate both using $\bar{d}_m$ and $\bar{d}$: it holds that $\bar{d}_m \leq m\bar{d}$ (cf. Lemma 5).

Conditions ensuring ergodicity require more than continuity. One is that of summable variations. We will need variations later : put

$$var_m(g) = \max_{|w|=m} \sup_{x,y \in [w]} |g(x) - g(y)|.$$

2.3 : Statement of results.

We assume $0 < \lambda < 1/2$ is such that $g, \tilde{g} > \lambda$. We let μ_λ denote the Bernoulli measure $\mathcal{B}(2\lambda, 1-2\lambda)$ on X. We denote by

$$X_p = \{z \in X : 1 \leq l \leq p-1 \Rightarrow z[l, l+m-1] \neq 0^m\}.$$

We let $m \geq 1$, we let E be an at most countable index set, and for each $i \in E$, p_i denotes an integer and $F_i \subset \{0,1\}^{m+p_i}$. We assume in the second statement of Theorem 1 below that

$$\Delta := \{g \neq \tilde{g}\} \subset \cup_i \cup_{v \in F_i} [v].$$

Theorem 1. *For any $K \geq m+1$,*

$$\begin{aligned}&\max\{\bar{d}_m(\mu_g, \mu_{\tilde{g}}), \bar{d}(\mu_g, \mu_{\tilde{g}})\} \\ &\leq 2\left[(K-1)\left(var_{m+1}(g) + (1-2\lambda)\mu_{\tilde{g}}(\Delta)\right) + \textstyle\sum_{k>K}(k-K)\mu_\lambda(X_k)\right].\end{aligned}$$

As a consequence, with $K = um+1$, $u \geq 1$,

$$\begin{aligned}&\max\{\bar{d}_m(\mu_g, \mu_{\tilde{g}}), \bar{d}(\mu_g, \mu_{\tilde{g}})\} \\ &\leq 2m\left[u\left(var_{m+1}(g) + (1-2\lambda)(\textstyle\sum_i \#F_i(1-\lambda)^{m+p_i}\right) + m\tfrac{(1-(2\lambda)^m)^{u-1}}{(2\lambda)^{2m}}\right].\end{aligned}$$

Comments 2. *In the first estimate of Theorem 1, we see a quantitative meaning of "$\tilde g$ is a small local perturbation of g" : that is $\mu_{\tilde g}(\{\tilde g \neq g\})$ is small.*

Under the assumption of the second statement ($\{\tilde g \neq g\} \subset \cup_{i\in E} \cup_{v\in F_i} [v]$), we have another illustration of this : enough is that $\sum_i \# F_i (1-\lambda)^{m+p_i}$ be small.

To deduce stability under local perturbations -for certain energies- from Theorem 1, second statement, take, for example, a g such that $var_{m+1}(g) = 0$ (that is g depends only on the first $m+1$ coordinates). Then pick a very large u first to make $m^2 u \frac{(1-(2\lambda)^m)^{u-1}}{(2\lambda)^{2m}}$ small ; then take $\tilde g$ so that $um\mu_{\tilde g}(\{\tilde g \neq g\})$ is small also, and conclude that $\bar d(\mu, \mu_{\tilde g})$ is small (Example 3 below falls into that case).

We emphasize here that the produced bounds require the sequence $(var_m(g))_{m\geq 1}$ to decrease very rapidly to 0. This is due to the nature of the proof : our result produces universal bounds for ergodic g satisfying $\lambda \leq g \leq 1-\lambda$. Now as revealed in [7] a proof technic keeping more information on g yields better bounds.

Now we produce the simplest example we found that illustrates how Theorem 1 even gives some weak-$\star$ convergence of equilibrium, and for which classical theory does not apply because $\| \tilde g - g \|_\infty$ remains bigger than or equal to 1/4.

Example 3. *Consider $\Pi : x \mapsto \sum_{i\geq 0} \frac{x_i}{2^{i+1}} \bmod 1$. This is the factor map to the transformation $M : x \mapsto 2x \bmod 1$. By [6] the same notions of $\mathcal{G}$-functions and equilibrium measures can be developed for M on the torus. Call $\mathcal{G}_M$ the corresponding set. Then any $g \in \mathcal{G}_T$ is such that $g \circ \Pi \in \mathcal{G}$. But the reverse is false : take Bernoulli measure $\mu_0 = \mathcal{B}(\frac{3}{4}, \frac{1}{4})$. It corresponds to $g_0 \in \mathcal{G}$ with $g_0(0x) = 3/4$ and $g_0(1x) = 1/4$.*

Consider $g_\eta \in \mathcal{G}_M$ to be such that for $0 < \eta < 1/4$, $g_\eta \circ \Pi$ and g_0 coincide outside balls of radius η centered at 0^∞, 1^∞, 01^∞ and 10^∞, and otherwise let g_η be ergodic and $\geq 1/4$.

Then though $\| g_0 - g_\eta \circ \Pi \|_\infty \geq 1/4$ for all η, our result shows that as $\eta \to 0$,

$$\mu_{g_\eta} \overset{weak\text{-}\star}{\longrightarrow} \mu_{g_0},$$

even in the $\bar d$-distance. Hence weak-$\star$ convergence may hold in absence of $\| \ \|_\infty$ accuracy.

3 : Sketch of proof of Theorem 1

The proof of Theorem 1 is quite simple and develops along the following four steps :

($\bullet^1$) : let $Y = X \times X$, $T = S \times S$, and pick a $\tau \in \mathcal{G}_T$ which is a $\mathcal{G}_T$-function for (Y, T) and satisfies for any $(x,y) \in Y$ and $i, j = 0, 1$,

$$\begin{cases} \sum_i \tau(ix, jy) = \tilde g(jy), \\ \sum_j \tau(ix, jy) = g(ix). \end{cases}$$

Then by ergodicity assumption, and these two properties, any ν such that $\mathcal{L}_\tau^\star \nu = \nu$ belongs to $J(\mu_g, \mu_{\tilde g})$. Here pay attention to choose τ charging the entry to diagonal as much as possible.

$(\bullet^2)$: pick a τ-measure ν, and define for $q \geq 1$,

$$A_q = \{(x, x') \in Y : i < q \Rightarrow x_i = x'_i\}.$$

Then show that

$$\begin{cases} \bar{d}_q(\mu_g, \mu_{\tilde{g}}) \leq 2\nu(A_q^c), \\ \bar{d}(\mu_g, \mu_{\tilde{g}}) \leq \nu(A_q^c). \end{cases}$$

In the sequel we estimate on $\nu(A_m^c)$, for $m \geq 1$ as in Theorem 1.

$(\bullet^3)$: introduce $n_m(z) = \min\{k \geq 1 : T^k z \in A_m\}$. First show, using attractivity - [4], that $\nu(\{n_m < \infty\}) = 1$, whence deduce by Kac's return time theorem [5] that

$$\sum_{k \geq 1} k\nu(A_m \cap \{n_m = k\}) = 1.$$

Remark 4. *There always exists an ergodic τ-measure ν, for which $\nu(\{n_m < \infty\}) = 1$ as soon as $\nu(\{n_m < \infty\}) > 0$, using invariance, and ergodicity. However we think the proof that this holds true for any τ-measure (for chosen τ) is interesting enough, and relevant for the understanding of the treatment of the tail series in $(\bullet^4)$.*

$(\bullet^4)$: observe that $\nu(A_m \cap \{n_m = 1\}) \leq \nu(A_m)$, and using stationarity of ν that for $k \geq 2$, $\nu(A_m \cap \{n_m = k\}) \geq \nu(A_m^c \cap T^{-1} A_m)$. Conclude using attractivity again that the tail series in $(\bullet^3)$ is small, that its first term ($k = 1$) is about $\nu(A_m)$, and that the intermediate terms are each about $\nu(A_m^c \cap T^{-1} A_m)$: whence if this one is really small we get estimates on the effect on equilibrium of local perturbations of g.

4 : PROOF OF THEOREM 1.

4.1 : The maximal coupling - $(\bullet^1)$.

We let $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Define the $\mathcal{G}_T$-function τ for (Y, T) - continuous but not strictly positive - by (cf. [2])

$$\tau(ix, jy) = \begin{cases} \min\{g(ix), \tilde{g}(jy)\} \text{ if } \ i = j; \\ \\ \frac{(g(ix) - \tilde{g}(iy))^+ (\tilde{g}(jy) - g(jx))^+}{(g(0x) - \tilde{g}(0y))^+ + (g(1x) - \tilde{g}(1y))^+} \ \text{ if } i \neq j \text{ and } g(1x) \neq \tilde{g}(1y); \\ \\ 0 \ \text{ otherwise.} \end{cases}$$

A few minutes require to check out the required properties of this τ as stated in $(\bullet^1)$. Notice that

$$\tau(1x, 1y) + \tau(0x, 0y) \geq 2\lambda.$$

By [6] (same argument as the one sketched in the introduction) there is at least one τ-measure. Pick one and call it ν.

By ergodicity assumptions on g and $\tilde{g}$, and the properties $(\bullet^1)$, $\pi_1 \nu = \mu_g$ and $\pi_2 \nu = \mu_{\tilde{g}}$: whence we have the following diagram of measure-theoretical factors :

$$\begin{array}{ccc} & (Y, T, \nu) & \\ & \pi_1 \swarrow \quad \searrow \pi_2 & \\ (X, S, \mu_g) & & (X, S, \mu_{\tilde{g}}) \end{array}$$

4.2 : Variation inequalities - ($\bullet^2$).

We let $A_q = \{(x,y) \in Y : i < q \Rightarrow x_i = y_i\}$, as in ($\bullet^2$), and relate the distance between μ_g and $\mu_{\tilde{g}}$ to the quantity $\nu(A_q)$:

Lemma 5. *Let $f \in \mathcal{C}(X)$ and $q \geq 1$. Then*

$$|\mu_g(f) - \mu_{\tilde{g}}(f)| \leq 2 \parallel f \parallel_\infty (1 - \nu(A_q)) + var_q(f)\nu(A_q).$$

Hence it follows that

$$\bar{d}_q(\mu_g, \mu_{\tilde{g}}) \leq 2(1 - \nu(A_q)).$$

Notice also that since $\nu(A_q) \leq \nu(A_1)$, it follows by ($\bullet^1$) that

$$\bar{d}(\mu_g, \mu_{\tilde{g}}) \leq 1 - \nu(A_q).$$

Finally, $\bar{d}_q \leq 2q\bar{d}$.

Proof. We decompose along cylinders and use ($\bullet^1$) to go from integrating on X to integrating on Y $(= X \times X)$:

$$\begin{array}{rcl} |\mu_g(f) - \mu_{\tilde{g}}(f)| & \leq & \sum_{|w|=q} | \int_{[w]} f d\mu_g - \int_{[w]} f d\mu_{\tilde{g}}| \\ & = & \sum_{|w|=q} | \int_{[w]\times X} f \circ \pi_1 d\nu - \int_{X\times[w]} f \circ \pi_2 d\nu| \\ & \leq & \sum_{\substack{|v|=|w|=q, \\ v \neq w}} \int_{[w]\times[v]} |f \circ \pi_1 - f \circ \pi_2| d\nu \\ & & \qquad + \sum_{|w|=q} \int_{[w]\times[w]} |f \circ \pi_1 - f \circ \pi_2| d\nu \\ & \leq & 2 \parallel f \parallel_\infty (1 - \nu(A_q)) + var_q(f)\nu(A_q). \end{array}$$

For the second statement we take f constant on cylinders of length q, and on each such $[w]$ equal to $sign(\mu_g([w]) - \mu_{\tilde{g}}([w]))$.

For the third observation of the lemma we notice additionally that $\bar{d}(\mu_g, \mu_{\tilde{g}}) \leq 1 - \nu(A_1)$.

Next $\bar{d}_q \leq 2\nu(A_q^c) \leq 2\sum_{i=0}^{q-1} \nu(T^{-i}A_1^c) = 2q\nu(A_1^c)$ by stationarity of ν. Passing to the infimum over $J(\mu_g, \mu_{\tilde{g}})$ in this last inequality, we deduce the last statement of the Lemma. ∎

4.3 : Attractivity and Kac's return time theorem - ($\bullet^3$).

Define $\delta : Y \to X$ by

$$\delta(x,y) = (|x_i - y_i|)_{i\geq 0}.$$

This is a shift commuting topological factor map.

On X, define the partial order $x \preceq y \Leftrightarrow \forall i,\ x_i \leq y_i$. Say an $f : X \to \mathbb{R}$ is **increasing** if

$$x \preceq y \Rightarrow f(x) \leq f(y).$$

We define μ_λ as in Theorem 1. Then $g_\lambda(1.)$ is increasing (constant) and satisfies as already noticed

$$\sum_{i\neq j} \tau(ix, jy) \leq 1 - 2\lambda = g_\lambda(1\delta(x,y)).$$

Using δ and $\preceq$, heavily inspired by [4, Lemma 4.1], we have the following :

Lemma 6. *Let $\Phi \in \mathcal{C}(Y)$, $f \in \mathcal{C}(X)$ be increasing and together satisfy*

$$\Phi(x,y) \leq f(\delta(x,y)).$$

Then for all $n \geq 0$, $\mathcal{L}_\tau^n \Phi(x,y) \leq \mathcal{L}_{g_\lambda}^n f(\delta(x,y))$, hence

$$\nu(\Phi) \leq \mu_\lambda(f).$$

Remark 7. *The same conclusion holds for g_λ replaced by an ergodic $\bar{g} \in \mathcal{G}$ such that $\bar{g}(1.)$ is increasing - use a remark on the characterization of ergodicity in [4].*

Proof. Set $\tau_{x,y}(i \neq j) = \sum_{i \neq j} \tau(ix, jy)$, and let $\tau_{x,y}(i = j) = 1 - \tau_{x,y}(i \neq j)$. Then compute using the hypothesis of the Lemma and that $g_\lambda(1.)$ increases :

$$\begin{array}{rcl}
\mathcal{L}_\tau \Phi(x,y) & = & \left\{\sum_{i\neq j} + \sum_{i=j}\right\} \tau(ix,jy)\Phi(ix,jy) \\
& \leq & \tau_{x,y}(i \neq j) f(1\delta(x,y)) + \tau_{x,y}(i = j) f(0\delta(x,y)) \\
& = & \tau_{x,y}(i \neq j) f(1\delta(x,y)) + (g_\lambda(1\delta(x,y)) - \tau_{x,y}(i \neq j)) f(0\delta(x,y)) \\
& & \qquad\qquad + g_\lambda(0\delta(x,y)) f(0\delta(x,y)) \\
& \leq & \mathcal{L}_{g_\lambda} f(\delta(x,y)).
\end{array}$$

By [4, Lemma 2.1], $\mathcal{L}_{g_\lambda} f^n$ is increasing for any $n \geq 0$, whence repeated application of the preceding computations yield that for any such n,

$$\mathcal{L}_\tau^n \Phi(x,y) \leq \mathcal{L}_{g_\lambda}^n f(\delta(x,y)).$$

By [6], since g_λ is Lipschitz, we have that $\mathcal{L}_{g_\lambda}^n f(z) \to \mu_\lambda(f)$ uniformly in z as n goes to infinity. Whence

$$\nu(\Phi) = \nu(\mathcal{L}_\tau^n \Phi) \leq \nu(\mathcal{L}_{g_\lambda}^n f) \to \nu(\mu_\lambda(f)) = \mu_\lambda(f).$$

■

Now we pass to Kac's theorem : we take notations from ($\bullet^3$). Then

$$\nu(A_m) \geq \nu([0^m] \times [0^m]) \geq (2\lambda)^m > 0.$$

Hence Kac's theorem applies :

Theorem 8 (Kac). *[5,10]. $\sum_{k\geq 1} k\nu(A_m \cap \{n_m = k\}) = \nu(n_m < \infty)$.*

We may use attractivity (Lemma 6) to prove that

Lemma 9. $\nu(n_m < \infty) = 1$.

Proof. Take $q \geq 1$ and put

$$\begin{cases} Y(q) = \{n_m \geq qm + 1\}; \\ X(q) = \{x \in X : 1 \leq u \leq m \Rightarrow x[um, (u+1)m[\neq 0^m\}. \end{cases}$$

Then define $\Phi_q(x, y) = \mathbf{1}_{Y(q)}(x, y)$ and $f_q(z) = \mathbf{1}_{X(q)}(z)$. It follows that Φ_q and f_q satisfy the conditions for Lemma 6 to apply : whence $\nu(Y(q)) \leq \mu_\lambda(X(q))$, but since μ_λ is Bernoulli with parameter $1 - 2\lambda$, we get

$$\nu(Y(q)) \leq (1 - (2\lambda)^m)^q.$$

Now $\{n_m = \infty\} \subset Y(q)$ whence $0 \leq \nu(\{n_m = \infty\}) \leq \liminf \nu(Y(q)) = 0$. ■

4.4 : End of Proof of Theorem 1 - ($\bullet^4$).

By stationarity, $\nu(A_m) = \nu(T^{-1}A_m) = \nu(A_m \cap \{n_m = 1\}) + \nu(A_m^c \cap T^{-1}A_m)$. Hence we deduce from ($\bullet^3$) that

$$(\star^1) \qquad \nu(A_m^c) = \sum_{k \geq 2} k\nu(A_m \cap \{n_m = k\}) \; - \nu(A_m^c \cap T^{-1}A_m).$$

Using stationarity of ν and Lemma 9 it follows that for $k \geq 2$,

$$(\star^2) \qquad \nu(A_m^c \cap T^{-1}A_m) = \sum_{k \geq 2} \nu(A_m \cap \{n_m = k\}).$$

Moreover, $A_m \cap \{n_m = k\} = \emptyset$ for $k < m + 1$. From ($\star^{1,2}$) we get

$$\nu(A_m^c) = m\nu(A_m^c \cap T^{-1}A_m) + \sum_{k \geq m+2} (k - m - 1)\nu(A_m \cap \{n_m = k\}).$$

Now for $K \geq m + 1$, the same argument with "$\leq$" instead of "$=$" yields to

$$(\star^3) \qquad \nu(A_m^c) \leq (K-1)\nu(A_m^c \cap T^{-1}A_m) + \sum_{k > K} (k - K)\nu(A_m \cap \{n_m = k\}).$$

Estimating $\nu(A_m^c \cap T^{-1}A_m)$.

Remember $\Delta = \{g \neq \tilde{g}\}$ ($= \cup_i \cup_{v \in F_i} [v]$ in the second statement of Theorem 1). Then decompose

$$\begin{aligned} A_m &= \cup_{|w|=m}\big(([w] \times ([w] \cap \Delta)) \cup ([w] \times ([w] \setminus \Delta))\big) \\ &= (A_m \cap (X \times \Delta)) \cup (A_m \cap (X \times (X \setminus \Delta))). \end{aligned}$$

Let us first consider $(x,y) \in A_m \cap (X \times (X \setminus \Delta))$: then $|g(ix) - \tilde{g}(iy)| \leq var_{m+1}(g)$, whence $\sum_{i=j} \tau(ix, jy) \geq 1 - var_{m+1}(g)$ and $\sum_{i \neq j} \tau(ix, jy) \leq var_{m+1}(g)$. Hence

$$(\star^4) \qquad \begin{array}{rcl} \nu(A_m^c \cap T^{-1}A_m) & = & \int_{A_m} \sum_{i \neq j} \tau(ix, jy) d\nu(x,y) \\ & = & \int_{A_m \cap (X \times \Delta)} \sum_{i \neq j} \tau(ix, jy) d\nu(x,y) \\ & & \quad + \int_{A_m \cap (X \times (X \setminus \Delta))} \sum_{i \neq j} \tau(ix, jy) d\nu(x,y) \\ & \leq & var_{m+1}(g)\nu(A_m) + (1-2\lambda)\mu_{\tilde{g}}(\Delta) \\ & \leq & var_{m+1}(g) + (1-2\lambda)\mu_{\tilde{g}}(\Delta) \\ & \big(\leq & var_{m+1}(g) + (1-2\lambda)\big(\sum_i \# F_i (1-\lambda)^{m+p_i}\big)\big). \end{array}$$

Estimating the tail series in $(\star^3)$**.**

Define $B_k = T^{-1}A_m^c \cap \ldots \cap T^{-k+1}A_m^c$. Then $A_m \cap \{n_m = k\} \subset B_k$.

Next set $Z_k = \{z \in X : 1 \leq p \leq k-1 \Rightarrow z[p, p+m[\neq 0^m\}$. Denote $\Psi_k = \mathbf{1}_{B_k}$ and $f_k = \mathbf{1}_{X_k}$. Then they satisfy conditions for Lemma 6 and therefore we obtain

$$\nu(A_m \cap \{n_m = k\}) \leq \nu(B_k) \leq \mu_\lambda(X_k).$$

Combining this with $(\star^{3,4})$, and Lemma 5, we deduce the first statement of Theorem 1.

To get the second one we first choose $K = um + 1$ for some $u \geq 1$. By stationarity we get that for each $v \geq u$,

$$\sum_{t=1}^{m} \mu_\lambda(X_{vm+t+1}) \leq m\mu_\lambda(T^{-m}X_{vm+1}) = m\mu_\lambda(X_{vm+1}).$$

We input this in the tail series of $(\star^3)$ to get with $(\star^4)$ that

$$(\star^5) \qquad \begin{array}{rcl} \nu(A_m^c) & \leq & um\big(var_{m+1}(g) + (1-2\lambda)\big(\sum_i \# F_i(1-\lambda)^{m+p_i}\big)\big) \\ & & \quad + \sum_{v \geq u} m^2 (v-u+1)\mu_\lambda(X_{vm+1}). \end{array}$$

To end with we compute that by definition of μ_λ,

$$\mu_\lambda(X_{vm+1}) \leq (1-(2\lambda)^m)^v, \quad \text{and} \quad \sum_{v \geq u} (v+1-u)m^2(1-(2\lambda)^m)^v = m^2 \frac{q^u}{q(1-q)^2}$$

with $q = 1 - (2\lambda)^m$. ■

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