M P E J

Mathematical Physics Electronic Journal

ISSN 1086-6655
Volume 12, 2006

Paper 4
Received: Jun 5, 2005, Accepted: Aug 7, 2006
Editor: R. Benguria

On the heat trace of the magnetic Schrödinger operators on the hyperbolic plane

Yuzuru INAHAMA *
and
Shin-ichi SHIRAI †

Abstract

In this paper we study the heat trace of the magnetic Schrödinger operator

$$H_V(\mathbf{a}) = \frac{1}{2}y^2\left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x} - a_1(x,y)\right)^2 + \frac{1}{2}y^2\left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial y} - a_2(x,y)\right)^2 + V(x,y)$$

on the hyperbolic plane $\mathbb{H} = \{z = (x,y)|x \in \mathbb{R}, y > 0\}$. Here $\mathbf{a} = (a_1, a_2)$ is a magnetic vector potential and V is a scalar potential on $\mathbb{H}$. Under some growth conditions on $\mathbf{a}$ and V at infinity, we derive an upper bound of the difference $\operatorname{Tr} e^{-tH_V(\mathbf{0})} - \operatorname{Tr} e^{-tH_V(\mathbf{a})}$ as $t \to +0$.

As a byproduct, we obtain the asymptotic distribution of eigenvalues less than λ as $\lambda \to +\infty$ when V has exponential growth at infinity (with respect to the Riemannian distance on $\mathbb{H}$). Moreover, we obtain the asymptotics of the logarithm of the eigenvalue counting function as $\lambda \to +\infty$ when V has polynomial growth at infinity. In both cases we assume that $\mathbf{a}$ is weaker than V in an appropriate sense.

*Partially supported by JSPS Fellowships for Young Scientists.
E-mail: `inahama@math.titech.ac.jp`
†E-mail: `shirai@ge.oit.ac.jp`

1 Introduction and Results

We study the short time asymptotics of the trace of the heat semi-group for the magnetic Schrödinger operator

$$H_V(\mathbf{a}) = \frac{1}{2}y^2\left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x} - a_1(x,y)\right)^2 + \frac{1}{2}y^2\left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial y} - a_2(x,y)\right)^2 + V(x,y) \tag{1.1}$$

on the hyperbolic plane $\mathbb{H} = \{z = (x,y) | x \in \mathbb{R}, y > 0\}$, and we obtain the asymptotic distribution of large eigenvalues of $H_V(\mathbf{a})$ under some growth conditions on V and $\mathbf{a}$. Here, V is a scalar potential and $\mathbf{a} = (a_1, a_2)$ is a magnetic vector potential on $\mathbb{H}$. Throughout this paper we identify the vector potential $\mathbf{a}$ with the 1-form $a_1dx + a_2dy$ on $\mathbb{H}$.

The Riemannian measure $m(dz)$ on $\mathbb{H}$ is given by $m(dz) = dxdy/y^2$ and the Riemannian distance $d(z, z')$ between $z = (x,y)$ and $z' = (x', y')$ is given by

$$\cosh(d(z,z')) = \frac{(x-x')^2 + y^2 + (y')^2}{2yy'}.$$

In what follows we canonically identify $\mathbb{H}$ with the complex upper-half plane via the correspondence $(x,y) \leftrightarrow x + y\sqrt{-1}$.

In what follows we write ∂_x for $\partial/\partial x$, etc., and we use the multi-index notation like $\partial^\alpha f(z)$ to denote $\partial_x^{\alpha_1}\partial_y^{\alpha_2} f(z)$ for any multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}\times\mathbb{N}$. Here, $\mathbb{N}$ stands for the set of *non-negative* integers. We denote by $C^k(M,N)$ the space of all N-valued, C^k-functions on M for $k \in \mathbb{N}\cup\{\infty\}$, and by $C(M,N)$ the space of all N-valued, continuous functions on M. We denote by $C_0^\infty(M,N)$ the space of all smooth functions with compact support, etc. When $N = \mathbb{C}$, we write $C^k(M)$ for $C^k(M,\mathbb{C})$, etc. The notation $A_1 + \ldots + A_n =: B_1 + \ldots + B_n$ means that $B_1, \ldots, B_n$ stand for $A_1, \ldots, A_n$, respectively. Throughout the paper we use the symbol $|\cdot|$ to denote the Euclidean norms.

To formulate the results, we introduce a class of functions. For a continuous function $\tilde{a}$ defined on $[0,\infty)$, we say that $\tilde{a}$ *belongs to the class* $\mathcal{G}$ if, for any δ satisfying $0 < \delta < 1$, there exists a positive number C_δ such that

$$\tilde{a}(\rho + \rho') \le C_\delta \tilde{a}\left((1+\delta)\rho\right) \exp\left(C_\delta(\rho')^2\right) \tag{1.2}$$

holds for any positive ρ and ρ'.

For example, if $c \ge 0$, $C \ge 0$ and $0 < \beta \le 2$, the polynomials $\tilde{a}(\rho) = C\rho^c$ and the exponentials $\tilde{a}(\rho) = C\exp\left(c\rho^\beta\right)$ belong to $\mathcal{G}$. We note that if $\tilde{a}$ belongs to $\mathcal{G}$, the functions $C\,\tilde{a}(c\rho)^\alpha$ also belong to $\mathcal{G}$ for any $c, C > 0$ and $\alpha > 0$. We also note that if $\tilde{a} \in \mathcal{G}$ then $\tilde{a}$ has a Gaussian bound $\tilde{a}(\rho) \le C\exp\left(c\rho^2\right)$ for some $c, C > 0$.

We now make the following conditions (A) for vector potentials and (V) for scalar potentials.

(A) The vector potential $\mathbf{a} = (a_1, a_2)$ belongs to $C^2(\mathbb{H}, \mathbb{R}^2)$. Moreover, there exists $\tilde{a} \in \mathcal{G}$ such that

$$\sum_{0\le|\alpha|\le 2}\left(y^{|\alpha|+1}|\partial^\alpha a_1(z)| + y^{|\alpha|+1}|\partial^\alpha a_2(z)|\right) \le \tilde{a}(d(z,\sqrt{-1})) \tag{1.3}$$

holds for all $z \in \mathbb{H}$.

(V) The scalar potential V belongs to $C(\mathbb{H},\mathbb{R})$. Moreover, there exist $\varepsilon > 0$ and $C > 0$ such that

$$C^{-1} d(z,\sqrt{-1})^{1+\varepsilon} \le V(z) \le C\exp\left(Cd(z,\sqrt{-1})^2\right)$$

holds outside some compact subset of $\mathbb{H}$.

It is known that the operator $H_V(\mathbf{a})$ is essentially self-adjoint on $C_0^\infty(\mathbb{H})$ and has the unique self-adjoint realization acting on $L^2(\mathbb{H})$ under the conditions (A) and (V) (See Shubin [15]). In what follows we identify any essentially self-adjoint operator with its operator closure.

Theorem 1.1 *Assume (A) and (V). Then the operator $e^{-tH_V(\mathbf{a})}$ is of trace class for all $t > 0$. Moreover, for any δ satisfying $0 < \delta < 1$, there exists $C_\delta > 0$ such that*

$$\begin{aligned} 0 &\le \operatorname{Tr} e^{-tH_V(\mathbf{0})} - \operatorname{Tr} e^{-tH_V(\mathbf{a})} \\ &\le C_\delta\, t \int_{\mathbb{H}} \tilde{a}\left((1+\delta)d(z,\sqrt{-1})\right)^2 e^{-tV_\delta^-(z)} m(dz) + C_\delta t^{11/8} \end{aligned}$$

holds if $0 < t \le 1/C_\delta$. Here, we set

$$V_\delta^-(z) = \inf\{V(z') | d(z,z') \le \delta d(z,\sqrt{-1})\}.$$

Remark 1.2 *The base point $z = \sqrt{-1}$ in (V) and (A) can be replaced by any fixed point $z_0 \in \mathbb{H}$ by the homogeneity of $\mathbb{H}$.*

The condition (A) is described in terms of the magnetic vector potential $\mathbf{a} = a_1 dx + a_2 dy$, which can be regarded as a connection 1-form on (the trivial Hermitian line bundle over) $\mathbb{H}$. From a physical view point, we prefer to make assumptions for the corresponding magnetic field $\omega = d\mathbf{a} = (\partial_x a_2 - \partial_y a_1) dx \wedge dy$ rather than for $\mathbf{a}$ itself, where d stands for the usual exterior derivative. In Appendix, we give a condition on the magnetic field ω which implies the existence of a vector potential $\mathbf{a}$ satisfying (A).

As a byproduct of Theorem 1.1, the standard Tauberian argument yields the asymptotic distribution of the number of large eigenvalues of $H_V(\mathbf{a})$ under some restrictive condition on the growth of V at infinity. We denote by $N(T < \lambda)$ the number of eigenvalues (counting multiplicities) of a self-adjoint operator T less than λ.

Corollary 1.3 *Let V belong to $C(\mathbb{H},\mathbb{R})$. Assume that there exist positive constants A and α such that*

$$\lim_{d(z,\sqrt{-1})\to\infty} \frac{V(z)}{A\exp\left(\alpha d(z,\sqrt{-1})\right)} = 1. \tag{1.4}$$

Assume that there exist positive numbers C and β satisfying $\beta < \alpha$ such that the condition (A) holds for $\tilde{a}(\rho) = C\exp(\beta\rho)$. Then we have the eigenvalue asymptotics

$$\begin{aligned} \lim_{\lambda\to\infty} N(H_V(\mathbf{a}) < \lambda)/\lambda^{1+1/\alpha} &= \lim_{\lambda\to\infty} N(H_V(\mathbf{0}) < \lambda)/\lambda^{1+1/\alpha} \\ &= \frac{1}{2}\frac{\alpha}{\alpha+1} A^{-1/\alpha}. \end{aligned}$$

If the scalar potential V has polynomial growth (with respect to the Riemannian distance) at infinity, we can obtain the asymptotics of the logarithm of the eigenvalue counting function $N(H_V(\mathbf{a}) < \lambda)$ as $\lambda \to \infty$.

Corollary 1.4 *Let V belong to $C(\mathbb{H}, \mathbb{R})$. Assume that there exist $A > 0$ and $\alpha > 1$ such that*

$$\lim_{d(z,\sqrt{-1})\to\infty} \frac{V(z)}{A\, d(z, \sqrt{-1})^{\alpha}} = 1. \tag{1.5}$$

Assume that (A) holds with $\tilde{a}(\rho) = C(\rho^{\beta}+1)$ for some $C > 0$ and β satisfying $0 < 2\beta < \alpha$. Then we have the asymptotic relation

$$\begin{aligned}\lim_{\lambda\to\infty} \lambda^{-1/\alpha} \log N(H_V(\mathbf{a}) < \lambda) &= \lim_{\lambda\to\infty} \lambda^{-1/\alpha} \log N(H_V(\mathbf{0}) < \lambda) \\ &= A^{-1/\alpha}.\end{aligned}$$

Remark 1.5 *In [12] and [13], Matsumoto studied the short time asymptotics of the (difference of) heat traces for the magnetic Schrödinger operators. In particular, Matsumoto [13] established the short time asymptotics of heat trace even in the case where the strength of magnetic fields is "stronger" than the scalar potential. (As a byproduct, the large eigenvalue asymptotics is also obtained by Tauberian argument.)*

However, the hyperbolic spaces are not included in the class of Riemannian manifolds under Matsumoto's consideration.

The organization of this paper is as follows: In Section 2, we recall some basic facts from stochastic analysis and introduce the Brownian motion on $\mathbb{H}$. Also we formulate some preparatory results on the pinned Wiener measures. In Section 3, we give proofs of theorem 1.1, Corollary 1.3 and Corollary 1.4, accepting two propositions (Proposition 4.1 and Proposition 3.6). In Section 4 and Section 5, we give a proof of Proposition 3.6 and of Proposition 4.1, respectively. In Appendix we rewrite the condition (A) for the vector potential $\mathbf{a}$ to the condition for the corresponding magnetic field.

Acknowledgment. The authors thank Professor Yuji KASAHARA for his advice on Tauberian theorems of exponential type.

2 Results from stochastic analysis

2.1 Generalized expectations

In this subsection we recall some basic definitions and results from the Malliavin calculus along the line of Ikeda and Watanabe [8], Chapter V, Sections 8 and 9. For any nonnegative integer d, we denote by $(W^{(d)}, H^{(d)}, P^{(d)})$ the d-dimensional Wiener space. Let $W^{(d)} = \{w \in C([0,\infty), \mathbb{R}^d) | w(0) = 0\}$ be the d-dimensional Wiener space. Let

$$H^{(d)} = \{h \in W^{(d)} | h \text{ is absolutely continuous and } \|h\|^2_{H^{(d)}} = \textstyle\int_0^\infty |\dot{h}(s)|^2 ds < \infty\}$$

be the Cameron-Martin subspace, where $\dot{h}$ denotes the derivative of h, and let $P^{(d)}$ be the Wiener measure on $W^{(d)}$. As usual we denote by $E^{(d)}[\,\cdot\,]$ the integration with respect to $P^{(d)}$. We denote by $w_t = (w_t^1, \ldots, w_t^d)$ $(t \geq 0)$ the canonical realization of the Wiener process. (We often drop the superscript (d) if there is no fear of confusion.)

For any $h \in H$, we define the measurable linear functional $[h](w) = \sum_{i=1}^d \int_0^\infty \dot{h}^i(s) \cdot dw_s^i$. The law of $[h]$ is the Gaussian measure with mean 0 and variance $\|h\|_H^2$. For any orthonormal elements $h_1, \ldots, h_n \in H$ and $f \in \mathcal{S}(\mathbb{R}^d)$, a function of the form $F(w) = f([h_1](w), \ldots, [h_n](w))$ is called *a cylindrical function*. The Ornstein-Uhlenbeck operator L is defined by

$$LF(w) = \sum_{i=1}^n \left(\frac{\partial^2 f}{\partial x_i^2}\big([h_1](w), \ldots, [h_n](w)\big) - [h_i](w) \cdot \frac{\partial f}{\partial x_i}\big([h_1](w), \ldots, [h_n](w)\big) \right)$$

on the space of all cylindrical functions, which is a core for L. For any $p \in (1, \infty)$ and $r \in R$, the Sobolev space $\mathbf{D}_{p,r}$ is the completion of the space of cylindrical functions on W with respect to the norm $\|F\|_{p,r} = \|(I-L)^{r/2} F\|_{L^p(W,P)}$. The spaces of *test functionals* and of *generalized Wiener functionals* are defined by $\mathbf{D}_\infty = \cap_{p>1, r\in\mathbb{R}} \mathbf{D}_{p,r}$ and $\mathbf{D}_{-\infty} = \cup_{p>1, r\in\mathbb{R}} \mathbf{D}_{p,r}$, respectively. (For a separable Hilbert space K, the Sobolev spaces of K-valued function(al)s are defined in a similar way. In that case we write $\mathbf{D}_{p,r}(K)$, $\mathbf{D}_\infty(K)$, $H(K)$, etc.) Note that for any non-negative integer r and for any $p > 1$ there exists a positive constant $C_{p,r}$ such that Meyer's equivalence

$$C_{p,r}^{-1} \|F\|_{p,r} \leq \sum_{j=0}^r \|D^j F\|_{L^p} \leq C_{p,r} \|F\|_{p,r}$$

holds for all $F \in \mathbf{D}_\infty$. Here D denotes the H-derivative, i.e., the Gâtaux derivative in H-direction.

The paring of $F \in \mathbf{D}_\infty$ and $\Psi \in \mathbf{D}_{-\infty}$ is defined in a canonical way and is denote by $E[\Psi \cdot F]$. (We often write as $E[\Psi(w)F(w)]$.) The pairing $E[\Psi \cdot 1]$ is often denoted simply by $E[\Psi]$ or formally by $\int_W \Psi(w) P(dw)$. We call $E[\,\cdot\,]$ *the generalized expectation.*

For any $F = (F^1, \ldots, F^d) \in \mathbf{D}_\infty(\mathbb{R}^d)$, we say that F *is non-degenerate in the sense of Malliavin* if

$$\det\Big(\langle DF^i, DF^j \rangle_H \Big)_{i,j=1,\ldots,d}^{-1}$$

belongs to $\cap_{p>1} L^p(W, P)$. If F is non-degenerate in the sense of Malliavin, then, for any Schwartz distribution $\psi \in \mathcal{S}'(\mathbb{R}^d)$, the composition $\psi \circ F$ is well-defined and belongs to $\mathbf{D}_{-\infty}$. In fact, the mapping $\psi \mapsto \psi \circ F$ is bounded from $\mathcal{S}_{-2k}$ to $\mathbf{D}_{p,-2k}$ for every $p \in (1, \infty)$ and $k = 1, 2, \ldots$ (See [8], Chapter V, Section 9, for detailed information on the pullback of the Schwartz distributions).

It is known (See Sugita [16]) that, for every positive generalized Wiener functional Ψ, there exists a unique positive finite measure μ^Ψ on W such that

$$E[\Psi \cdot F] = \int_W \tilde{F}(w) \mu^\Psi(dw)$$

holds for any $F \in \mathbf{D}_\infty$, where $\tilde{F}$ stands for the $\mathbf{D}_\infty$-quasi continuous modification of F (See Malliavin [10] Chapter IV, Section 2, p.94). Note the $\tilde{F}(w)$ is uniquely defined up to the measure μ^Ψ.

2.2 Brownian motion on the hyperbolic plane

In this subsection we introduce the Brownian motion on $\mathbb{H}$ and the heat kernel for $H_0(\mathbf{0}) = -\triangle_{\mathbb{H}}/2$. In what follows we identify $z = (x, y) \in \mathbb{H}$ with $z = x + y\sqrt{-1}$ the inclusion $\mathbb{H} \hookrightarrow \mathbb{C} \cong \mathbb{R}^2$.

Let $w = (w^1, w^2) \in W^{(2)}$. We consider the following stochastic differential equation on $\mathbb{H}$:

$$dX(t) = Y(t)dw_t^1, \qquad dY(t) = Y(t)dw_t^2, \tag{2.1}$$

with the initial condition $(X(0), Y(0)) = z = (x, y)$. The solution is explicitly written as follows (See Ikeda and Matsumoto [7], p. 69):

$$\begin{aligned} X(t,z,w) &= x + y\int_0^t \exp(w_s^2 - s/2)dw_s^1, \\ Y(t,z,w) &= y\exp(w_t^2 - t/2). \end{aligned} \tag{2.2}$$

We write $Z(t,z,w) = (X(t,z,w), Y(t,z,w))$ and write $Z(t,w) = (X(t,w), Y(t,w))$ when $z = \sqrt{-1}$. Using Itô's formula, one can find that $\{Z(t,z,w)\}_{t\geq 0}$ is the Brownian motion on $\mathbb{H}$, i.e., a diffusion process whose generator is $\triangle_{\mathbb{H}}/2$. One can easily to see that $Z(t,z,w)$ is non-degenerate in the sense of Malliavin (See also [9]).

The integral kernel $p_{\mathbf{00}}(t,z,z')$ of $e^{-t\triangle_{\mathbb{H}}/2}$ is given by

$$p_{\mathbf{00}}(t,z,z') = \frac{\sqrt{2}e^{-t/8}}{(2\pi t)^{3/2}}\int_d^\infty \frac{be^{-b^2/2t}}{\sqrt{\cosh b - \cosh d}}db \tag{2.3}$$

with $d = d(z,z')$ (See, e.g., Terras [17]), and we have the following estimate

$$ck(t,z,z') \leq p_{\mathbf{00}}(t,z,z') \leq Ck(t,z,z') \tag{2.4}$$

holds for some constants $c, C > 0$ independent of z, z', t, where

$$k(t,z,z') = \frac{1}{2\pi t}\frac{1 + d(z,z')}{\sqrt{1 + d(z,z') + t/2}}\exp\left(-t/8 - d(z,z')/2 - d(z,z')^2/(2t)\right)$$

(See Theorem 5.7.2 in Davies [5]).

It is well-known (See [8], Chapter 5, Section 3) that $p_{\mathbf{00}}(t,z,z') = E[\tilde{\delta}_{z'}(Z(t,z,\cdot))]$ and the law of $Z(t,z,w)$ on $\mathbb{H}$ is given by $p_{\mathbf{00}}(t,z,z')m(dz)$. Here $\tilde{\delta}$ denotes the Dirac delta function on $\mathbb{H}$ with respect to the Riemannian measure $m(dz)$, i.e., $\tilde{\delta}_{z'}(z) = y^2\delta_{(x',y')}(x,y)$.

2.3 Pinned Wiener measures

In this subsection we introduce the pinned Wiener measure on $\mathbb{H}$ and recall some basic properties. Let $T > 0$ and $z, z' \in \mathbb{H}$. Set $W_T(\mathbb{H}) = C([0, T], \mathbb{H})$ and $\mathcal{L}_T^{z,z'}(\mathbb{H}) = \{l \in W_T(\mathbb{H}) | l_0 = z, l_T = z'\}$. We equip the space $W_T(\mathbb{H})$ with the distance $\tilde{d}(l, l') = \sup\{d(l_s, l'_s) | 0 \le s \le T\}$. Here d is the distance on $\mathbb{H}$. Then $W_T(\mathbb{H})$ is a complete separable metric space and $\mathcal{L}_T^{z,z'}(\mathbb{H})$ is a closed subspace. The pinned Wiener measure $P_T^{z,z'}$ on $\mathbb{H}$ is defined by the probability measure on $\mathcal{L}_T^{z,z'}(\mathbb{H})$ which satisfies

$$\int_{\mathcal{L}_T^{z,z'}(\mathbb{H})} \prod_{i=1}^{n} f_i(l_{t_i}) P_t^{z,z'}(dl) = p_0(t, z, z')^{-1} \int_{\mathbb{H}^d} \prod_{i=1}^{n} (m(dz_i) f_i(z_i)) \prod_{i=1}^{n+1} p_0(t_i - t_{i-1}, z_{i-1}, z_i) \tag{2.5}$$

for any partition $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$ of $[0, T]$ and any $f_1, \ldots, f_n \in C_0^\infty(\mathbb{H})$. Here we set $z_0 = z$ and $z_{n+1} = z'$.

We denote by $\mu_T^{z,z'}$ the probability measure which corresponds to the Wiener functional $\tilde{\delta}_{z'}(Z(T, z, w))/p_{\mathbf{00}}(T, z, z')$ by Sugita's theorem. It follows from Theorem 4.2 and Corollary 4.3 in Malliavin and Nualart [11] that there exists a process $\{\tilde{Z}(t, z, w)\}_{t \ge 0}$ which satisfies the following property: There exists a decreasing sequence $\mathcal{O}_j$ $(j = 1, 2, \ldots)$ of open subsets of W such that

1. For each $j = 1, 2, \ldots$, $(t, w) \mapsto \tilde{Z}(t, z, w)$ is continuous on $[0, T] \times \mathcal{O}_j^c$.
2. For each $p \in (1, \infty)$ and $r > 0$, $\mathrm{cap}_{p,r}(\mathcal{O}_j) \to 0$ as $j \to \infty$. Here $\mathrm{cap}_{p,r}$ denotes the (p, r)-capacity.
3. For each $t \in [0, T]$, $Z(t, z, w) = \tilde{Z}(t, z, w)$ outside a set of zero Wiener measure.

In particular, $\{\tilde{Z}(t, z, w)\}_{0 \le t \le T}$ is a well-defined $W_T(M)$-valued random variable on the measure space $(W, \mu_T^{z,z'})$. By using the Chapman-Kolmogorov formula, we can easily see that the image measure of $\mu_T^{z,z'}$ induced by the $W_T(M)$-valued Wiener functional $\{\tilde{Z}(t, z, w)\}_{0 \le t \le T}$ is the pinned Wiener measure $P_T^{z,z'}$. That is, for any bounded Borel function F on $W_T(M)$,

$$\int_W F(\tilde{Z}(\cdot, z, w)) \mu_T^{z,z'}(dw) = \int_{\mathcal{L}_T^{z,z'}(\mathbb{H})} F(l) P_T^{z,z'}(dl)$$

holds. Note that this fact can also be regarded as an existence theorem of the pinned Wiener measure on $\mathbb{H}$.

In what follows we denote by $E_T^{z,z'}$ the expectation with respect to the pinned Wiener measure $P_T^{z,z'}$. Also we denote by E^z the expectation with respect to the Wiener measure P^z on the space of all continuous paths starting at z.

We shall often use the fact that, for any s, T satisfying $0 < s < T$,

$$P_T^{z,z'}|_{\mathcal{B}_s} = \frac{p_{\mathbf{00}}(T - s, l_s, z')}{p_{\mathbf{00}}(T, z, z')} P^z|_{\mathcal{B}_s} \tag{2.6}$$

holds, where $\mathcal{B}_*$ stands for the natural filtration of $W_T(\mathbb{H})$.

3 Preliminary estimates

In this section we obtain some preliminary estimates concerning the pinned Wiener measure on $\mathbb{H}$.

Lemma 3.1 *Assume that $\tilde{a}$ belongs the class $\mathcal{G}$ defined in Section 1. Then, for any δ satisfying $0<\delta<1$, there exists $C_\delta>0$ such that the following assertions hold:*

1. *The estimate*

$$E^z[\tilde{a}(d(l_s,\sqrt{-1}))] \le C_\delta \tilde{a}\left((1+\delta)d(z,\sqrt{-1})\right) \tag{3.1}$$

holds for all s satisfying $0<s\le 1/C_\delta$ and for all $z\in\mathbb{H}$.

2. *The estimate*

$$E_t^{z,z}[\tilde{a}(d(l_s,\sqrt{-1}))] \le C_\delta \tilde{a}\left((1+\delta)d(z,\sqrt{-1})\right) \tag{3.2}$$

holds for all s,t with $0<s<t\le 1/C_\delta$ and for all $z\in\mathbb{H}$.

Proof. Using the relation (2.6) with $s=t/2$, we find that the left-hand side (lhs) of (3.2) is equal to

$$\begin{aligned}
& E^z[\tilde{a}\left(d(l_s,\sqrt{-1})\right)\frac{p_{\mathbf{00}}(t/2,l_{t/2},z)}{p_{\mathbf{00}}(t,z,z)}] \\
\le\ & E^z[\tilde{a}\left(d(l_s,\sqrt{-1})\right)\frac{p_{\mathbf{00}}(t/2,z,z)}{p_{\mathbf{00}}(t,z,z)}] \\
\le\ & CE^z[\tilde{a}\left(d(l_s,\sqrt{-1})\right)]
\end{aligned}$$

where we used the facts that $p_{\mathbf{00}}(t,z,z')\le p_{\mathbf{00}}(t,z,z)$ and that

$$p_{\mathbf{00}}(t,z,z)=p_{\mathbf{00}}(t,\sqrt{-1},\sqrt{-1})=(2\pi t)^{-1}(1+o(1)) \qquad \text{as} \qquad t\to +0$$

in the first and second inequality, respectively.

Thus it is enough to show only the assertion (3.1). By (1.2), for any δ satisfying $0<\delta<1$, there exists $C_\delta>0$ such that

$$\begin{aligned}
& E^z[\tilde{a}\left(d(l_s,\sqrt{-1})\right)] \\
=\ & \int_{\mathbb{H}} m(dz')\tilde{a}\left(d(z',\sqrt{-1})\right)p_{\mathbf{00}}(s,z,z') \\
\le\ & C_\delta\tilde{a}\left((1+\delta)d(z,\sqrt{-1})\right)\int_{\mathbb{H}} m(dz')\exp\left(C_\delta d(z,z')^2\right)p_{\mathbf{00}}(s,z',z) \\
=\ & C_\delta\tilde{a}\left((1+\delta)d(z,\sqrt{-1})\right)\int_{\mathbb{H}} m(dz)\exp\left(C_\delta d(z,\sqrt{-1})^2\right)p_{\mathbf{00}}(s,z,\sqrt{-1}) \\
\le\ & C_\delta s^{-1}\tilde{a}\left((1+\delta)d(z,\sqrt{-1})\right) \\
& \quad\times\int_{\mathbb{H}} m(dz)\exp\left(C_\delta d(z,\sqrt{-1})^2\right)\exp\left(-d(z,\sqrt{-1})^2/(2s)\right)
\end{aligned} \tag{3.3}$$

holds, where we used the $SL(2,\mathbb{R})$-invariance of d and $m(dz)$ in the third equality and used (2.4) in the last inequality.

Since $m(dz)$ is expressed as $\sinh\rho\ d\rho d\theta$ in the geodesic polar coordinates $(\rho,\theta) \in [0,\infty)\times[0,2\pi)$ (See, e.g., Terras [17]), we have

$$
\begin{aligned}
& \int_{\mathbb{H}} m(dz) \exp\left(C_\delta d(z,\sqrt{-1})^2\right) \exp\left(-d(z,\sqrt{-1})^2/(2s)\right) \\
= \;& 2\pi \int_0^\infty \exp\left(-\rho^2/(2s) + C_\delta \rho^2\right) \sinh\rho\ d\rho \\
= \;& 2\pi\sqrt{2s} \int_0^\infty \exp\left(-y^2 + C_\delta(\sqrt{2s}y)^2\right) \sinh(\sqrt{2s}y) dy \\
\leq \;& 2\pi s^{1/2} \int_0^{1/\sqrt{2s}} \exp\left(-y^2 + 2C_\delta s y^2\right) \frac{\sinh(\sqrt{2s}y)}{\sqrt{2s}y} \sqrt{2s} y dy \\
& +2\pi s^{1/2} \int_{1/\sqrt{2s}}^\infty \exp\left(-y^2 + 2C_\delta s y^2\right) e^{\sqrt{2s}y} dy \\
\leq \;& Cs \int_0^{1/\sqrt{2s}} \exp\left(-(1-2C_\delta s)y^2\right) y dy \\
& +Cs^{1/2}e^{2s} \int_{1/\sqrt{2s}}^\infty \exp\left(-\left(\frac{3}{4} - 2C_\delta\, s\right) y^2\right) dy \\
\leq \;& Cs \int_0^{1/\sqrt{2s}} y dy \\
& +Cs^{1/2}e^{2s} \exp\left(-\left(\frac{3}{4} - 2C_\delta\, s\right)\frac{1}{4s}\right) \int_{1/\sqrt{2s}}^\infty \exp\left(-\left(\frac{3}{4} - 2C_\delta\, s\right) y^2/2\right) \\
\leq \;& C'_\delta\, s
\end{aligned} \tag{3.4}
$$

holds for some $C'_\delta > 0$ if $0 < s < 1/(8C_\delta)$, where we changed the variable $y = \rho/\sqrt{2s}$ in the second equality and used the elementary facts that $\sinh x/x$ is bounded if $0 < x \leq 1$ and that $\sqrt{2s}y \leq 2s + y^2/4$ holds, in the second inequality.

Then we can derive the estimate (3.2) from (3.3) and (3.4) by choosing C_δ larger, so (3.1) also follows as we mentioned above. ∎

Remark 3.2 *The above proof shows that the assertion of Lemma 3.1 is still valid for $\delta = 0$ if $\tilde{a}(\rho) = C'e^{c\rho}$ at infinity.*

Lemma 3.3 *Let $t > 0$, $z, z' \in \mathbb{H}$ and $N > 0$. Then there exists a positive number $C = C(t,z,z',N)$ such that*

$$
E_t^{z,z'}[d(l_s,z)^{2N}] \leq Cs^N \tag{3.5}
$$

holds for all s with $0 \leq s \leq t$.

Moreover, if $z = z'$, the constant C is bounded uniformly in $z \in \mathbb{H}$ and t satisfying $0 < t \leq 1$.

Proof. First, we consider the case $0 < s \le t/2$. As in the proof of Lemma 3.1, using (2.6) with $s = t/2$, we see that the lhs of (3.5) is less than or equal to

$$\begin{aligned} & E^z[d(l_s, z)^{2N} \frac{p_{\mathbf{00}}(t/2, l_s, z')}{p_{\mathbf{00}}(t, z, z')}] \\ \le\ & \frac{p_{\mathbf{00}}(t/2, z', z')}{p_{\mathbf{00}}(t, z, z')} \int_{\mathbb{H}} m(d\zeta) d(\zeta, z)^{2N} p_{\mathbf{00}}(s, z, \zeta) \\ \le\ & \frac{p_{\mathbf{00}}(t/2, z', z')}{p_{\mathbf{00}}(t, z, z')} s^N \int_{\mathbb{H}} m(d\zeta) \left(d(\zeta, z)^2/s \right)^N p_{\mathbf{00}}(s, z, \zeta). \end{aligned} \tag{3.6}$$

By the same argument as in the proof of Lemma 3.1, we can deduce that the integral on the right-hand side (rhs) of (3.6) is bounded uniformly in s, z. Moreover, it follows from (3.2) that

$$\frac{p_{\mathbf{00}}(t/2, z', z')}{p_{\mathbf{00}}(t, z, z')} \le C \exp\left(d(z, z')/2 + d(z, z')^2/(2t) \right).$$

Then, taking the obvious fact $d(z, z) = 0$ into account, we deduce the assertion when $0 < s \le t/2$.

Next, we consider the case of $t/2 < s < t$. Using the triangle equality $d(l_s, z) \le d(l_s, z') + d(z, z')$ and the fact that $t - s \le t/2$ if $t/2 < s < t$, we find that the lhs of (3.5) is less than or equal to

$$\begin{aligned} & C_N \left(E_t^{z,z'}[d(l_s, z')^{2N}] + d(z, z')^{2N} \right) \\ \le\ & C_N C \left((t-s)^{2N} + d(z, z')^{2N} \right) \\ =\ & C_N C \left(1 + \frac{d(z, z')^{2N}}{(t/2)^{N/2}} \right) (t/2)^N \\ =\ & C'_N \left(1 + \frac{d(z, z')^{2N}}{(t/2)^{N/2}} \right) s^N, \end{aligned} \tag{3.7}$$

where we used the fact that $\hat{l}_u = l_{t-u}$ is the law of $P_t^{z',z}$ and the assertion for the case $0 < s < t/2$ in the second inequality. Note that the coefficient of s^N on the rhs of (3.7) has the desired boundedness. Thus the lemma obeys. ∎

Lemma 3.4 *Let $\alpha > 0$, $z \in \mathbb{H}$ and $0 < t \le 1$. Then*

$$E_t^{z,z}[|l_s^2|^\alpha] = y^\alpha E_t^{\sqrt{-1},\sqrt{-1}}[|l_s^2|] \tag{3.8}$$

is finite and bounded uniformly in s with $0 < s \le t$. Here we write $z = (x, y)$ and $l_s = (l_s^1, l_s^2)$.

Proof. For each $z_0 \in \mathbb{H}$, the map $g : z = (x, y) \to (x_0 + y_0 y, y_0 y)$ defines an isometric isomorphism on $\mathbb{H}$. Then by considering the finite dimensional distribution, we can deduce that

$$E_t^{z,z'}[f(l)] = E_t^{g(z),g(z')}[f(g^{-1}(l))]$$

holds for any nice function f and for any $z, z' \in \mathbb{H}$. Setting $z = z' = \sqrt{-1}$ and $f(z) = y^\alpha$, we have the equality (3.8) since $g(\sqrt{-1}) = z_0$.

Since elementary calculation shows that $y + 1/y \leq C \exp(Cd(z, \sqrt{-1}))$, the finiteness of the integral follows from Lemma 3.1. ∎

Lemma 3.5 *We set*

$$\Omega_{\varepsilon,t,z} = \{l \in \mathcal{L}_t^{z,z}(\mathbb{H}) | \sup_{0 \leq s \leq t} d(l_s, z) \geq \varepsilon\}.$$

There exist positive constants c, C' such that

$$P_t^{z,z}(\Omega_{\varepsilon,t,z}) \leq C' t^{-1/4} \exp(-c\varepsilon^2/t)$$

holds for any $z \in \mathbb{H}$, $\varepsilon > 0$ and $t > 0$ satisfying the relations $0 < t < 1$ and $\varepsilon^2/t \geq 1$.

Proof. The estimate has been shown by Eberle [6], Proposition 3.1 (i) in the case of $t = 1$. Replacing the partition $i2^{-k}$ $(0 \leq i \leq 2^k)$ of $[0, 1]$ used in [6] by $i2^{-k}t$ $(0 \leq i \leq 2^k)$ of $[0, t]$, one can show the assertion for general t and ε in a similar way. ∎

The following result is crucial for the proof of the main theorem. We give a proof in Section 5.

Proposition 3.6 *Assume (A). Then for any $t > 0$, the Stratonovich integral*

$$\int_0^t \mathbf{a}(l_s) \circ dl_s = \int_0^t a_1(l_s) \circ dl_s^1 + \int_0^t a_2(l_s) \circ dl_s^2$$

defines a well-defined random variable on $\mathcal{L}_t^{z,z}$. Moreover, for any δ satisfying $0 < \delta < 1$, there exists $C_\delta > 0$ such that

$$E_t^{z,z}[\left| \int_0^t \mathbf{a}(l_s) \circ dl_s \right|^4] \leq C_\delta\, t^4 \tilde{a}\left(d(z, \sqrt{-1})\right)^4$$

holds for all $z \in \mathbb{H}$ and all $t > 0$ satisfying $0 < t < 1/C_\delta$. Here $\tilde{a}$ is as in (A).

4 Proof of the main results

4.1 Proof of Theorem 1.1

In this section we give a proof of the main theorem, following the line of argument as in the proof of Theorem 1 in Matsumoto [12].

We need the Feynman-Kac-Itô representation for the heat kernel of $H_V(\mathbf{a})$ (We give a proof in Section 4 below):

Proposition 4.1 *Assume (A) and (V). For any $t > 0$, the operator $e^{-tH_V(\mathbf{a})}$ has the continuous integral kernel $p_{\mathbf{a}V}(t,z,z')$. Moreover, $p_{\mathbf{a}V}(t,z,z')$ has the Feynman-Kac-Itô expression*

$$p_{\mathbf{a}V}(t,z,z') = p_{\mathbf{0}\mathbf{0}}(t,z,z')E_t^{z,z'}[\exp\left(-\sqrt{-1}\int_0^t \mathbf{a}(l_s)\circ dl_s - \int_0^t V(l_s)ds\right)]. \tag{4.1}$$

Then we can deduce from a result by Brislawn [4] that the heat trace of $H_V(\mathbf{a})$ is given by the integration of $p_{\mathbf{a}V}(t,z,z)$ with respect to z, as in Inahama and Shirai [9].

Lemma 4.2 *We set*

$$V_{(\varepsilon)}^-(z) = \inf\{V(z')|d(z,z') \le \varepsilon\}.$$

Then there exist positive constants C', c such that

$$p_{\mathbf{0}V}(t,z,z) \le C't^{-1}(t^{-1/4}e^{-c\varepsilon^2/t} + e^{-tV_\varepsilon^-(z)})$$

for any $z \in \mathbb{H}$, $\varepsilon > 0$ and $t > 0$ satisfying $0 < t < 1$ and $\varepsilon^2/t \ge 1$.

Proof. We use the Feynman-Kac representation (See [9]):

$$p_{\mathbf{0}V}(t,z,z) \quad = \quad p_{\mathbf{0}\mathbf{0}}(t,z,z)E_t^{z,z}[\exp\left(-\int_0^t V(l_s)ds\right)]. \tag{4.2}$$

Let $\Omega_{\varepsilon,t,z}$ be the set as in Lemma 6 and in this proof we denote simply by Ω. Then it follows that

$$\begin{aligned} E_t^{z,z}[\exp(-\int_0^t V(l_s)ds)] &= \left(\int_\Omega + \int_{\Omega^c}\right)\exp\left(-\int_0^t V(l_s)ds\right)P_t^{z,z}(dl) \\ &\le P_t^{z,z}(\Omega) + \int_{\Omega^c}\exp(-tV_{(\varepsilon)}^-(z))P_t^{z,z}(dl) \\ &\le C't^{-1/4}\exp(-c\varepsilon^2/t) + \exp(-tV_{(\varepsilon)}^-(z)), \end{aligned} \tag{4.3}$$

where we used Lemma 3.5 in the last inequality. Then the lemma follows from (4.2) and (4.3) since $p_{\mathbf{0}\mathbf{0}}(t,z,z) = O(1/t)$ holds as $t \to +0$ uniformly in z. ▮

Lemma 4.3 *Let $\tilde{a}$ be any continuous function satisfying the condition (A). We set*

$$V_\kappa^-(z) = \inf\{V(z')|d(z,z') \le \kappa d(z,\sqrt{-1})\}.$$

Then there exists $C>0$ such that, for any δ satisfying $0<\delta<1$,

$$\begin{aligned}
&\int_{\mathbb{H}} \tilde{a}\left(d(z,\sqrt{-1})\right) p_{\mathbf{0}V}(t,z,z)^{1/2} m(dz) \\
\le\ & Ct^{-1/2}\int_{\mathbb{H}} \tilde{a}\left(d(z,\sqrt{-1})\right) e^{-tV_\kappa^-(z)/2} m(dz) + Ct^{-1/8}
\end{aligned}$$

holds if $0<t\le\delta^2$.

Proof. Let $0<\delta<1$. Applying Lemma 4.2 with $\varepsilon = \delta d(z,\sqrt{-1})$, we find that

$$p_{\mathbf{0}V}(t,z,z)^{1/2} \le Ct^{-1/2}\left(t^{-1/8}\exp\left(-c\delta^2 d(z,\sqrt{-1})^2/(2t)\right) + \exp\left(-tV_{(\varepsilon)}^-(z)/2\right)\right)$$

holds if $\varepsilon^2/t = \delta^2 d(z,\sqrt{-1})^2/t \ge 1$, where we used the inequality $(a+b)^{1/2} \le a^{1/2}+b^{1/2}$. Then we find that

$$\begin{aligned}
&\int_{\mathbb{H}} \tilde{a}(d(z,\sqrt{-1})) p_{\mathbf{0}V}(t,z,z)^{1/2} m(dz) \\
=\ & \int_{d(z,\sqrt{-1})\ge\sqrt{t}/\delta} + \int_{d(z,\sqrt{-1})\le\sqrt{t}/\delta} \\
\le\ & Ct^{-5/8}\int_{\mathbb{H}} \exp\left(Cd(z,\sqrt{-1})^2\right)\exp\left(-c\delta^2 d(z,\sqrt{-1})^2/(2t)\right) m(dz) \\
& + Ct^{-1/2}\int_{\mathbb{H}} \tilde{a}(d(z,\sqrt{-1})) e^{-tV_{(\varepsilon)}^-(z)/2} m(dz) \\
& + \int_{d(z,\sqrt{-1})\le\sqrt{t}/\delta} \tilde{a}(d(z,\sqrt{-1})) p_{\mathbf{0}V}(t,z,z)^{1/2} m(dz)
\end{aligned} \tag{4.4}$$

holds for any $t>0$, where we used the Gaussian bound for $\tilde{a}\in\mathcal{G}$ in the inequality.

Then we find that the first term on the rhs of (4.4) is less than or equal to

$$\begin{aligned}
& Ct^{-5/8}\int_0^\infty \exp\left(C\rho^2 - c\rho^2/(2t)\right)\sinh\rho\, d\rho \\
\le\ & C't^{-5/8}\int_0^\infty \exp\left(-(c/(2t)-C-1)\rho^2\right) d\rho \\
=\ & C't^{-5/8}(c/(2t)-C-1)^{-1/2}\int_0^\infty e^{-y^2}dy \\
\le\ & C''t^{-1/8}
\end{aligned}$$

holds if $0<t<c/(4(C+1))$, where we changed the variable $y = (c/(2t)-C-1)^{1/2}\rho$.

Since $\tilde{a}$ is continuous near $\rho = 0$, we find that the third term on the rhs of (4.4) is less than or equal to

$$
\begin{aligned}
& Ct^{-1/2}\left(\sup_{0\le\rho\le 1}\tilde{a}(\rho)\right)\int_0^{\sqrt{t}/\delta}\sinh\rho\, d\rho \\
\le\ & C'\left(\sup_{0\le\rho\le 1}\tilde{a}(\rho)\right)\delta^{-2}t^{1/2}
\end{aligned}
$$

holds if $0 < t < \delta^2$, where we used the fact that $p_{\mathbf{0}V}(t,z,z) \le p_{\mathbf{00}}(t,z,z) = O(t^{-1})$ holds as $t \to +0$ by the Feynman-Kac formula. This completes the proof. ▌

Using the continuity and the reality of the kernel $p_{\mathbf{a}V}(t,z,z)$, which follows from Proposition 4.1 and the self-adjointness of $H_V(\mathbf{a})$, respectively, we obtain

$$
p_{\mathbf{a}V}(t,z,z) = p_{\mathbf{00}}(t,z,z)E_t^{z,z}[\cos\left(\int_0^t \mathbf{a}(l_s)\circ dl_s\right)\exp\left(-\int_0^t V(l_s)ds\right)]
$$

as in the proof of Theorem 1 in Matsumoto [12]. This expression of the kernel leads us to the trace version dia-magnetic inequality $\operatorname{Tr} e^{-tH_V(\mathbf{a})} \le \operatorname{Tr} e^{-tH_V(\mathbf{0})}$. Then, using Schwarz' inequality, we have, for any δ satisfying $0 < \delta < 1$,

$$
\begin{aligned}
& |p_{\mathbf{a}V}(t,z,z) - p_{\mathbf{0}V}(t,z,z)| \\
\le\ & p_{\mathbf{00}}(t,z,z)E_t^{z,z}[\frac{1}{2}\left|\int_0^t \mathbf{a}(l_s)\circ dl_s\right|^2 \exp(-\int_0^t V(l_s)ds)] \\
\le\ & \frac{1}{2}p_{\mathbf{00}}(t,z,z)\left(E_t^{z,z}[\left|\int_0^t \mathbf{a}(l_s)\circ dl_s\right|^4]\right)^{1/2}\left(E_t^{z,z}[\exp\left(-\int_0^t 2V(l_s)ds\right)]\right)^{1/2} \\
=\ & \frac{1}{2}p_{\mathbf{00}}(t,z,z)^{1/2}\left(E_t^{z,z}[\left|\int_0^t \mathbf{a}(l_s)\circ dl_s\right|^4]\right)^{1/2} p_{\mathbf{0},2V}(t,z,z)^{1/2} \\
\le\ & C_\delta t^{3/2}\tilde{a}\left((1+\delta)d(z,\sqrt{-1})\right)^2 p_{\mathbf{0},2V}(t,z,z)^{1/2} \qquad (4.5)
\end{aligned}
$$

for some $C_\delta > 0$, where we used the estimate $|\cos x - 1| \le |x|^2/2$ in the first inequality and used Proposition 3.6 and the estimate $p_{\mathbf{00}}(t,z,z) = O(t^{-1})$ in the third inequality.

Since the function $\tilde{a}(c\rho)^2$ belongs to the class $\mathcal{G}$, it follows from Lemma 4.3 (replaced $\tilde{a}(\rho)$ and V by $\tilde{a}((1+\delta)\rho)^2$ and $2V$, respectively) that, for any δ satisfying $0 < \delta < 1$, there exist $C_\delta > 0$ such that

$$
\begin{aligned}
& \int_{\mathbb{H}} |p_{\mathbf{a}V}(t,z,z) - p_{\mathbf{0}V}(t,z,z)|m(dz) \\
\le\ & C_\delta t^{3/2}\int_{\mathbb{H}}\tilde{a}\left((1+\delta)d(z,\sqrt{-1})\right)^2 p_{\mathbf{0},2V}(t,z,z)^{1/2}m(dz) \\
\le\ & (C_\delta)^2 t^{3/2}\left(t^{-1/2}\int_{\mathbb{H}}\tilde{a}\left((1+\delta)d(z,\sqrt{-1})\right)^2 e^{-tV_\delta^-(z)}m(dz) + t^{-1/8}\right)
\end{aligned}
$$

holds if $0 < t < 1/(C_\delta)^2$. Then we complete the proof of Theorem 1.1 by choosing C_δ larger.

4.2 Proof of Corollary 1.3

Before proceeding to the proof, we make a reduction. The condition (1.4) implies that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$(1-\varepsilon)A\exp\left(\alpha\, d(z,\sqrt{-1})\right) - C_\varepsilon \le V(z) \le (1+\varepsilon)A\exp\left(\alpha\, d(z,\sqrt{-1})\right) + C_\varepsilon$$

holds for all $z \in \mathbb{H}$. We set $W_A^\alpha(z) = A\exp\left(\alpha\, d(z,\sqrt{-1})\right)$ for any $A > 0$ and $\alpha > 0$. Then the min-max principle (Reed and Simon [14]) yields that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$N(H_{W^\alpha_{(1+\varepsilon)A}}(\mathbf{a}) < \lambda - C_\varepsilon) \le N(H_V(\mathbf{a}) < \lambda) \le N(H_{W^\alpha_{(1-\varepsilon)A}}(\mathbf{a}) < \lambda + C_\varepsilon) \tag{4.6}$$

holds for any vector potentials $\mathbf{a}$ satisfying the condition (A). If we assume that Corollary 1.3 holds for $V = A\exp\left(\alpha\, d(z,\sqrt{-1})\right)$ for any $A > 0$ and $\alpha > 1$, it follows from (4.6) that, for any $\varepsilon > 0$,

$$\begin{aligned}
& N(H_{W^\alpha_{(1+\varepsilon)A}}(\mathbf{a}) < \lambda - C_\varepsilon)/(\lambda - C_\varepsilon)^{1+1/\alpha} \cdot \frac{(\lambda - C_\varepsilon)^{1+1/\alpha}}{\lambda^{1+1/\alpha}} \\
\le\ & N(H_V(\mathbf{a}) < \lambda)/\lambda^{1+1/\alpha} \\
\le\ & N(H_{W^\alpha_{(1-\varepsilon)A}}(\mathbf{a}) < \lambda + C_\varepsilon)/(\lambda + C_\varepsilon)^{1+1/\alpha} \cdot \frac{(\lambda + C_\varepsilon)^{1+1/\alpha}}{\lambda^{1+1/\alpha}}
\end{aligned}$$

holds, from which we obtain the results for general Vs by taking a limit $\varepsilon \to +0$ after $\lambda \to \infty$. Therefore, in the rest of this subsection, we may assume that $V(z) = A\exp\left(\alpha\, d(z,\sqrt{-1})\right)$ for all z.

One can also verify that the assumptions (A.1) and (A.2) of the main theorem in Inahama and Shirai [9] are fulfilled in our situation (by a direct computation, or see Section 7 in [9]). So, the main theorem in [9] tells us that

$$\lim_{\lambda\to\infty} N(H_V(\mathbf{0}) < \lambda)/\lambda^{1+1/\alpha} = \frac{1}{2}\frac{\alpha}{\alpha+1}A^{-1/\alpha},$$

or equivalently, by Karamata's Tauberian theorem,

$$\begin{aligned}
\lim_{t\to+0} t^{1+1/\alpha}\operatorname{Tr} e^{-tH_V(\mathbf{0})} &= \lim_{\lambda\to\infty}\frac{1}{(2\pi)^2\lambda^{1+1/\alpha}}\left|\{(z,\xi)\in \mathrm{T}^*\mathbb{H} | y^2|\xi|^2/2 + V(z) < \lambda\}\right| \\
&= \frac{1}{2}\frac{\alpha}{\alpha+1}A^{-1/\alpha}\Gamma(2+\frac{1}{\alpha}),
\end{aligned} \tag{4.7}$$

where $|\cdot|$ denotes the four dimensional Lebesgue measure and Γ is the Gamma function.

By Remark 3.2 after the proof of Lemma 3.1 and Theorem 1.1 with $\tilde{a}(\rho) = Ce^{\beta\rho}$, we find that, for any $\delta > 0$ satisfying $0 < \delta < 1$, there exists $C_\delta > 0$ such that

$$\begin{aligned}
0 &\le \operatorname{Tr} e^{-tH_V(\mathbf{0})} - \operatorname{Tr} e^{-tH_V(\mathbf{a})} \\
&\le C_\delta t\int_{\mathbb{H}} \exp\left(2\beta d(z,\sqrt{-1})\right)e^{-tV_\delta^-(z)}m(dz) + C_\delta t^{11/8}
\end{aligned} \tag{4.8}$$

if $0 < t < 1/C_\delta$.

Using the triangle inequality, we observe that the condition $d(z,z') \le \delta d(z,\sqrt{-1})$ implies that $d(z',\sqrt{-1}) \ge d(z,\sqrt{-1}) - d(z,z') \ge (1-\delta)d(z,\sqrt{-1})$, so we obtain the lower bound $V_\delta^-(z) \ge A\exp\big((1-\delta)\alpha d(z,\sqrt{-1})\big)$. Then it follows that, for $N>0$ suitably chosen below, the first term on the rhs of (4.8) is less than or equal to

$$\begin{aligned}
& C_\delta t \int_0^\infty \exp(2\beta\rho)\exp(-tA\exp((1-\delta)\alpha\rho))\sinh\rho\, d\rho \\
= \;& C_\delta t \int_0^\infty \exp(2\beta\rho)\,(tA\exp((1-\delta)\alpha\rho))^{-N} \\
& \qquad \times (tA\exp((1-\delta)\alpha\rho))^N \exp(-tA\exp((1-\delta)\alpha\rho))\sinh\rho\, d\rho \\
\le \;& C_\delta A^{-N}\left(\sup_{0\le X} X^N e^{-X}\right) t^{1-N}\int_0^\infty \exp\big[(2\beta+1-N(1-\delta)\alpha)\,\rho\big]\, d\rho.
\end{aligned}$$

The last integral converges if we set $N = \frac{2\beta+1}{(1-\delta)\alpha} + \delta$. Then, since

$$1 - N = -\left(1+\frac{1}{\alpha}\right) + \frac{2(\alpha-\beta)}{(1-\delta)\alpha} - \left(\frac{2\alpha+1}{(1-\delta)\alpha}+1\right)\delta,$$

we can choose $\delta>0$ so small that $1-N > -(1+1/\alpha)$ because of the assumption $\alpha>\beta$. Then we conclude that the rhs of (4.8) is of order $o(t^{-1-1/\alpha})$ as $t\to+0$ for any fixed δ. Hence, by Karamata's Tauberian theorem and (4.7), we completes the proof of Corollary 1.3.

4.3 Proof of Corollary 1.4

We may assume that $V(z) = A\,d(z,\sqrt{-1})^\alpha$ for all $z\in\mathbb{H}$ without loss of generality. To see this, we observe that the condition (1.5) implies that, for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$(1-\varepsilon)A\,d(z,\sqrt{-1})^\alpha - C_\varepsilon \le V(z) \le (1+\varepsilon)A\,d(z,\sqrt{-1})^\alpha + C_\varepsilon$$

holds for all $z\in\mathbb{H}$. Then we can show the claim by the min-max argument as in the reduction at the beginning of the preceding subsection.

We formulate Kohlbecker's Tauberian theorem following Bingham, Goldie and Teugels [3]. For any $\rho\ge 0$ and any positive function f on $[0,\infty)$, we say that f *belongs to the class* R_ρ if $\lim_{x\to\infty} f(\lambda x)/f(x) = \lambda^\rho$ holds for each $\lambda>0$. For any locally bounded function on $[0,\infty)$ satisfying the condition $\lim_{x\to\infty} f(x)=\infty$, we set $f^{\leftarrow}(x) = \inf\{y\ge 0 | f(y) > x\}$. We write $f(x)\sim g(x)$ as $x\to\infty$ if $\lim_{x\to\infty} f(x)/g(x)=1$ holds. For any $f\in R_\rho$, there exists $g\in R_{1/\rho}$ such that $f(g(x))\sim g(f(x))\sim x$ holds as $x\to\infty$, and g is unique in the sense of the asymptotic equivalence $\sim$. Moreover, $f^{\leftarrow}$ is one version of g (See Theorem 1.5.12 in [3]). Then Kohlbecker's Tauberian theorem ([3], Theorem 4.12.1) is formulated as follows.

Theorem 4.4 *Let μ be a measure on $[0,\infty)$, which is finite on every compact sets. Let $\alpha>1$, $c>0$ and $\phi\in R_\alpha$. We set $\psi(x)=\phi(x)/x$. The the following two statements 1 and 2 are equivalent.*

1. $\log \mu([0,\lambda)) \sim c\phi^{\leftarrow}(\lambda)$ *holds as* $\lambda \to \infty$.

2. $\log \mathcal{L}\mu(t) \sim (\alpha-1)(c/\alpha)^{\alpha/(\alpha-1)}\psi^{\leftarrow}(1/t)$ *holds as* $t \to +0$.

Here, $\mathcal{L}\mu(t) = \int_0^\infty e^{-t\lambda}\mu(d\lambda)$ *is the Laplace transform of* μ.

Lemma 4.5 *Let* $\alpha > 0$. *Then*

$$\int_0^R \rho^\alpha \sinh\rho\, d\rho = R^\alpha \cosh R - \frac{\alpha}{2}R^{\alpha-1}e^R(1+o(1))$$

holds as $R \to \infty$.

In addition, if $\alpha > 1$, *we have*

$$\int_0^R \rho^\alpha \cosh\rho\, d\rho = R^\alpha \sinh R - \frac{\alpha}{2}R^{\alpha-1}e^R(1+o(1))$$

as $R \to \infty$.

Proof. Using integration by parts, we find that

$$\begin{aligned}\int_0^R \rho^\alpha \sinh\rho\, d\rho &= \rho^\alpha \cosh\rho\big|_0^R - \alpha\int_0^R \rho^{\alpha-1}\cosh\rho\, d\rho \\ &= R^\alpha\cosh R - \alpha R^{\alpha-1}\sinh R + \alpha(\alpha-1)\int_0^R \rho^{\alpha-2}\sinh\rho\, d\rho. \quad (4.9)\end{aligned}$$

We note that the last integral is finite since $\alpha > 0$. The last integral in (4.9) is equal to

$$\begin{aligned}&\int_0^{R/2} \rho^{\alpha-2}\sinh\rho\, d\rho + \int_{R/2}^R \rho^{\alpha-2}\sinh\rho\, d\rho \\ \le\ & \sinh(R/2)\int_0^{R/2}\rho^{\alpha-2}d\rho + \rho^{\alpha-2}\cosh\rho\big|_{R/2}^R - (\alpha-2)\int_{R/2}^R \rho^{\alpha-3}\cosh\rho\, d\rho \\ \le\ & O(R^{\alpha-2}e^R) + |\alpha-2|\cosh R\int_{R/2}^R \rho^{\alpha-3}d\rho \\ =\ & O(R^{\alpha-2}e^R)\end{aligned}$$

as $R \to \infty$, where we used an integration by parts in the first inequality. Note that all the integrals above are finite since $\alpha > 0$ and $\sinh\rho = O(\rho)$ near $\rho = 0$. The rest of assertion follows from the relation

$$\int_0^R \rho^\alpha\cosh\rho\, d\rho = \rho^\alpha\sinh\rho\big|_0^R - \alpha\int_0^R \rho^{\alpha-1}\sinh\rho\, d\rho$$

and the first assertion. ∎

We introduce the volume function

$$\mu_{sc}(\lambda) = \frac{1}{(2\pi)^2} \left|\{(z,\xi) \in \mathrm{T}^*\mathbb{H} | y^2|\xi|^2/2 + V(z) < \lambda\}\right|.$$

Here $|\cdot|$ stands for the four dimensional Lebesgue measure.

Lemma 4.6 *Let $\alpha > 1$ and $V(z) = A\, d(z, \sqrt{-1})^\alpha$. Then we have*

$$\lim_{\lambda\to\infty} \lambda^{-1/\alpha} \log \mu_{sc}(\lambda) = A^{-1/\alpha}$$

and

$$\lim_{t\to+0} t^{1/(\alpha-1)} \log \mathcal{L}\mu_{sc}(t) = (\alpha-1)\alpha^{-\alpha/(\alpha-1)} A^{-1/(\alpha-1)}.$$

Proof. A direct computation leads us to

$$\mu_{sc}(\lambda) = \frac{1}{2\pi} \int_{\{V<\lambda\}} (\lambda - V(z)) m(dz). \tag{4.10}$$

Thus we may assume that $A = 1$ considering the scaling $\lambda \to \lambda/A$. Then it follows that the rhs of (4.10) is equal to

$$\int_0^{\lambda^{1/\alpha}} (\lambda - \rho^\alpha) \sinh\rho\, d\rho = \frac{\alpha}{2} \lambda^{(\alpha-1)/\alpha} e^{\lambda^{1/\alpha}} (1 + o(1))$$

as $\lambda \to \infty$, where we used Lemma 4.5 with $R = \lambda^{1/\alpha}$. The first assertion follows just by taking logarithm.

The second assertion follows from Kohlbecker's Tauberian theorem 4.4 with $\mu = \mu_{sc}$, $c = A^{-1/\alpha}$, $\phi(x) = x^\alpha$, $\phi^\leftarrow(x) = x^{1/\alpha}$, $\psi(x) = x^{\alpha-1}$ and $\psi^\leftarrow(x) = x^{1/(\alpha-1)}$. ∎

For any $\varepsilon > 0$, we introduce the auxiliary potential

$$V^+_{(\varepsilon)}(z) = \sup\{V(z') | d(z,z') \le \varepsilon\}$$

and the associated volume function

$$\mu_{sc,\varepsilon}(\lambda) = \frac{1}{(2\pi)^2} \left|\left\{(z,\xi) \in \mathrm{T}^*\mathbb{H} | y^2|\xi|^2/2 + V^+_{(\varepsilon)}(z) < \lambda\right\}\right|.$$

Lemma 4.7 *Let $\mu_{sc,\varepsilon}$ be as above. Under the same assumption as in Lemma 4.6, we have*

$$\lim_{\varepsilon\to+0} \lim_{t\to+0} t^{1/(\alpha-1)} \log \mathcal{L}\mu_{sc,\varepsilon}(t) = (\alpha-1)\alpha^{-\alpha/(\alpha-1)} A^{-1/(\alpha-1)}.$$

Proof. We first show that $\lim_{\lambda\to\infty}\lambda^{-1/\alpha}\log\mu_{sc,\varepsilon}(\lambda)$ exists for all $\varepsilon>0$, and

$$\lim_{\varepsilon\to+0}\lim_{\lambda\to\infty}\lambda^{-1/\alpha}\log\mu_{sc,\varepsilon}(\lambda)=A^{-1/\alpha}.$$

We may consider the case of $A=1$ by considering the scaling $\lambda\to\lambda/A$. By the triangle inequality, one can find that $V^+_{(\varepsilon)}(z)$=$\inf\{d(z',\sqrt{-1})^\alpha|d(z,z')\le\varepsilon\}$=$(d(z,\sqrt{-1})+\varepsilon)^\alpha$ for large z and small $\varepsilon>0$. Then we have

$$\begin{aligned}\mu_{sc,\varepsilon}(\lambda) &= \int_0^{\lambda^{1/\alpha}-\varepsilon}(\lambda-(\rho+\varepsilon)^\alpha)\sinh\rho d\rho\\ &= \int_\varepsilon^{\lambda^{1/\alpha}}(\lambda-\rho^\alpha)\sinh(\rho-\varepsilon)d\rho\\ &= \int_0^{\lambda^{1/\alpha}}(\lambda-\rho^\alpha)\sinh(\rho-\varepsilon)d\rho\\ &\quad -\int_0^{\varepsilon}(\lambda-\rho^\alpha)\sinh(\rho-\varepsilon)d\rho, \end{aligned}\tag{4.11}$$

where we changed the variable $\rho+\varepsilon\to\rho$ in the second inequality. The second integral on the rhs of (4.11) is of order $O(\lambda)$ as $\lambda\to\infty$. Using the elementary relation $\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$, by Lemma 4.5, we find that the first integral on the rhs of (4.11) is equal to

$$\begin{aligned}&\cosh\varepsilon\int_0^{\lambda^{1/\alpha}}(\lambda-\rho^\alpha)\sinh\rho d\rho-\sinh\varepsilon\int_0^{\lambda^{1/\alpha}}(\lambda-\rho^\alpha)\cosh\rho d\rho\\ =\ &\cosh\varepsilon\mu_{sc}(\varepsilon)-\sinh\varepsilon\cdot O(\lambda^{(\alpha-1)/\alpha}e^{\lambda^{1/\alpha}})\end{aligned}$$

as $\lambda\to\infty$.

Finally, Kohlbecker's Tauberian theorem 4.4 completes the proof as in the proof of Lemma 4.6. ▌

Lemma 4.8 *Let $V(z)\in C(\mathbb{H},\mathbb{R})$. Assume that there exists $f\in C([0,\infty),\mathbb{R})$ such that $\lim_{\rho\to\infty}f(\rho)=\infty$ holds and $V(z)\ge d(z,\sqrt{-1})f(d(z,\sqrt{-1}))$ holds for all $z\in\mathbb{H}$. Then the integral $\int_{\mathbb{H}}e^{-tV(z)}m(dz)$ if finite for any $t>0$.*

Proof. We put $R_t=\inf\{\rho>0|f(\rho)>2/t\}$ for any $t>0$. Using the expression $m(dz)=\sinh\rho\,d\rho d\theta$ in the geodesic polar coordinate, we have

$$\begin{aligned}\int_{\mathbb{H}}e^{-tV(z)}m(dz) &\le 2\pi\int_0^\infty e^{-t\rho f(\rho)}\sinh\rho\,d\rho\\ &\le \pi\int_0^\infty e^{-t\rho f(\rho)+\rho}\,d\rho\\ &= \pi\int_0^{R_t}e^{-t\rho f(\rho)+\rho}\,d\rho+\pi\int_{R_t}^\infty e^{-t\rho f(\rho)+\rho}\,d\rho\\ &\le C\,e^{R_t}+\int_{R_t}^\infty e^{-\rho}\,d\rho\\ &\le C\,e^{R_t}+C'\end{aligned}$$

for some $C, C' > 0$ independent of $t > 0$. ▮

The conclusion of the above lemma is identical to the condition (A.1) in [9], in particular, the assumption (A.1) in [9] is fulfilled for $V(z) = A\, d(z, \sqrt{-1})^\alpha$ if $\alpha > 1$. Then we can deduce that, for any $\delta > 0$, there exists $T_\delta > 0$ such that the inequality

$$\mathcal{L}\mu_{sc,\varepsilon}(t)(1-\delta) \leq \operatorname{Tr} e^{-tH_V(\mathbf{0})} \leq \mathcal{L}\mu_{sc}(t)(1+\delta) \tag{4.12}$$

holds if $0 < t \leq T_\delta$. In fact, this has been proven in subsections 5.2 and 5.3 in [9] (See Lemma 5.2, Lemma 5.3, Lemma 5.5 and Lemma 5.6 in [9]) with no assumption but (A.1).

Lemma 4.9 *Under the same assumption as in Lemma 4.6, we have the asymptotics*

$$\lim_{t\to+0} t^{1/(\alpha-1)} \log \operatorname{Tr} e^{-tH_V(\mathbf{0})} = (\alpha-1)\alpha^{-\alpha/(\alpha-1)} A^{-1/(\alpha-1)},$$

or equivalently,

$$\lim_{\lambda\to\infty} \lambda^{-1/\alpha} \log N(H_V(\mathbf{0}) < \lambda) = A^{-1/\alpha}.$$

Proof. We take logarithm of (4.12) and take a limit $\varepsilon \to +0$ after a limit $t \to +0$. Then the first asymptotic formula follows from Lemma 4.6 and Lemma 4.7. The second formula is a consequence of Kohlbecker's theorem. ▮

Lemma 4.10 *Assume that $\tilde{a}$ belongs to the class $\mathcal{G}$ and $\tilde{a}$ is sub-exponential, i.e., for any $\delta > 0$, there exists $C_\delta > 0$ such that $\tilde{a}(\rho) \leq C_\delta e^{\delta\rho}$ holds for any $\rho > 0$. Then, for any δ satisfying $0 < \delta < 1$, there exists $C_\delta > 0$ such that*

$$0 \leq \operatorname{Tr} e^{-tH_V(\mathbf{0})} - \operatorname{Tr} e^{-tH_V(\mathbf{a})} \leq C_\delta (At)^{-1/\alpha} \exp\left(C(\alpha,\delta)(1-\delta)^{\frac{1+\alpha}{1-\alpha}}(At)^{-1/(\alpha-1)}\right)$$

holds if $0 < t \leq 1/C_\delta$, where we set

$$C(\alpha,\delta) = (\alpha-1)\left(\frac{1+\delta}{\alpha}\right)^{\alpha/(\alpha-1)}.$$

Proof. Let V_δ^- be as above. Using the triangle inequality, one can observe that

$$\begin{aligned} V_\delta^- &\geq \inf\{A\left(d(z,\sqrt{-1}) - d(z,z')\right)^\alpha \,|\, d(z,z') \leq \delta\, d(z,\sqrt{-1})\} \\ &\geq A(1-\delta)^\alpha d(z,\sqrt{-1})^\alpha. \end{aligned}$$

By Theorem 1.1 and the assumption on $\tilde{a}(\rho)$, we find that, for any δ satisfying $0 < \delta < 1$,

$$\begin{aligned} 0 &\leq \operatorname{Tr} e^{-tH_V(\mathbf{0})} - \operatorname{Tr} e^{-tH_V(\mathbf{a})} \\ &\leq C_\delta \int_0^\infty \tilde{a}((1+\delta)\rho)^2 e^{-tA(1-\delta)^\alpha\rho^\alpha} \sinh\rho\, d\rho \\ &\leq C_\delta \int_0^\infty \exp\left(-tA(1-\delta)^\alpha\rho^\alpha + (1+\delta)\rho\right) d\rho \end{aligned} \tag{4.13}$$

holds for small $t > 0$, where we used the fact that the sub-exponential property of $\tilde{a}$ yields that $\tilde{a}((1+\delta)\rho) \le C_\delta e^{\delta\rho}$ holds, in the second inequality.

We use the following elementary inequality

$$(1+\delta)\rho \le \varepsilon\rho^\alpha + (\alpha-1)\left(\frac{1+\delta}{\alpha}\right)^{\alpha/(\alpha-1)} \varepsilon^{-1/(\alpha-1)}$$

for any $\rho > 0$, $\varepsilon > 0$, $\delta > 0$ and $\alpha > 1$. We can show this by considering the minimum of the function $f(\rho) = \varepsilon\rho^\alpha - (1+\delta)\rho$. Then it follows that, for small $\varepsilon > 0$ suitably chosen below, the first integral on the rhs of (4.13) is less than or equal to

$$\begin{aligned} & C_\delta \exp\left(C(\alpha,\delta)\varepsilon^{-1/(\alpha-1)}\right) \int_0^\infty \exp\left(-tA(1-\delta)^\alpha\rho^\alpha + \varepsilon\rho^\alpha\right) d\rho \\ = \; & C_\delta \exp\left(C(\alpha,\delta)\varepsilon^{-1/(\alpha-1)}\right)(tA(1-\delta)^\alpha - \varepsilon)^{-1/\alpha}\left(\int_0^\infty e^{-y^\alpha} dy\right), \end{aligned}$$

where the constant $C(\alpha,\delta)$ is as above and we changed the variable $y^\alpha = (tA(1-\delta)^\alpha - \varepsilon)\rho^\alpha$ in the second equality. Finally we have the lemma by setting $\varepsilon = tA(1-\delta)^{\alpha+1}$. ∎

Lemma 4.11 *Let V and $\mathbf{a}$ satisfy the same assumption as in Corollary 1.4. Then, for any small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that the inequality $\operatorname{Tr} e^{-(1+\varepsilon)tH_{W_\varepsilon}(\mathbf{0})} \le \operatorname{Tr} e^{-tH_V(\mathbf{a})}$ holds for any $t > 0$, where the potential W_ε is given by $W_\varepsilon(z) = A\, d(z, \sqrt{-1})^\alpha + C_\varepsilon$.*

Proof. Because of the min-max theorem, it suffices to show that

$$(f, H_V(\mathbf{a})f) \le (1+\varepsilon)(f, H_{W_\varepsilon}(\mathbf{0})f)$$

holds for any $f \in C_0^\infty(\mathbb{H})$.

We denote $-\sqrt{-1}\partial_x$ and $-\sqrt{-1}\partial_y$ by D_x and D_y, respectively. Let $\mathbf{a} = (a_1, a_2)$ as before. In this proof we often use the fact that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|XY| \le \varepsilon X^2 + C_\varepsilon Y^2$ holds for all $X, Y \in \mathbb{R}$.

We first claim that, for any small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|(f, y^2 a_1 D_x f)| \le \varepsilon(f, y^2 D_x^2 f) + C_\varepsilon \|y a_1 f\|^2, \tag{4.14}$$
$$|(f, y^2 a_2 D_y f)| \le \varepsilon(f, y^2 D_y^2 f) + C_\varepsilon \|y a_2 f\|^2 + C_\varepsilon \|f\|^2 \tag{4.15}$$

hold for any $f \in C_0^\infty(\mathbb{H})$. Indeed, writing $|(f, y^2 a_1 D_x f)| \le \|y D_x f\| \|y a_1 f\| =: XY$, we can obtain (4.14) by the elementary inequality stated above. Similarly, for any $\varepsilon > 0$, we have

$$\begin{aligned} \|y D_y f\|^2 &= (f, D_y y^2 D_y f) \le 2|(f, y D_y f)| + (f, y^2 D_y^2 f) \\ &\le \varepsilon \|y D_y f\|^2 + (f, y^2 D_y^2 f) + C_\varepsilon \|f\|^2 \end{aligned}$$

for some $C_\varepsilon > 0$, from which we have

$$\|yD_y f\|^2 \le \frac{1}{1-\varepsilon}(f, y^2 D_y^2 f) + \frac{C_\varepsilon}{1-\varepsilon}\|f\|^2. \tag{4.16}$$

Then we obtain that

$$\begin{aligned}
|(f, y^2 a_2 D_y f)| &\le \|y a_2 f\| \|yD_y f\| \\
&\le \varepsilon \|yD_y f\|^2 + C_\varepsilon \|y a_2 f\|^2 \\
&\le \frac{\varepsilon}{1-\varepsilon}(f, y^2 D_y^2 f) + \frac{\varepsilon C_\varepsilon}{1-\varepsilon}\|f\|^2 + C_\varepsilon \|y a_2 f\|^2,
\end{aligned}$$

where we used (4.16) in the last inequality. Thus, by setting $\varepsilon(1-\varepsilon)^{-1} = \varepsilon'$, we have shown the claim.

If we denote by $\Re$ the real part of a complex number, then

$$\begin{aligned}
(f, H_V(\mathbf{a})f) &= \Re(f, H_V(\mathbf{a})f) \\
&= -\frac{1}{2}(f, \triangle_{\mathbb{H}} f) - \Re(f, y^2 a_1 D_x f) - \Re(f, y^2 a_2 D_y f) \\
&\quad -\frac{1}{2}\Re(f, y^2 \left((D_x a_1) + (D_y a_2)\right) f) + \frac{1}{2}(f, y^2 \left(a_1^2 + a_2^2\right) f) + (f, Vf) \\
&= -\frac{1}{2}(f, \triangle_{\mathbb{H}} f) - \Re(f, y^2 a_1 D_x f) - \Re(f, y^2 a_2 D_y f) \\
&\quad +\frac{1}{2}(f, y^2 \left(a_1^2 + a_2^2\right) f) + (f, Vf)
\end{aligned}$$

holds for any $f \in C_0^\infty(\mathbb{H})$. Then it follows from (4.14), (4.15) that

$$\begin{aligned}
H_V(\mathbf{a}) &\le -(1+\varepsilon)\frac{1}{2}\triangle_{\mathbb{H}} + V(z) + C_\varepsilon \left(y^2(a_1^2 + a_2^2) + 1\right) \\
&\le -(1+\varepsilon)\frac{1}{2}\triangle_{\mathbb{H}} + A\, d(z, \sqrt{-1})^\alpha + C_\varepsilon \left(d(z, \sqrt{-1})^{2\beta} + 1\right) \\
&\le -(1+\varepsilon)\frac{1}{2}\triangle_{\mathbb{H}} + (1+\varepsilon) A\, d(z\sqrt{-1})^\alpha + C_\varepsilon
\end{aligned}$$

holds on $C_0^\infty(\mathbb{H})$, since the assumption $2\beta < \alpha$ implies that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $\rho^{2\beta} \le \varepsilon \rho^\alpha + C_\varepsilon$ holds for any $\rho \ge 0$. This completes the proof. ▌

By the same argument at the beginning of this subsection, one can observe that Lemma 4.9 holds also for the potential of the form $V(z) = A\, d(z, \sqrt{-1})^\alpha + C$ for any $C > 0$. We set $C(\alpha, 0) = (\alpha-1)\alpha^{-\alpha/(\alpha-1)}$. Let W_ε be as in the previous lemma. Applying Lemma 4.9 to $V = W_\varepsilon$, we find that, for any small $\delta > 0$, $\varepsilon > 0$ fixed,

$$\operatorname{Tr} e^{-(1+\varepsilon)tH_{W_\varepsilon}} \ge e^{-t(1+\varepsilon)C_\varepsilon} \exp\left(C(\alpha, 0)\left(A(1+\varepsilon)t\right)^{-1/(\alpha-1)}(1-\delta)\right)$$

holds for small $t > 0$. Then it follows from Lemma 4.10 and Lemma 4.11 that, for any small $\delta > 0$ and $\varepsilon > 0$ fixed,

$$\begin{aligned}
&\left|\frac{\operatorname{Tr} e^{-tH_V(\mathbf{0})} - \operatorname{Tr} e^{-tH_V(\mathbf{a})}}{\operatorname{Tr} e^{-(1+\varepsilon)tH_{W_\varepsilon}(\mathbf{0})}}\right| \\
&\le e^{t(1+\varepsilon)C_\varepsilon} C_\delta (At)^{-1/\alpha} \exp\left(\left(\frac{C(\alpha,\delta)}{(1-\delta)^{\frac{1+\alpha}{\alpha-1}}} - \frac{(1-\delta)C(\alpha,0)}{(1+\varepsilon)^{1/(\alpha-1)}}\right)(At)^{-1/(\alpha-1)}\right)
\end{aligned} \tag{4.17}$$

holds for small $t > 0$.

Then, using the inequality $\log(1+X) \le \log 2 + |\log X|$ for any $X > 0$, we find that, for any small $\delta > 0$ and $\varepsilon > 0$ fixed,

$$
\begin{aligned}
0 &\le \log \operatorname{Tr} e^{-tH_V(\mathbf{0})} - \log \operatorname{Tr} e^{-tH_V(\mathbf{a})} \\
&= \log\left(1 + \frac{\operatorname{Tr} e^{-tH_V(\mathbf{0})} - \operatorname{Tr} e^{-tH_V(\mathbf{a})}}{\operatorname{Tr} e^{-tH_V(\mathbf{a})}}\right) \\
&\le \log\left(1 + \frac{\operatorname{Tr} e^{-tH_V(\mathbf{0})} - \operatorname{Tr} e^{-tH_V(\mathbf{a})}}{\operatorname{Tr} e^{-(1+\varepsilon)tH_{W_\varepsilon}(\mathbf{0})}}\right) \\
&\le \log 2 + \left|\log\left(\frac{\operatorname{Tr} e^{-tH_V(\mathbf{0})} - \operatorname{Tr} e^{-tH_V(\mathbf{a})}}{\operatorname{Tr} e^{-(1+\varepsilon)tH_{W_\varepsilon}(\mathbf{0})}}\right)\right| \\
&\le O(|\log t|) + \left|\frac{C(\alpha,\delta)}{(1-\delta)^{\frac{1+\alpha}{\alpha-1}}} - \frac{(1-\delta)C(\alpha,0)}{(1+\varepsilon)^{1/(\alpha-1)}}\right| (At)^{-1/(\alpha-1)} \qquad (4.18)
\end{aligned}
$$

holds for all small $t > 0$, where we used Lemma 4.11 in the second inequality and used (4.17) in the last inequality. Then we conclude that

$$\lim_{t\to+0} t^{1/(\alpha-1)} \left|\log \operatorname{Tr} e^{-tH_V(\mathbf{0})} - \log \operatorname{Tr} e^{-tH_V(\mathbf{a})}\right| = 0$$

by taking a limit $\delta, \varepsilon \to +0$ after taking $\limsup_{t\to+0}$, since the coefficient of $t^{-1/(\alpha-1)}$ on the rhs of (4.18) tends to zero as $\delta, \varepsilon \to +0$. Hence, by Lemma 4.9, we obtain the asymptotic relation

$$\lim_{t\to+0} t^{1/(\alpha-1)} \log \operatorname{Tr} e^{-tH_V(\mathbf{a})} = (\alpha-1)\alpha^{-\alpha/(\alpha-1)} A^{-1/(\alpha-1)},$$

from which Corollary 1.4 follows via Kohlbecker's theorem.

5 Proof of Proposition 3.6

In this section we give a proof of Proposition 3.6 accepted in Section 3 and Section 4.

To the end of this section, we write $p(t,z,z')$ for the heat kernel $p_{\mathbf{00}}(t,z,z')$ for simplicity. The notation $\nabla \log p(t,z_0,z')$ stands for $(\partial_x \log p(t,\cdot,z')dx + \partial_y \log p(t,\cdot,z')dy)|_{z=z_0}$, i.e., the exterior differentiation with respect to the second variable of p. We shall use the notations $\partial_x \log p(t,z,z')$, $\partial_y \log p(t,z,z')$ in a similar manner.

Lemma 5.1 *Let $T > 0$. Then there exists $C_T > 0$ such that*

$$\|\nabla \log p(t,z,z')\| \le C_T\,(1 + d(z,z')/t)$$

holds for any $(t,z,z') \in (0,T] \times \mathbb{H} \times \mathbb{H}$. Here $\|\cdot\|$ is the norm on $\mathrm{T}^*_z\mathbb{H}$ as in Section 1.

Proof. This is a special case of Theorem 4.4 (1) in Aida [1]. ∎

We consider $\mathbb{H}$ as the upper-half space of $\mathbb{R}^2$. Let $z, z' \in \mathbb{H}$, $T > 0$ and let $w = (w^1, w^2) \in W^{(2)}$. We consider the following SDE on $\mathbb{R}^2$ for $\{\zeta_t\}_{0\le t<T}$=$\{(\xi_t, \eta_t)\}_{0\le t<T}$:

$$\text{(SDE)} \begin{cases} d\xi_t &= \eta_t \left(dw_t^1 + \eta_t \partial_x \log p(T-t, \zeta_t, z')dt\right), \\ d\eta_t &= \eta_t \left(dw_t^2 + \eta_t \partial_y \log p(T-t, \zeta_t, z')dt\right) \end{cases}$$

with initial condition $\zeta_0 = z$.

Lemma 5.2 *The following assertions hold:*

1. *The solution $\{\zeta_t\}_{0\le t<T}$ exists as a stochastic process on $\mathbb{H}$.*
2. *We have the limit $\lim_{t\nearrow T} \zeta_t = z'$ a.s.*
3. *The law of $\{\zeta_t\}_{0\le t\le T}$ is $P_T^{z,z'}$.*

Proof. Let $z, z' \in \mathbb{H}$. Up to the explosion time $\tau_\infty = \lim_{N\to\infty} \inf\{t | 0 \le t \le T, d(\zeta_t, \sqrt{-1}) \ge N\}$, the above (SDE) has the existence and uniqueness of solutions, since the coefficients of (SDE) are locally Lipschitz. Now we show that $\tau_\infty \ge T'$ for any positive $T'(< T)$ given, and the law of $\{\zeta_t\}_{0\le t\le T'}$ is $P_T^{z,z'}|_{\mathcal{B}_{T'}}$.

Let $Z_t = (X_t, Y_t) = Z(t, z, w)$ be the solution of the SDE (2.1). Put $e_t = p(T - t, Z_t, z')/p(T, z, z')$ and

$$K_t = \int_0^t Y_s \partial_x \log p(T-s, Z_s, z')dw_s^1 + \int_0^t Y_s \partial_y \log p(T-s, Z_s, z')dw_s^2.$$

We use Girsanov's theorem. Recall that $P^{(2)}$ is the Wiener measure on $W^{(2)}$. We define a new measure $\hat{P}$ by $\hat{P} = e_{T'} P^{(2)}$. Applying Itô's formula to $\log e_t = \log p(T-t, Z_t, z') - \log p(T, z, z')$, we have

$$\begin{aligned} d\log e_t &= Y_t \partial_x \log p(T-t, Z_t, z')dw_t^1 + Y_t \partial_y \log p(T-t, Z_t, z')dw_t^2 \\ &\quad -\frac{1}{2}\|\nabla \log p(T-t, Z_t, z')\|^2 dt \\ &= dK_t - \frac{1}{2} d\langle K\rangle_t, \end{aligned}$$

where we used the easily verified formula

$$\partial_t \log p = \frac{1}{2}\triangle_{\mathbb{H}} \log p + \frac{1}{2}\|\nabla p\|^2$$

for the heat kernel p. Then we conclude that $e_t = \exp\left(K_t - \langle K\rangle_t/2\right)$ since $\log e_0 = 0$. We note that since $\{e_t\}_{0\le t\le T'}$ is a bounded, local martingale, so $\{e_t\}_{0\le t\le T'}$ is a martingale. Then Girsanov's theorem says that the process $\{\tilde{w}_t\}_{0\le t\le T'}$ defined by

$$\begin{cases} d\tilde{w}_t^1 &= dw_t^1 - Y_t \partial_x \log p(T-t, Z_t, z')dt, \\ d\tilde{w}_t^2 &= dw_t^2 - Y_t \partial_y \log p(T-t, Z_t, z')dt \end{cases}$$

is a two dimensional Brownian motion under $\hat{P}$. Hence it follows from the SDE (2.1) that $\{Z_t\}_{0\le t\le T'}$ is a solution to (SDE) under $\hat{P}$. This shows that $\tau_\infty \ge T'$, and it is well-known that, under $\hat{P}$, the law of $\{Z_t\}_{0\le t\le T'}$ is $P_T^{z,z'}|_{\mathcal{B}_{T'}}$. In particular, the solution $\{\zeta_t\}_{0\le t<T}$ exists since τ_∞ also has the unique law.

To complete the proof, it is suffice to show that the assertion 2, from which the assertion 3 obeys because of the above argument. This follows from the facts that the map $(w \to \lim_{t\nearrow T}\zeta_t(w))$ is $\sigma(\zeta_t|t<T)$-measurable and that we have the limit $\lim_{t\nearrow T} l_t = z'$ a.s. under $P_T^{z,z'}$. ▮

Lemma 5.3 *Let $\{\zeta_t\}_{0\le t\le T}$ be the solution to (SDE) as above. Let $\mathbf{a}\in C^1(\mathbb{H},\mathbb{R}^2)$. Then we have the following assertions:*

1. *The Stratonovich integral $\int_0^T \mathbf{a}(\zeta_t)\circ d\zeta_t$ is a well-defined, real-valued random variable, or equivalently, $\int_0^T \mathbf{a}(l_t)\circ dl_t$ is well-defined under $P_t^{z,z'}$.*
2. *Let $\{\mathbf{a}_j\}_{j=1}^\infty \subset C_0^\infty(\mathbb{H},\mathbb{R}^2)$. Assume that $\mathbf{a}_j \to \mathbf{a}$ and $\nabla\mathbf{a}_j \to \nabla\mathbf{a}$ as $j\to\infty$ uniformly on every compact subsets. Then $\int_0^T \mathbf{a}_j(\zeta_t)\circ d\zeta_t$ converges to $\int_0^T \mathbf{a}(\zeta_t)\circ d\zeta_t$ in probability as $j\to\infty$.*

Proof. Since the well-definedness of the Stratonovich integral $\int_0^{T'} \mathbf{a}(\zeta_t)\circ d\zeta_t$ follows from Lemma 5.2 as long as $0<T'<T$, it suffice to show the well-definedness near $t=T$. Recall that the law of $\{\zeta_t\}_{0\le t\le T}$ is $P_T^{z,z'}$ by Lemma 5.2, and note that the law of the time-reverse process $\{l_{T-t}\}_{0\le t\le T}$ is $P_T^{z',z}$ because of the anti-symmetricity of the Stratonovich integral with respect to the time-reversal. Then this time-reverse $t\to T-t$ transfers the problem near $t=T$ to that near $t=0$, so the assertion 1 obeys.

We show the assertion 2. Using (SDE), we find that

$$\begin{aligned}\int_0^T a_1(\zeta_t)\circ d\xi_t &= \int_0^T a_1(\zeta_t)d\xi_t + \frac{1}{2}\int_0^T \eta_t^2(\partial_x a_1)(\zeta_t)ds \\ &= \int_0^T \eta_t a_1(\zeta_t)dw_t^1 + \int_0^T a_1(\zeta_t)(\eta_t)^2\partial_x \log p(T-t,\zeta_t,z')dt \\ &\quad + \frac{1}{2}\int_0^T \eta_t^2(\partial_x a_1)(\zeta_t)dt \end{aligned} \tag{5.1}$$

for any $a_1\in C^2(\mathbb{H},\mathbb{R}^2)$. We consider the first term on the rhs of (5.1) since a similar argument works for the second and third terms.

Let $\delta>0$. We introduce a first exit time $\tau_N = \inf\{t|0\le t\le T, d(\zeta_t,\sqrt{-1})\ge N\}$. Then for any $\varepsilon>0$ there exists $N_0>0$ such that $P(\{w|\tau_{N_0}<T\})\le\varepsilon$, since the image $\{\zeta_t\}_{0\le t\le T}$ on $\mathbb{H}$ is compact a.s. by Lemma 5.2. So, it follows that

$$\begin{aligned}& P\left(\{w|\left|\int_0^T \eta_t a_1(\zeta_t)dw_t^1 - \int_0^T \eta_t a_{1,j}(\zeta_t)dw_t^1\right| > \delta\}\right) \\ \le\ & P\left(\{w|\left|\int_0^T \eta_t a_1(\zeta_t)dw_t^1 - \int_0^T \eta_t a_{1,j}(\zeta_t)dw_t^1\right| > \delta, \tau_{N_0}\ge T\}\right) + \varepsilon,\end{aligned} \tag{5.2}$$

where $a_{1,j}$ stands for the first component of $\mathbf{a}_j$.

On the other hand, for any $\varepsilon > 0$, there exists $j_0 > \mathbb{N}$ such that

$$\sup\{|ya_1(z) - ya_{1,j}(z)||d(z, \sqrt{-1}) \le N_0\} \le \varepsilon.$$

Then Burkholder's inequality yields that

$$\begin{aligned} & E[\left|\int_0^{T\wedge\tau_{N_0}} (\eta_t a_1(\zeta_t) - \eta_t a_{1,j}(\zeta_t))\, dw_t^1\right|^2] \\ \le\ & E[\int_0^{T\wedge\tau_{N_0}} |\eta_t a_1(\zeta_t) - \eta_t a_{1,j}(\zeta_t)|^2\, dt] \\ \le\ & \varepsilon^2 T \end{aligned}$$

for all $j \ge j_0$. Hence, Chebyshev's inequality yields that

$$\begin{aligned} & P\left(\{w|\left|\int_0^T \eta_t a_1(\zeta_t) dw_t^1 - \int_0^T \eta_t a_{1,j}(\zeta_t) dw_t^1\right| > \delta, \tau_{N_0} \ge T\}\right) \\ =\ & P\left(\{w|\left|\int_0^{T\wedge\tau_{N_0}} \eta_t a_1(\zeta_t) dw_t^1 - \int_0^{T\wedge\tau_{N_0}} \eta_t a_{1,j}(\zeta_t) dw_t^1\right|^2 > \delta^2, \tau_{N_0} \ge T\}\right) \\ \le\ & \varepsilon^2 T/\delta^2 \end{aligned} \tag{5.3}$$

for all $j \ge j_0$. Finally it follows from (5.2) and (5.3) that, for any $\varepsilon > 0$, there exists j_0 such that

$$P\left(\{w|\left|\int_0^T \eta_t a_1(\zeta_t) dw_t^1 - \int_0^T \eta_t a_{1,j}(\zeta_t) dw_t^1\right| > \delta\}\right) \le \varepsilon^2 T/\delta^2 + \varepsilon$$

holds for all $j \ge j_0$. Then the arbitrariness of $\varepsilon > 0$ shows that $\int_0^T a_{1,j}(\zeta_t) d\xi_t$ converges to $\int_0^T a_1(\zeta_t) d\xi_t$ in probability as $j \to \infty$. The same conclusion holds for the case of $\int_0^T a_2(\zeta_t) \circ d\zeta_t$. We completes the proof. ▌

Let $0 < \varepsilon \le 1$ and $z \in \mathbb{H}$. We consider the following SDE for $\zeta_t^\varepsilon = (\xi_t^\varepsilon, \eta_t^\varepsilon)$:

$$(\mathrm{SDE})_\varepsilon \begin{cases} d\xi_t^\varepsilon &= \eta_t^\varepsilon \left(\varepsilon dw_t^1 + \varepsilon^2 \eta_t^\varepsilon \partial_x \log p(\varepsilon^2(1-t), \zeta_t^\varepsilon, z) dt\right), \\ d\eta_t^\varepsilon &= \eta_t^\varepsilon \left(\varepsilon dw_t^2 + \varepsilon^2 \eta_t^\varepsilon \partial_y \log p(\varepsilon^2(1-t), \zeta_t^\varepsilon, z) dt\right) \end{cases}$$

with initial condition $\zeta_0^\varepsilon = z$.

Lemma 5.4 *Let $\{\zeta_t\}_{0\le t\le \varepsilon^2}$ be the solution to (SDE) with $z' = z$ and $T = \varepsilon^2$. Then we have the following assertions:*

1. *The process $\{\zeta_{\varepsilon^2 t}\}_{0\le t\le 1}$ has the same law as $\{\zeta_t^\varepsilon\}_{0\le t\le 1}$. In particular, the condition $\zeta_0^\varepsilon = \zeta_1^\varepsilon = z$ is fulfilled.*

2. *Let* $\mathbf{a} \in C^1(\mathbb{H}, \mathbb{R}^2)$. *Then* $\int_0^{\varepsilon^2} \mathbf{a}(\zeta_t) \circ d\zeta_t$ *and* $\int_0^1 \mathbf{a}(\zeta_t^\varepsilon) \circ d\zeta_t^\varepsilon$ *have the same law. In particular, we have*

$$E[\left|\int_0^{\varepsilon^2} \mathbf{a}(\zeta_t) \circ d\zeta_t\right|^4] = E_{\varepsilon^2}^{z,z}[\left|\int_0^{\varepsilon^2} \mathbf{a}(l_t) \circ dl_t\right|^4] = E[\left|\int_0^1 \mathbf{a}(\zeta_t^\varepsilon) \circ d\zeta_t^\varepsilon\right|^4].$$

Proof. The pathwise uniqueness of solutions to $(\mathrm{SDE})_\varepsilon$ can be verified as in the case of (SDE). Applying the Brownian scaling property $t \to \varepsilon^2 t$ to (SDE), we have the assertion 1.

We show the assertion 2. Let $\mathcal{P}$ stand for the partition $0 = t_0 < T_1 < \ldots < t_n = \varepsilon^2$ of $[0, \varepsilon^2]$ and $|\mathcal{P}|$ the size of mesh of $\mathcal{P}$. Obviously, $\mathcal{P}' = (0 = t_0/\varepsilon^2 < t_1/\varepsilon^2 \ldots < t_n/\varepsilon^2 = 1)$ is a partition of $[0, 1]$. Then we find that

$$\begin{aligned}\int_0^{\varepsilon^2} \mathbf{a}(\zeta_t) \circ d\zeta_t &= \lim_{|\mathcal{P}| \to +0} \sum_{\mathcal{P}} \frac{\mathbf{a}(\zeta_{t_i}) + \mathbf{a}(\zeta_{t_{i-1}})}{2} (\zeta_{t_i} - \zeta_{t_{i-1}}) \\ &= \lim_{|\mathcal{P}'| \to +0} \sum_{\mathcal{P}'} \frac{\mathbf{a}(\zeta_{\varepsilon^2(t_i/\varepsilon^2)}) + \mathbf{a}(\zeta_{\varepsilon^2(t_{i-1}/\varepsilon^2)})}{2} (\zeta_{\varepsilon^2(t_i/\varepsilon^2)} - \zeta_{\varepsilon^2(t_{i-1}/\varepsilon^2)}) \qquad (5.4)\end{aligned}$$

is well-defined by Lemma 5.3. Here, lim above denotes limit in probability.

On the other hand, it follows from the assertion 1 that the law of the rhs of (5.4) coincides with that of

$$\int_0^1 \mathbf{a}(\zeta_t^\varepsilon) \circ d\zeta_t^\varepsilon = \lim_{|\mathcal{P}'| \to +0} \sum_{\mathcal{P}'} \frac{\mathbf{a}(\zeta_{t_i}^\varepsilon) + \mathbf{a}(\zeta_{t_{i-1}}^\varepsilon)}{2} (\zeta_{t_i}^\varepsilon - \zeta_{t_{i-1}}^\varepsilon).$$

Here, lim above denotes limit in probability. This proves the assertion 2. ▮

Lemma 5.5 *Let* $z = (x, y) \in \mathbb{H}$ *and let* $\{\zeta_t^\varepsilon\}_{0 \le t \le 1}$ *be the solution to* $(SDE)_\varepsilon$ *as above. Let* $\tilde{a}$ *be as in Section 1. Assume that* $b \in C(\mathbb{H})$ *and the estimate* $y|b(z)| \le \tilde{a}(d(z, \sqrt{-1}))$ *holds for all* z. *Then, for any* δ *satisfying* $0 < \delta < 1$ *and for any* $n \in \mathbb{N}$, *there exists* $C_\delta > 0$ *independent of* z *such that*

$$E[\int_0^1 |b(\zeta_t^\varepsilon)|^n dt] \le C_\delta y^{-n} \tilde{a}((1 + \delta) d(z, \sqrt{-1}))^n$$

holds if $0 < \varepsilon < 1/C_\delta$.

Proof. Let $0 \le t \le 1$, $n \in \mathbb{N}$ and $0 < \delta < 1$. By Schwarz' inequality, we find that

$$\begin{aligned}E[|b(\zeta_t^\varepsilon)|^n] &= E[|\eta_t^\varepsilon b(\zeta_t^\varepsilon)|^n (\eta_t^\varepsilon)^{-n}] \\ &\le E[|\eta_t^\varepsilon b(\zeta_t^\varepsilon)|^{2n}]^{1/2} E[(\eta_t^\varepsilon)^{-2n}]^{1/2} \\ &\le C y^{-n} E[\tilde{a}(d(\zeta_t^\varepsilon, \sqrt{-1})^{2n}]^{1/2} \\ &\le C_\delta y^{-n} \tilde{a}((1 + \delta) d(z, \sqrt{-1}))^n,\end{aligned}$$

where we used the assumption on b and Lemma 3.4 in the second inequality and used Lemma 3.1 in the last inequality. This shows the lemma. ▮

Lemma 5.6 *Let $n \in \mathbb{N}$ and let ζ_t^ε be as above. Assume that a stochastic process $\{b_t\}_{0\le t\le 1}$ satisfies the condition $\sup_{0\le t\le 1} E[|b_t|^{2n}] < \infty$. Then*

$$E\left[\left|\int_0^1 dt\ b_t\ \eta_t^\varepsilon \partial \log p(\varepsilon^2(1-t), \zeta_t^\varepsilon, z)\right|^n\right] \le C\varepsilon^{-n} \sup_{0\le t\le 1} E[|b_t|^{2n}]^{1/2} \tag{5.5}$$

holds for some $C > 0$, independent of z, ε. Here ∂ stands for ∂_x or ∂_y.

Proof. By Lemma 5.1, we have

$$\begin{aligned} |\eta_t^\varepsilon \partial \log p(\varepsilon^2(1-t), \zeta_t^\varepsilon, z)| &\le \|\nabla \log p(\varepsilon^2(1-t), \zeta_t^\varepsilon, z)\| \\ &\le C\left(1 + \frac{d(\zeta_t^\varepsilon, z)}{\varepsilon^2(1-t)}\right). \end{aligned}$$

Then

$$\begin{aligned} \text{the lhs of (5.5)} &\le CE\left[\left|\int_0^1 |b_t| dt\right|^n\right] + C\varepsilon^{-2n} E\left[\left|\int_0^1 dt |b_t| \frac{d(\zeta_t^\varepsilon, z)}{1-t}\right|^n\right] \\ &\le C\int_0^1 dt\ E[|b_t|^n] + C\varepsilon^{-2n} E\left[\left|\int_0^1 dt |b_t| \frac{d(\zeta_t^\varepsilon, z)}{1-t}\right|^n\right], \end{aligned} \tag{5.6}$$

where we used Jensen's inequality in the second inequality. The first term on the rhs of (5.6) is bounded uniformly in ε by the assumption of the lemma.

We consider the second term on the rhs of (5.6). Hölder's inequality with exponents $p = n$, $q = n/(n-1)$ yields that

$$\begin{aligned} \int_0^1 dt |b_t| \frac{d(\zeta_t^\varepsilon, z)}{1-t} &= \int_0^1 dt |b_t| \frac{d(\zeta_t^\varepsilon, z)}{(1-t)^{\frac{n+1}{2n}}} \frac{1}{(1-t)^{\frac{n-1}{2n}}} \\ &\le \left(\int_0^1 dt |b_t|^n \frac{d(\zeta_t^\varepsilon, z)^n}{(1-t)^{\frac{n+1}{2}}}\right)^{1/n} \left(\int_0^1 \frac{dt}{(1-t)^{1/2}}\right)^{(n-1)/n} \\ &\le C\left(\int_0^1 dt |b_t|^n \frac{d(\zeta_t^\varepsilon, z)^n}{(1-t)^{\frac{n+1}{2}}}\right)^{1/n}. \end{aligned}$$

Then it follows that the second term on the rhs of (5.6) is less than or equal to

$$\begin{aligned} & C\varepsilon^{-2n} \int_0^1 dt\ E[|b_t|^n d(\zeta_t^\varepsilon, z)^n](1-t)^{-(n+1)/2} \\ \le\ & C\varepsilon^{-2n} \int_0^1 dt\ E[|b_t|^{2n}]^{1/2} E[d(\zeta_t^\varepsilon, z)^{2n}]^{1/2}(1-t)^{-(n+1)/2} \\ \le\ & C\varepsilon^{-2n} \int_0^1 dt\ E[|b_t|^{2n}]^{1/2} \left(\varepsilon^2(1-t)\right)^{n/2} (1-t)^{-(n+1)/2} \\ =\ & C\varepsilon^{-n} \int_0^1 dt\ E[|b_t|^{2n}]^{1/2}(1-t)^{-1/2}, \end{aligned}$$

where we used Schwarz' inequality in the second inequality and used Lemma 3.3, Lemma 5.4 and the invariance of the pinned Brownian motion with respect to time reversal $t \to 1-t$ in the third inequality. This completes the proof. ▌

To prove Proposition 3.6, it is enough to consider $E[\left|\int_0^1 \mathbf{a}(\zeta_t^\varepsilon) \circ d\zeta_t^\varepsilon\right|^4]$ because of Lemma 5.4 (2). Moreover, we consider only the integral $E[\left|\int_0^1 a_1(\zeta_t^\varepsilon) \circ d\xi_t^\varepsilon\right|^4]$ in the following, since $\mathbf{a}(\zeta_t^\varepsilon) \circ d\zeta_t^\varepsilon = a_1(\zeta_t^\varepsilon) \circ d\xi_t^\varepsilon + a_2(\zeta_t^\varepsilon) \circ d\eta_t^\varepsilon$ by definition and the same argument works for $E[\left|\int_0^1 a_2(\zeta_t^\varepsilon) \circ d\eta_t^\varepsilon\right|^4]$.

Itô's formula yields that

$$\begin{aligned} a_1(\zeta_t^\varepsilon) &= a_1(z) + \int_0^t (\partial_x a_1)(\zeta_s^\varepsilon) d\xi_s^\varepsilon + \int_0^t (\partial_y a_1)(\zeta_s^\varepsilon) d\eta_s^\varepsilon + \frac{\varepsilon^2}{2}\int_0^t (\triangle_{\mathbb{H}} a_1)(\zeta_s^\varepsilon) ds \\ &=: a_1(z) + D_1(t) + D_2(t) + D_3(t). \end{aligned}$$

Lemma 5.7 *Let $\tilde{a}$ be as in Section 1. For any small $\varepsilon > 0$ and $\delta > 0$, there exists $C_\delta > 0$ such that*

$$E[\left|\int_0^1 D_3(t) \circ d\xi_t^\varepsilon\right|^4] \le C_\delta\ \varepsilon^{12} \tilde{a}\left((1+\delta) d(z, \sqrt{-1})\right)^4$$

holds for all $z \in \mathbb{H}$. Here, C_δ is independent of ε, z.

Proof. Using $(\mathrm{SDE})_\varepsilon$, we find that the integral $\int_0^1 D_3(t) \circ d\xi_t^\varepsilon$ is equal to

$$\begin{aligned} \int_0^1 D_3(t) d\xi_t^\varepsilon &= \frac{\varepsilon^3}{2}\int_0^1 \left(\int_0^t (\triangle_{\mathbb{H}} a_1)(\zeta_s^\varepsilon) ds\right) \eta_t^\varepsilon dw_t^1 \\ &\quad + \frac{\varepsilon^4}{2}\int_0^1 \left(\int_0^t (\triangle_{\mathbb{H}} a_1)(\zeta_s^\varepsilon) ds\right) (\eta_t^\varepsilon)^2 \partial_x \log p(\varepsilon^2(1-t), \zeta_t^\varepsilon, z)\ dt. \end{aligned} \tag{5.7}$$

Then it follows that

$$\begin{aligned} & E[|\text{the first term on the rhs of (5.7)}|^4] \\ \le\ & \varepsilon^{12} E[\left|\int_0^1 \left(\int_0^t (\triangle_{\mathbb{H}} a_1)(\zeta_s^\varepsilon) ds\right) \eta_t^\varepsilon dw_t^1\right|^4] \\ \le\ & C\ \varepsilon^{12} E[\left|\int_0^1 \left(\int_0^t (\triangle_{\mathbb{H}} a_1)(\zeta_s^\varepsilon) ds\right)^2 (\eta_t^\varepsilon)^2 dt\right|^2] \\ \le\ & C\ \varepsilon^{12} E[\int_0^1 \left(\int_0^t (\triangle_{\mathbb{H}} a_1)(\zeta_s^\varepsilon) ds\right)^4 (\eta_t^\varepsilon)^4 dt] \\ \le\ & C\ \varepsilon^{12} \int_0^1 dt E[\int_0^1 |(\triangle_{\mathbb{H}} a_1)(\zeta_s^\varepsilon)|^8 ds]^{1/2} E[(\eta_t^\varepsilon)^8]^{1/2} \\ \le\ & C_\delta\ \varepsilon^{12} \tilde{a}\left((1+\delta) d(z, \sqrt{-1})\right), \end{aligned} \tag{5.8}$$

where we used Burkholder's inequality in the second inequality, Jensen's in the third inequality, Schwarz' and Jensen's in the fourth inequality and used Lemma 3.4 with $\alpha = 8$, Lemma 5.5 with $b = \triangle_{\mathbb{H}} a_1$, $n = 8$ in the fifth inequality.

Applying Lemma 5.6 to the case of $b_t = (\int_0^t (\triangle_{\mathbb{H}} a_1)(\zeta_s^\varepsilon) ds)\eta_t^\varepsilon$ and $n = 4$, we obtain

$$E[\text{the second term on the rhs of (5.7)}] \le C\varepsilon^{12} \sup_{0\le t\le 1} E[|b_t|^8]^{1/2}.$$

On the other hand, the same argument we have used to prove (5.8) shows that, for small $\delta > 0$, $E[|b_t|^8]^{1/2} \le C_\delta \tilde{a}((1+\delta)d(z,\sqrt{-1}))^4$ holds for some $C_\delta > 0$. This completes the proof. ▌

Lemma 5.8 *Let $\tilde{a}$ be as in Section 1. Let $j = 1, 2$. For any small $\varepsilon > 0$ and $\delta > 0$, there exists $C_\delta > 0$ such that*

$$E[\left| \int_0^1 D_j(t) \circ d\xi_t^\varepsilon \right|^4] \le C_\delta \ \varepsilon^8 \tilde{a} \left((1+\delta) d(z, \sqrt{-1}) \right)^4$$

holds for all $z \in \mathbb{H}$. Here, C_δ is independent of ε, z.

Proof. We show the assertion only for the case $j = 1$ since the proof is similar in the case of $j = 2$. Using $(\mathrm{SDE})_\varepsilon$, we find that

$$\begin{aligned}
& \int_0^1 D_1(t) \circ d\xi_t^\varepsilon \\
= \; & \int_0^1 \left(\int_0^t (\partial_x a_1)(\zeta_s^\varepsilon) d\xi_s^\varepsilon \right) d\xi_t^\varepsilon + \frac{\varepsilon^2}{2} \int_0^1 (\partial_x a_1)(\zeta_t^\varepsilon)(\eta_t^\varepsilon)^2 \, dt \\
= \; & \varepsilon^2 \int_0^1 \left(\int_0^t (\partial_x a_1)(\zeta_s^\varepsilon) \eta_s^\varepsilon dw_s^1 \right) \eta_t^\varepsilon dw_t^1 \\
& + \varepsilon^3 \int_0^1 \left(\int_0^t (\partial_x a_1)(\zeta_s^\varepsilon) \eta_s^\varepsilon dw_s^1 \right) (\eta_t^\varepsilon)^2 \partial_x \log p(\varepsilon^2(1-t), \zeta_t^\varepsilon, z) dt \\
& + \varepsilon^3 \int_0^1 \left(\int_0^t (\partial_x a_1)(\zeta_s^\varepsilon)(\eta_s^\varepsilon)^2 \partial_x \log p(\varepsilon^2(1-s), \zeta_s^\varepsilon, z) ds \right) \eta_t^\varepsilon dw_t^1 \\
& + \varepsilon^4 \int_0^1 \left(\int_0^t (\partial_x a_1)(\zeta_s^\varepsilon)(\eta_s^\varepsilon)^2 \partial_x \log p(\varepsilon^2(1-s), \zeta_s^\varepsilon, z) \, ds \right) \\
& \qquad \times (\eta_t^\varepsilon)^2 \partial_x \log p(\varepsilon^2(1-t), \zeta_t^\varepsilon, z) \, dt \\
& + \frac{\varepsilon^2}{2} \int_0^1 (\partial_x a_1)(\zeta_s^\varepsilon)(\eta_t^\varepsilon)^2 \, dt \\
=: \; & J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned} \tag{5.9}$$

We estimate J_1. By Burkholder's inequality, we obtain

$$\begin{aligned}
E[|J_1|^4] &\leq C\,\varepsilon^8 E\Big[\Big|\int_0^1 \Big(\int_0^1 (\partial_x a_1)(\zeta_s^\varepsilon)\eta_s^\varepsilon dw_s^1\Big)^2 (\eta_t^\varepsilon)^2 dt\Big|^2\Big] \\
&\leq C\,\varepsilon^8 \int_0^1 dt\; E\Big[\Big(\int_0^1 (\partial_x a_1)(\zeta_s^\varepsilon)\eta_s^\varepsilon dw_s^1\Big)^4 (\eta_t^\varepsilon)^4\Big] \\
&\leq C\,\varepsilon^8 \int_0^1 dt\; E\Big[\Big(\int_0^1 (\partial_x a_1)(\zeta_s^\varepsilon)\eta_s^\varepsilon dw_s^1\Big)^8\Big]^{1/2} E[(\eta_t^\varepsilon)^8]^{1/2} \\
&\leq C\,\varepsilon^8 y^4\; E\Big[\Big|\int_0^1 |(\partial_x a_1)(\zeta_s^\varepsilon)\eta_s^\varepsilon|^2 ds\Big|^4\Big]^{1/2} \\
&\leq C_\delta\;\varepsilon^8 \tilde{a}\,\big((1+\delta)d(z,\sqrt{-1})\big)^4, \qquad (5.10)
\end{aligned}$$

where we used Jensen's inequality in the second inequality, Schwarz' in the third inequality, Burkholder's and Lemma 3.4 with $\alpha = 8$ in the fourth inequality, and used Jensen's and Lemma 5.5 with $b(z) = y\partial_x a_1(z)$, $n = 8$ in the last inequality.

We estimate J_2. We use Lemma 5.6 with $b_t = (\int_0^t (\partial_x a_1)(\zeta_s^\varepsilon)\eta_s^\varepsilon dw_s^1)\eta_t^\varepsilon$ and $n = 4$. First we find that

$$\begin{aligned}
E[|b_t|^8] &\leq E\Big[\Big|\int_0^1 (\partial_x a_1)(\zeta_s^\varepsilon)\eta_s^\varepsilon dw_s^1\Big|^{16}\Big]^{1/2} E[(\eta_t^\varepsilon)^{16}]^{1/2} \\
&\leq C y^8 E\Big[\int_0^1 |(\partial_x a_1)(\zeta_s^\varepsilon)\eta_s^\varepsilon|^{16} ds\Big]^{1/2} \\
&\leq C_\delta \tilde{a}\,\big((1+\delta)d(z,\sqrt{-1})\big)^8,
\end{aligned}$$

where we used Schwarz' inequality in the first inequality, Burkholder's, Jensen's and Lemma 3.4 with $\alpha = 16$ in the second and used Lemma 5.5 with $b(z) = y(\partial_x a_1)(z)$ and $n = 16$ in the last. Next we apply Lemma 5.6 and obtain

$$\begin{aligned}
E[|J_2|^4] &\leq C\varepsilon^8 \sup_{0\leq t\leq 1} E[|b_t|^8]^{1/2} \\
&\leq C_\delta\;\varepsilon^8 \tilde{a}\,\big((1+\delta)d(z,\sqrt{-1})\big)^4. \qquad (5.11)
\end{aligned}$$

We estimate J_3. By Burkholder's inequality, we have

$$\begin{aligned}
E[|J_3|^4] &\leq C\varepsilon^{12} E\Big[\Big|\int_0^1 \Big(\int_0^t (\partial_x a_1)(\zeta_s^\varepsilon)(\eta_s^\varepsilon)^2 \partial_x \log p(\varepsilon^2(1-s),\zeta_s^\varepsilon,z)ds\Big)^2 (\eta_t^\varepsilon)^2 dt\Big|^2\Big] \\
&\leq C\varepsilon^{12} \int_0^1 dt\; E\Big[\Big(\int_0^t (\partial_x a_1)(\zeta_s^\varepsilon)(\eta_s^\varepsilon)^2 \partial_x \log p(\varepsilon^2(1-s),\zeta_s^\varepsilon,z)ds\Big)^8\Big]^{1/2} E[(\eta_t^\varepsilon)^8]^{1/2} \\
&\leq C\varepsilon^8 y^4 \sup_{0\leq t\leq 1} E[|b_t|^{16}]^{1/4},
\end{aligned}$$

where we used Jensen's and Schwarz' inequality in the second inequality, and used Lemma 3.4 with $\alpha = 8$ and Lemma 5.6 with $b_t = (\partial_x a_1)(\zeta_t^\varepsilon)\eta_t^\varepsilon$, $n = 8$ in the third inequality. Using

Lemma 5.5 with $b(z) = y(\partial_x a_1)(z)$ and $n = 16$, we obtain

$$E[|b_t|^{16}]^{1/4} \leq C_\delta y^{-4} \tilde{a} \left((1+\delta) d(z, \sqrt{-1})\right)^4,$$

hence we have

$$E[|J_3|^4] \leq C_\delta \, \varepsilon^8 \tilde{a} \left((1+\delta) d(z, \sqrt{-1})\right)^4. \tag{5.12}$$

We estimate J_4. By Lemma 5.6 with

$$b_t = \left(\int_0^t (\partial_x a_1)(\zeta_s^\varepsilon)(\eta_s^\varepsilon)^2 \partial_x \log p(\varepsilon^2(1-s), \zeta_s^\varepsilon, z) ds \right) \eta_t^\varepsilon$$

and $n = 4$, we obtain

$$\begin{aligned}
E[|J_4|^4] &\leq C\varepsilon^{12} \sup_{0 \leq t \leq 1} E[|b_t|^8]^{1/2} \\
&\leq C\varepsilon^{12} E[\left| \int_0^t (\partial_x a_1)(\zeta_s^\varepsilon)(\eta_s^\varepsilon)^2 \partial_x \log p(\varepsilon^2(1-s), \zeta_s^\varepsilon, z) ds \right|^{16}]^{1/4} E[(\eta_t^\varepsilon)^{16}]^{1/4} \\
&\leq C\varepsilon^8 y^{-4} \sup_{0 \leq t \leq 1} E[|(\partial_x a_1)(\zeta_s^\varepsilon)(\eta_s^\varepsilon)|^{32}]^{1/8} \\
&\leq C_\delta \, \varepsilon^8 \tilde{a} \left((1+\delta) d(z, \sqrt{-1})\right)^4,
\end{aligned} \tag{5.13}$$

where we used Schwarz' inequality in the second inequality, Lemma 3.4 with $\alpha = 16$, Lemma 5.6 with $n = 16$ in the third and used Lemma 5.5 with $b(z) = y(\partial_x a_1)(z)$, $n = 32$ in the last.

We estimate J_5. Similarly we have

$$\begin{aligned}
E[|J_5|^4] &\leq C\varepsilon^8 \int_0^1 dt \; E[|(\partial_x a_1)(\zeta_t^\varepsilon)|^4 (\eta_t^\varepsilon)^8] \\
&= C\varepsilon^8 \int_0^1 dt \; E_{\varepsilon^2}^{z,z}[|(\partial_x a_1)(l_{\varepsilon^2 t})|^4 (l_{\varepsilon^2 t}^2)^8] \\
&\leq C_\delta \, \varepsilon^8 \tilde{a} \left((1+\delta) d(z, \sqrt{-1})\right)^4,
\end{aligned} \tag{5.14}$$

where we used Lemma 3.1 and the assumption on ε in the last inequality.

Then the lemma follows from (5.9)–(5.14) since $\int_0^1 a_1(z) \circ d\xi_t^\varepsilon = a_1(z)(\xi_1^\varepsilon - \xi_0^\varepsilon) = 0$. ∎

Thus we have proved the following assertion: For $\delta > 0$ small enough, there exists $C_\delta > 0$ such that

$$E[\left| \int_0^1 a_1(\zeta_t^\varepsilon) \circ d\xi_t^\varepsilon \right|^4] \leq C_\delta \, \varepsilon^8 \tilde{a} \left((1+\delta) d(z, \sqrt{-1})\right)^4$$

holds for all $z \in \mathbb{H}$ and all $\varepsilon > 0$ satisfying $0 < \varepsilon \leq 1/C_\delta$.

Then we can deduce Proposition 3.6 as we mentioned after the proof of Lemma 5.6 above.

6 Continuity of the heat kernel of $H_V(\mathbf{a})$

In this section we give a proof of Proposition 4.1 accepted in Section 3. We follow the same argument as in the proof of Proposition 6.1 in Broderix, Hundertmark and Leschke [2].

We introduce some notations. Let $W_T(\mathbb{H})$ be the space of all continuous paths from $[0,T]$ to $\mathbb{H}$. Define a transformation $\hat{\cdot}$ by $\hat{l}_s = l_{T-s}$ for any $l \in W_T(\mathbb{H})$. Let P^z be the diffusion measure corresponding to $\triangle_{\mathbb{H}}/2$ with starting point $z \in \mathbb{H}$. Then $\{P^z\}_{z\in\mathbb{H}}$ is reversible with respect to $m(dz)$, i.e.,

$$\int_{\mathbb{H}} m(dz) \int_{W_T(\mathbb{H})} P^z(dl)F(l) = \int_{\mathbb{H}} m(dz) \int_{W_T(\mathbb{H})} P^z(dl)F(\hat{l}) \tag{6.1}$$

for all non-negative Borel function F on $W_T(\mathbb{H})$, from which it follows that $m(dz)$ is invariant measure for $\{P^z\}_{z\in\mathbb{H}}$, i.e.,

$$\int_{\mathbb{H}} m(dz) \int_{W_T(\mathbb{H})} P^z(dl)g(l_t) = \int_{\mathbb{H}} g(z)m(dz) \tag{6.2}$$

for all $t \geq 0$ and all bounded Borel function g on $\mathbb{H}$. In the following we refer the words 'reversibility' and 'invariance' to the facts (6.1) and (6.2), respectively.

For any $t, s > 0$ and $z \in \mathbb{H}$, we define the functionals

$$S(s,t;z,w) \;=\; -\sqrt{-1}\int_s^t \mathbf{a}(Z(u,z,w)) \circ dZ(u,z,w) - \int_s^t V(Z(u,z,w))du$$

on $(W^{(2)}, P^{(2)})$ and

$$\mathcal{S}(s,t;l) \;=\; -\sqrt{-1}\int_s^t \mathbf{a}(l_u) \circ dl_u - \int_s^t V(l_u)du \tag{6.3}$$

on $(W_T(\mathbb{H}), P^z)$ or on $(W_T(\mathbb{H}), P_t^{z,z'})$.

Let the potentials $\mathbf{a}$ and V satisfy the assumptions (A) and (V), respectively. For any $t > 0$, we define an operator T_t by

$$(T_t f)(z) = E^{(2)}[e^{S(0,t;z,w)} f(Z(t,z,w))]$$

on $L^2(\mathbb{H})$.

Lemma 6.1 *Let* $\mathbf{a}$ *and* V *be as above. For any* $t \geq 0$*, the inequality* $\|T_t f\| \leq \|f\|$ *for all* $L^2(\mathbb{H})$*. Moreover,* $\lim_{t\to+0} \|T_t f - f\| = 0$ *and*

$$\lim_{t\to+0} \left\| \frac{T_t f - f}{t} + H_V(\mathbf{a})f \right\| = 0 \tag{6.4}$$

hold for all $f \in \mathrm{Dom}(H_V(\mathbf{a}))$.

Proof. The well-definedness of T_t on $L^2(\mathbb{H})$ follows from Proposition 3.6. The first assertion of the lemma follows from a simple dia-magnetic argument. We show (6.4). The rest of the assertion follows in a similar way. It follows from Itô's formula that

$$\begin{aligned} e^{S(0,t;z,w)}f(Z(t,z,w)) &= f(z)+\int_0^t e^{S(0,s;z,w)}(-H_V(\mathbf{a})f)(Z(s,z,w))ds \\ &\quad + \text{a martingale} \end{aligned}$$

for all $f \in C_0^\infty(\mathbb{H})$. In the rest of the proof we denote $-H_V(\mathbf{a})f$ simply by g. Note that $g \in C_0^\infty(\mathbb{H})$. Then we have

$$\begin{aligned} \left\| \frac{T_t f - f}{t} - g \right\| &= \int_{\mathbb{H}} m(dz) \left| \frac{T_t f(z) - f(z)}{t} - g(z) \right|^2 \\ &\le \int_{\mathbb{H}} m(dz) \left| E^{(2)}[\frac{1}{t}\int_0^t ds(e^{S(0,s;z,w)}-1)g(Z(s,z,w))] \right|^2 \\ &\quad + \int_{\mathbb{H}} m(dz) \left| E^{(2)}[\frac{1}{t}\int_0^t ds(g(Z(s,z,w))-g(z))] \right|^2 \\ &\le \int_{\mathbb{H}} m(dz) E^{(2)}[\frac{1}{t}\int_0^t |e^{S(0,s;z,w)}-1|^2 |g(Z(s,z,w))|^2] \\ &\quad + \int_{\mathbb{H}} m(dz) E^{(2)}[\frac{1}{t}\int_0^t ds |g(Z(s,z,w))-g(z)|^2] \end{aligned} \tag{6.5}$$

Then the first term of the rhs of (6.5) is equal to

$$\begin{aligned} &\frac{1}{t}\int_0^t ds \int_{\mathbb{H}} m(dz) E^z\left[|e^{\mathcal{S}(0,s;l)}-1|^2 |g(l_s)|^2\right] \\ =\ &\frac{1}{t}\int_0^t ds \int_{\mathbb{H}} m(dz) E^z\left[|e^{\mathcal{S}(0,s;\hat{l})}-1|^2 |g(\hat{l}_s)|^2\right] \\ =\ &\frac{1}{t}\int_0^t ds \int_{\mathbb{H}} m(dz) E^z\left[|e^{\tilde{\mathcal{S}}(0,s;l)}-1|^2 |g(z)|^2\right] \\ =\ &\frac{1}{t}\int_0^t ds \int_{\mathbb{H}} m(dz) |g(z)|^2 E^z\left[|e^{\tilde{\mathcal{S}}(0,s;l)}-1|^2\right], \end{aligned} \tag{6.6}$$

where we used the invariance in the second equality and we set

$$\tilde{\mathcal{S}}(s,t;l) = \sqrt{-1}\int_s^t \mathbf{a}(l_u)\circ dl_u - \int_s^t V(l_u)du. \tag{6.7}$$

Since $|e^{\tilde{\mathcal{S}}(0,s;l)}-1| \to 0$ as $s \to +0$ a.s. l for each z, by Lebesgue's dominated convergence theorem, we deduce that (6.6) tends to zero as $t \to +0$.

The second term of the rhs of (6.5) is equal to

$$
\begin{aligned}
& \frac{1}{t}\int_0^t ds \int_{\mathbb{H}} m(dz) E^z \left[|g(l_s)-g(l_0)|^2\right] \\
= & \frac{1}{t}\int_0^t ds \int_{\mathbb{H}} m(dz) E^z \left[(g(l_s)-g(l_0))\,\overline{g(l_s)} - (g(l_s)-g(l_0))\,\overline{g(l_0)}\right] \\
= & \frac{1}{t}\int_0^t ds \int_{\mathbb{H}} m(dz) E^z \left[-2\,(g(l_s)-g(l_0))\,\overline{g(l_0)}\right] \\
= & -\frac{2}{t}\int_0^t ds \int_{\mathbb{H}} m(dz)\overline{g(z)} E^z \left[(g(l_s)-g(l_0))\right], \qquad (6.8)
\end{aligned}
$$

where we used the reversibility in the second inequality, the invariance in the third equality. By Lebesgue's theorem, (6.8) tends to zero as $t \to +0$. Then the assertion follows from a simple density argument. ∎

Lemma 6.2 *Let* $\mathbf{a}$ *and* V *be as above. For any* $t \geq 0$*, the operator* T_t *is self-adjoint. Moreover, the family of operators* $\{T_t\}_{t\geq 0}$ *defines a semigroup, i.e.,* $T_tT_s = T_{t+s}$ *holds for all* $t, s \geq 0$*.*

Proof. First we show the self-adjointness. The reversibility implies that

$$
\begin{aligned}
\int_{\mathbb{H}} m(dz) T_t f(z)\overline{g(z)} &= \int_{\mathbb{H}} m(dz) E^z[e^{\mathcal{S}(0,t;l)} f(l_t)\overline{g(l_0)}] \\
&= \int_{\mathbb{H}} m(dz) E^z[e^{\mathcal{S}(0,t;\hat{l})} f(\hat{l}_t)\overline{g(\hat{l}_0)}] \\
&= \int_{\mathbb{H}} m(dz) f(z)\overline{T_t g(z)}
\end{aligned}
$$

for any $f, g \in C_0^\infty(\mathbb{H})$.

Next, using the Markov property of the diffusion, we have

$$
\begin{aligned}
T_{t+s} f(z) &= E^z\left[e^{\mathcal{S}(0,t+s;l)} f(l_{t+s})\right] \\
&= E^z\left[E^z[e^{\mathcal{S}(0,t+s;l)} f(l_{t+s})|\mathcal{B}_t](\tilde{l})\right] \\
&= E^z\left[e^{\mathcal{S}(0,t;\tilde{l})} E^z[e^{\mathcal{S}(t,t+s;l)} f(l_{t+s})|\mathcal{B}_t](\tilde{l})\right] \\
&= E^z\left[e^{\mathcal{S}(0,t;l)} E^{\tilde{l}_t}[e^{\mathcal{S}(0,s;l)} f(l_s)]\right] \\
&= E^z\left[e^{\mathcal{S}(0,t;\tilde{l})} (T_s f)(\tilde{l}_t)\right] \\
&= T_t T_s f(z)
\end{aligned}
$$

for all $f \in C_0^\infty(\mathbb{H})$, where we used the fact that $E[E[f|\mathcal{B}_t]] = E[f]$ in the second equality. ∎

Lemma 6.3 *Let* $\mathbf{a}$ *and* V *be as above. The operator* T_t *coincides with the heat operator* $e^{-tH_V(\mathbf{a})}$ *for any* $t \geq 0$ *and the integral kernel of* $e^{-tH_V(\mathbf{a})}$ *is given by*

$$p_{\mathbf{a}V}(t,z,z') = p_{\mathbf{0}0}(t,z,z')E_t^{z,z'}[e^{\mathcal{S}(0,t;l)}]. \tag{6.9}$$

Proof. By Lemmas 6.1 and 6.2, the domain of the infinitesimal generator of the contraction semigroup $\{T_t\}$ contains $C_0^\infty(\mathbb{H})$ and the generator is given by $-H_V(\mathbf{a})$ on $C_0^\infty(\mathbb{H})$. Since the operator $-H_V(\mathbf{a})$ is essentially self-adjoint on $C_0^\infty(\mathbb{H})$, we see that $T_t = e^{-tH_V(\mathbf{a})}$.

We first assume that $\mathbf{a} \in C_0^\infty(\mathbb{H}, \mathbb{R}^2)$ and $V \in C_0^\infty(\mathbb{H}, \mathbb{R})$. We note that the superposition $f = \int_{\mathbb{H}} \tilde{\delta}_z f(z)m(dz)$ holds for any $f \in \mathcal{S}'(\mathbb{R})$ with $\mathrm{supp}\,(f) \subset \mathbb{H}$ $(\subset \mathbb{R}^2)$. Then we can deduce from the same argument as in Ikeda and Watanabe [8], p.414 that

$$\begin{aligned} p_{\mathbf{a}V}(t,z,z') &= E^{(2)}[e^{S(0,t;z,w)}\tilde{\delta}_{z'}(Z(t,z,w))] \\ &= p_{\mathbf{0}0}(t,z,z')\int_{W^{(2)}} e^{\tilde{S}(0,t;z,w)}\mu_t^{z,z'}(dw) \\ &= p_{\mathbf{0}0}(t,z,z')E_t^{z,z'}[e^{\mathcal{S}(0,t;l)}], \end{aligned} \tag{6.10}$$

where $\tilde{S}(0,t;z,\cdot)$ stands for a quasi-continuous modification of $S(0,t;z,\cdot)$ and $\mu_t^{z,z'}$ is as in Subsection 2.3. Here, we used also the facts that, when the size of mesh of partition of $[0,t]$ tends to zero, $2^{-1}(\mathbf{a}(Z_{t_i}) + \mathbf{a}(Z_{t_{i-1}}))(Z_{t_i} - Z_{t_{i-1}})$ converges to $\int_0^t \mathbf{a}(Z_s) \circ dZ_s$ in $\mathbf{D}_\infty$ and that $2^{-1}(\mathbf{a}(l_{t_i}) + \mathbf{a}(l_{t_{i-1}}))(l_{t_i} - l_{t_{i-1}})$ converges to $\int_0^t \mathbf{a}(l_s) \circ dl_s$ in probability with respect to $P_t^{z,z'}$ in the second equality. Hence the assertion holds if $\mathbf{a}, V \in C_0^\infty$. Next we consider the general case. Given $\mathbf{a}, V$, we can find a sequences $\mathbf{a}_j \in C_0^\infty(\mathbb{H}, \mathbb{R}^2)$, $V_j \in C_0^\infty(\mathbb{H}, \mathbb{R})$ so that $\mathbf{a}_j$, $\partial\mathbf{a}_j$ and V_j converge to $\mathbf{a}$, $\partial\mathbf{a}$ and V on every compact subset of $\mathbb{H}$ as $j \to \infty$, respectively. Then we can deduce from Lemma 5.3 and Lebesgue's dominated convergence theorem that the rhs of (6.10) for a_j and V_j converges the one for $\mathbf{a}$ and V as $j \to \infty$. On the other hand, $H_{V_j}(\mathbf{a}_j)$ converges to $H_V(\mathbf{a})$ strongly, so $e^{-tH_{V_j}(\mathbf{a}_j)}$ converges to $e^{-tH_V(\mathbf{a})}$ strongly. Hence, we deduce that the equality (6.10) is still valid for the case of general $\mathbf{a}, V$. ∎

Lemma 6.4 *Let* K *be a compact subset of* $\mathbb{H}$*, The following two assertions hold:*

(1) Assume (V). Then we have

$$\lim_{t\to+0}\sup_{z\in K} E^z[\left|\int_0^t V(l_s)ds\right|] = 0. \tag{6.11}$$

(2) Assume (A). Then we have

$$\lim_{t\to+0}\sup_{z\in K} E^z[\left|\int_0^t \mathbf{a}(l_s) \circ dl_s\right|] = 0. \tag{6.12}$$

Proof. First we show the assertion (1), which is equivalent to

$$\lim_{t\to+0}\sup_{z\in K}E^{(2)}[\Big|\int_0^t V(Z(s,z,w))ds\Big|].$$

For any small $t>0$ and any $z\in\mathbb{H}$, the lhs of (6.11) is less than or equal to

$$\begin{aligned}
& \lim_{t\to+0}\sup_{z\in K}\int_0^t dsE^{(2)}[|V(Z(s,z,w))|] \\
= & \lim_{t\to+0}\sup_{z\in K}\int_0^t ds\int_{\mathbb{H}} m(dz')|V(z')|p_{\mathbf{00}}(s,z,z') \\
\leq & \lim_{t\to+0} Ct^{1/2}\sup_{z\in K}\exp\left(cd(z,\sqrt{-1})^2\right).
\end{aligned}$$

Here we can show the last estimate above as in the proof of Lemma 3.1 2, replaced $\tilde{a}$ by $C\exp\left(cd(z,\sqrt{-1})\right)$.

Next we show the assertion (2). It is enough to show that

$$\lim_{t\to+0}\sup_{z\in K}E^{(2)}[\Big|\int_0^t \mathbf{a}(Z(s,z,w))\circ dZ(s,z,w)ds\Big|]=0.$$

Taking (2.1) into account, using Itô's formula, we see that for $j=1,2$

$$\begin{aligned}
da_j(Z(s,z,w)) = {} & \partial_x a_j(Z(s,z,w))dX(s,z,w)+\partial_y a_j(Z(s,z,w))dY(s,z,w) \\
& +\frac{1}{2}\triangle_{\mathbb{R}^2}a_j(Z(s,z,w))Y(s,z,w)^2ds.
\end{aligned}$$

Thus we have

$$\begin{aligned}
& \int_0^t \mathbf{a}(Z(s,z,w))\circ dZ(s,z,w) \\
= & \int_0^t a_1(Z(s,z,w))dX(s,z,w)+\int_0^t a_2(Z(s,z,w))dY(s,z,w) \\
& +\frac{1}{2}\int_0^t \partial_x a_1(Z(s,z,w))d\langle X(s,z,w),X(s,z,w)\rangle \\
& +\frac{1}{2}\int_0^t \partial_y a_2(Z(s,z,w))d\langle Y(s,z,w),Y(s,z,w)\rangle \\
= & \int_0^t \mathbf{a}(Z(s,z,w))dZ(s,z,w)+\frac{1}{2}\int_0^t Y(s,z,w)^2(\partial_x a_1+\partial_y a_2)(Z(s,z,w))ds.
\end{aligned}$$

Finally, Burkholder's inequality yields that

$$
\begin{aligned}
& E^{(2)}\left[\left|\int_0^t \mathbf{a}(Z(s,z,w)) \circ dZ(s,z,w)\right|\right] \\
\leq\ & E^{(2)}\left[\left|\int_0^t \mathbf{a}(Z(s,z,w)) dZ(s,z,w)\right|\right] \\
& +\frac{1}{2}E^{(2)}\left[\left|\int_0^t Y(s,z,w)^2(\partial_x a_1+\partial_y a_2)(Z(s,z,w))ds\right|\right] \\
\leq\ & C\, E^{(2)}\left[\int_0^t Y(s,z,w)^2|\mathbf{a}(Z(s,z,w))|^2 ds\right] \\
& +\frac{1}{2}E^{(2)}\left[\int_0^t Y(s,z,w)^2|(\partial_x a_1+\partial_y a_2)(Z(s,z,w))|ds\right] \\
\leq\ & C'\int_0^t ds \int_{\mathbb{H}} m(dz')\left((y'|\mathbf{a}(z')|)^2+(y')^2|(\partial_x a_1+\partial_y a_2)(z')|\right) p_{\mathbf{00}}(s,z,z') \\
=\ & C'\int_0^t ds\ E^z[\tilde{a}(l_s)^2+\tilde{a}(l_s)],
\end{aligned}
$$

where $\tilde{a}$ is as in (A). Then the assertion follows from (3.1) since if $\tilde{a} \in \mathcal{G}$ then $\tilde{a}^2 \in \mathcal{G}$ as we mentioned in Section 1. ▮

Lemma 6.5 *Let $z, z'' \in \mathbb{H}$ and $s, t > 0$. Let K be a compact subset of $\mathbb{H}$. Then we have*

$$
\lim_{d(z',z'')\to 0}\sup_{z\in K}\int_{\mathbb{H}} m(dz_0)|p_{\mathbf{00}}(s,z_0,z')-p_{\mathbf{00}}(s,z_0,z'')|p_{\mathbf{00}}(t,z,z_0)=0. \tag{6.13}
$$

Proof. Fix $t, s > 0$. For any $R > 0$, we denote by $B_R(z'')$ the geodesic ball centered at z'' of radius R. We may assume that $z' \in B_1(z'')$. By (2.3) and (2.4), the value of $p_{\mathbf{00}}(s,z,z')$ tends to zero as $d(z,z') \to 0$. Thus, for any $\mathbf{a} > 0$, there exists $R > 0$ such that $|p_{\mathbf{00}}(s,z,z')| \leq \varepsilon$ for any $z, z' \in \mathbb{H}$ satisfying $d(z,z') \geq R$. If $d(z'',z_0) \geq R+1$, then $d(z',z_0) \geq R$. Hence we have

$$
\begin{aligned}
& \int_{B_{R+1}(z'')^c} m(dz_0)|p_{\mathbf{00}}(s,z_0,z')-p_{\mathbf{00}}(s,z_0,z'')|p_{\mathbf{00}}(t,z,z_0) \\
\leq\ & 2\varepsilon\int_{B_{R+1}(z'')^c} m(dz_0)p_{\mathbf{00}}(t,z,z_0) \leq 2\varepsilon.
\end{aligned} \tag{6.14}
$$

Here we used the fact that $p_{\mathbf{00}}(t,z,z_0)m(dz_0)$ is a probability measure.

Since the function $B_{R+1}(z'') \times B_1(z'') \ni (z_0,z') \mapsto p_{\mathbf{00}}(s,z_0,z')$ is uniformly continuous we see that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|p_{\mathbf{00}}(s,z_0,z')-p_{\mathbf{00}}(s,z_0,z'')| \leq \varepsilon$ if $d(z',z'') \leq \delta$. Hence, we have

$$
\begin{aligned}
& \int_{B_{R+1}(z'')} m(dz_0)|p_{\mathbf{00}}(s,z_0,z')-p_{\mathbf{00}}(s,z_0,z'')|p_{\mathbf{00}}(t,z,z_0) \\
\leq\ & \varepsilon\int_{B_{R+1}(z'')} m(dz_0)p_{\mathbf{00}}(t,z,z_0) \leq \varepsilon
\end{aligned} \tag{6.15}
$$

if $d(z', z'') \leq \min\{\delta, 1\}$, where we used the fact that $p_{\mathbf{00}}(t, z, z_0)m(dz_0)$ is a probability measure.

Then the result follows from (6.15) and (6.14). ▌

Lemma 6.6 *Let $z'' \in \mathbb{H}$ and let K be a compact subset of $\mathbb{H}$. Then we have*

$$\lim_{z' \to z''} \sup_{z \in K} |p_{\mathbf{a}V}(t, z, z') - p_{\mathbf{a}V}(t, z, z'')| = 0$$

holds for each $t > 0$.

Proof. We take and fix any compact set K and $t > 0$. For any $s > 0$ small enough, by (6.9), we calculate

$$\begin{aligned}
& p_{\mathbf{a}V}(t, z, z') - p_{\mathbf{a}V}(t, z, z'') \\
= \; & p_{\mathbf{00}}(t, z, z') E_t^{z,z'}[\big(e^{\mathcal{S}(0,t;l)} - e^{\mathcal{S}(0,t-s;l)}\big)] \\
& + p_{\mathbf{00}}(t, z, z'') E_t^{z,z''}[\big(e^{\mathcal{S}(0,t;l)} - e^{\mathcal{S}(0,t-s;l)}\big)] \\
& + \Big(p_{\mathbf{00}}(t, z, z') E_t^{z,z'}[e^{\mathcal{S}(0,t-s;l)}] - p_{\mathbf{00}}(t, z, z'') E_t^{z,z''}[e^{\mathcal{S}(0,t-s;l)}]\Big) \\
=: \; & I_1 + I_2 + I_3.
\end{aligned} \tag{6.16}$$

First, we consider the integral I_1 and I_2. Taking the form (6.3) and (6.7) into account, we estimate

$$\begin{aligned}
|I_1| \; & \leq \; p_{\mathbf{00}}(t, z, z') E_t^{z,z'}[\big|e^{\mathcal{S}(t-s,t;l)} - 1\big|] \\
& = \; p_{\mathbf{00}}(t, z, z') E_t^{z',z}[\Big|e^{\tilde{\mathcal{S}}(0,s;\hat{l})} - 1\Big|] \\
& \leq \; C p_{\mathbf{00}}(t, z, z') E_t^{z',z}[\Big|\tilde{\mathcal{S}}(0, s; l)\Big|] \\
& = \; C E^{z'}[|\tilde{\mathcal{S}}(0, s; l)| p_{\mathbf{00}}(t - s, l_s, z)] \\
& \leq \; C p_{\mathbf{00}}(t - s, z, z) E^{z'}[|\tilde{\mathcal{S}}(0, s; l)|] \\
& \leq \; C p_{\mathbf{00}}(t - s, z, z) E^{z'}[\Big|\int_0^s \mathbf{a}(l_u) \circ dl_u\Big| + \Big|\int_0^s V(l_u) du\Big|] \\
& \leq \; C p_{\mathbf{00}}(t - s, \sqrt{-1}, \sqrt{-1}) \sup_{z' \in K} E^{z'}[\Big|\int_0^s \mathbf{a}(l_u) \circ dl_u\Big| + \Big|\int_0^s V(l_u) du\Big|],
\end{aligned} \tag{6.17}$$

where we used the fact that

$$\begin{aligned}
\int_{t-s}^t V(l_u) du \; &= \; \int_0^s V(\hat{l}_u) du, \\
\int_{t-s}^t \mathbf{a}(l_u) \circ dl_u \; &= \; -\int_0^s \mathbf{a}(\hat{l}_u) \circ d\hat{l}_u
\end{aligned}$$

in the second equality and used the elementary inequality $|e^x - 1| \le C|x|$ if $\Re x \le 0$ in the second inequality and used (2.6) in the fourth equality. Note that the rhs of (6.17) is independent of z' and z. In the case of I_2, we can find a similar estimate.

Next we consider I_3. Then it follows from (2.6) that

$$\begin{aligned} |I_3| &= |E^z[e^{\mathcal{S}(0,t-s;l)}(p_{\mathbf{0}0}(s,l_{t-s},z') - p_{\mathbf{0}0}(s,l_{t-s},z''))]| \\ &\le E^z[|p_{\mathbf{0}0}(s,l_{t-s},z') - p_{\mathbf{0}0}(s,l_{t-s},z'')|] \\ &= E[|p_{\mathbf{0}0}(s,Z(t-s,z,w),z') - p_{\mathbf{0}0}(s,Z(t-s,z,w),z'')|] \\ &= \int_{\mathbb{H}} m(dz_0)|p_{\mathbf{0}0}(s,z_0,z') - p_{\mathbf{0}0}(s,z_0,z'')|p_{\mathbf{0}0}(t-s,z,z_0). \qquad (6.18) \end{aligned}$$

Hence it follows from (6.17) and (6.18) that

$$\begin{aligned} &\lim_{z'\to z''} \sup_{z\in K} |I_1 + I_2 + I_3| \\ \le\ & 2Cp_{\mathbf{0}0}(t-s,\sqrt{-1},\sqrt{-1}) \sup_{z'\in K} E^{z'}\Big[\Big|\int_0^s \mathbf{a}(l_u)\circ dl_u\Big| + \Big|\int_0^s V(l_u)du\Big|\Big] \\ &\quad + \lim_{z'\to z''} \sup_{z\in K} \int_{\mathbb{H}} m(dz_0)|p_{\mathbf{0}0}(s,z_0,z') - p_{\mathbf{0}0}(s,z_0,z'')|p_{\mathbf{0}0}(t-s,z,z_0). \quad (6.19) \end{aligned}$$

The first and the second terms of the rhs of (6.19) tend to zero as $s \to +0$ by Lemma 6.4 and by Lemma 6.5, respectively. This completes the proof. ▌

Lemma 6.7 *For each $t > 0$, the kernel $p_{\mathbf{a}V}(t,z,z')$ is continuous in $(z,z') \in \mathbb{H}\times\mathbb{H}$.*

Proof. We consider the continuity at (z_0, z_0'). Using the self-adjointness of $e^{-tH_V(\mathbf{a})}$, we have

$$\begin{aligned} & p_{\mathbf{a}V}(t,z,z') - p_{\mathbf{a}V}(t,z_0,z_0') \\ =\ & (p_{\mathbf{a}V}(t,z,z') - p_{\mathbf{a}V}(t,z,z_0')) + (\overline{p_{\mathbf{a}V}(t,z_0',z)} - \overline{p_{\mathbf{a}V}(t,z_0',z_0)}) \end{aligned}$$

for all z, z' and $t > 0$. Then, for any (z,z') near (z_0, z_0'), the estimate

$$\begin{aligned} & |p_{\mathbf{a}V}(t,z,z') - p_{\mathbf{a}V}(t,z_0,z_0')| \\ \le\ & \sup_{z\in B_1(z_0)} |p_{\mathbf{a}V}(t,z,z') - p_{\mathbf{a}V}(t,z,z_0')| \\ &\qquad + |p_{\mathbf{a}V}(t,z_0',z) - p_{\mathbf{a}V}(t,z_0',z_0)|, \end{aligned}$$

where we denote the compact set $\{z \in \mathbb{H} | d(z,z_0) \le 1\}$ by $B_1(z_0)$. Then the lemma follows from (6.16) and Lemma 6.6. ▌

Then Proposition 4.1 follows from Lemmas 6.1, 6.2, 6.3, 6.6 and 6.7.

7 Appendix

In this appendix we show the condition (B) below on the magnetic field ω implies the existence of a magnetic vector potential $\mathbf{a} = a_1dx + a_2dy$ satisfying the condition (A) in Section 1. (See Proposition 7.6).

As before we identify $\mathbf{a} = (a_1, a_2)$ with the 1-form $a_1dx + a_2dy$. For the notational simplicity, we use the symbol ∇ to denote not only the Levi-Civita connection on $\mathrm{T}\mathbb{H}$ but also to denote the induced connections on the tensor bundles. We denote by $\|\cdot\|$ the norms on the tensor bundles induced from the metric $g = y^{-2}dx\otimes dx + y^{-2}dy\otimes dy$ on $\mathbb{H}$. We denote by $\langle\cdot,\cdot\rangle$ the natural pairing between $\mathrm{T}^*_z\mathbb{H}$ and $\mathrm{T}_z\mathbb{H}$, etc. For any vector bundle E over $\mathbb{H}$, we denote by $C^k(E)$ the set of all C^k-sections of E. We adopt a convention that a connection on E is a map from $C^{k+1}(E)$ to $C^k(E\otimes \mathrm{T}^*\mathbb{H})$.

Lemma 7.1 *The condition (A) in Section 1 is equivalent to the following condition (A)':*

(A)' The vector potential $\mathbf{a} = a_1dx + a_2dy$ belongs to $C^2(\Lambda\mathrm{T}^*\mathbb{H})$. Moreover, there exists $\tilde{a}\in\mathcal{G}$ such that

$$\|a(z)\| + \|\nabla a(z)\| + \|\nabla^2 a(z)\| \le \tilde{a}(d(z,\sqrt{-1}))$$

holds for all $z\in\mathbb{H}$.

Remark 7.2 *We can choose $\tilde{a}$ in condition (A)' above so that it is a constant multiple of $\tilde{a}$ on the rhs of (1.3) in condition (A). We can easily check this from the proof of Lemma 7.1 below.*

Proof of Lemma 7.1. Note that if we write $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$ for $i,j,k = 1,2$, all the non-zero Christoffel symbols are $\Gamma^1_{12} = \Gamma^1_{21} = -1/y$, $\Gamma^2_{11} = 1/y$ and $\Gamma^2_{22} = -1/y$.

We fix a orthonormal basis $e_1 = y\partial_x$, $e_2 = y\partial_y$ for $\mathrm{T}_z\mathbb{H}$ and the dual basis $e_1^* = dx/y$, $e_2^* = dy/y$ for $\mathrm{T}^*_z\mathbb{H}$. Then we find that

$$\begin{aligned}\nabla\mathbf{a} &= y(a_2 + y\partial_x a_1)e_1^*\otimes e_1^* + y(a_1 + y\partial_x a_2)e_2^*\otimes e_1^* \\ &\quad + y(a_1 + y\partial_y a_1)e_1^*\otimes e_2^* + y(a_2 + y\partial_y a_2)e_2^*\otimes e_2^*,\end{aligned}$$

so we have

$$\begin{aligned}\|\nabla\mathbf{a}\|^2 &= y^2\left((a_2 + y\partial_x a_1)^2 + (a_1 + y\partial_x a_2)^2\right. \\ &\quad \left. + (a_1 + y\partial_y a_1)^2 + (a_2 + y\partial_y a_2)^2\right).\end{aligned}$$

If we define f_{ij} by $\nabla^2\mathbf{a} = \sum_{i,j=1}^2 f_{ij}\otimes e_i^*\otimes e_j^*$, we have $f_{ij} = \nabla^2_{e_i\otimes e_j}\mathbf{a} = \langle\nabla_{e_j}\nabla\mathbf{a}, e_i\rangle = \nabla_{e_j}\nabla_{e_i}\mathbf{a} - \langle\nabla\mathbf{a}, \nabla_{e_j}e_i\rangle$. By a direct computation shows that

$$\begin{aligned}f_{11} &= y(2y\partial_x a_2 - y\partial_y a_1 + y^2\partial_x^2 a_1)e_1^* + y(2y\partial_x a_1 - y\partial_y a_2 + y^2\partial_x^2 a_2)e_2^* \\ &=: f^1_{11}e_1^* + f^2_{11}e_2^*, \\ f_{12} &= y(a_2 + y\partial_x a_1 + y\partial_y a_2 + y^2\partial_x\partial_y a_1)e_1^* + y(a_1 + y\partial_y a_1 + y\partial_x a_2 + y^2\partial_x\partial_y a_2)e_2^* \\ &=: f^1_{12}e_1^* + f^2_{12}e_2^*, \\ f_{21} &= f_{12} =: f^1_{21}e_1^* + f^2_{21}e_2^*, \\ f_{22} &= y(a_1 + 3y\partial_y a_1 + y^2\partial_y^2 a_1)e_1^* + y(a_2 + 3y\partial_y a_2 + y^2\partial_y^2 a_2)e_2^* \\ &=: f^1_{22}e_1^* + f^2_{22}e_2^*,\end{aligned}$$

so we have $\|\nabla^2\mathbf{a}(z)\|^2 = \sum_{i,j,k=1}^2 |f_{ij}^k|^2$.

Hence. one can observe that the lhs of (1.3) defines a norm on $\mathrm{T}_z\mathbb{H}$ equivalent to the norm $\|\mathbf{a}(z)\| + \|\nabla\mathbf{a}(z)\| + \|\nabla^2\mathbf{a}(z)\|$. This proves the lemma ▌

In what follows we consider the problem in the geodesic (polar) coordinate $X = r\cos\theta, Y = r\sin\theta$ ($r \geq 0, 0 \leq \theta < 2\pi$) at the base point $\sqrt{-1}$, which is linked $z = (x, y)$ via $z = \tanh(r/2)\, e^{\sqrt{-1}\theta}$. In this coordinate the Riemannian metric is expressed as $g = dr \otimes dr + \sinh^2 r d\theta \otimes d\theta$.

In this coordinate, we write $\omega = B(X,Y)dX \wedge dY = B(r,\theta) r dr \wedge d\theta$ (Precisely, $B(r,\theta)$ on the rhs stands for $B(r\cos\theta, r\sin\theta)$). We may assume that B is real.

We introduce the following functions

$$R(r,\theta) = \sum_{0 \leq \alpha+\beta \leq 2} \frac{|\partial_r^\alpha \partial_\theta^\beta (rB)(r,\theta)|}{\sinh^{\beta+1} r}$$

and

$$R_{\alpha\beta}(r,\theta) = \frac{|\partial_r^\alpha \partial_\theta^\beta (rB)(r,\theta)|}{\sinh^{\beta+1} r}$$

for any α, β.

Lemma 7.3 *Assume that $\omega = Brdr \wedge d\theta$ belongs to $C^2(\Lambda^2\mathrm{T}^*\mathbb{H})$. Let R be as above and we write $z = \tanh(r/2)e^{\sqrt{-1}\theta}$ as before. Then there exists $C > 0$ such that the estimate $R(r,\theta) \leq C(\|\omega(z)\| + \|\nabla\omega(z)\| + \|\nabla^2\omega(z)\|)$ holds for any z with $r \geq 1$.*

Proof. Note that all the nonzero Christoffel symbols in the geometric polar coordinate are $\Gamma_{\theta\theta}^r = -\sinh r \cosh r$ and $\Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = 1/\tanh r$, so we find that $\nabla dr = \sinh r \cosh r d\theta \otimes d\theta$, $\nabla d\theta = -(1/\tanh r)(d\theta \otimes dr + dr \otimes d\theta)$, which follows from the fact that $\langle \nabla_V \omega, W\rangle = \langle d\langle \omega, W\rangle, V\rangle - \langle \omega, \nabla_V W\rangle$ holds for all $V, W \in C^1(\mathrm{T}\mathbb{H})$ and $\omega \in C^1(\mathrm{T}^*\mathbb{H})$.

Then, differentiating $\omega = Brdr \wedge d\theta$, we can obtain

$$\begin{aligned}
\nabla\omega &= (\partial_r - \frac{1}{\tanh r})(Br)(dr \wedge d\theta) \otimes dr + \partial_\theta(Br)(dr \wedge d\theta) \otimes d\theta, \\
\nabla^2\omega &= (\partial_r - \frac{1}{\tanh r})^2(Br)(dr \wedge d\theta) \otimes dr \otimes dr \\
&\quad + (\partial_r - \frac{2}{\tanh r})(\partial_\theta Br)(dr \wedge d\theta) \otimes (dr \otimes d\theta + d\theta \otimes dr) \\
&\quad + \left(\sinh r \cosh r(\partial_r - \frac{1}{\tanh r})(Br) + \partial_\theta^2(Br)\right)(dr \wedge d\theta) \otimes d\theta \otimes d\theta, \quad (7.1)
\end{aligned}$$

after straightforward calculations using the Leibniz formula $\nabla_X(Y_1 \otimes Y_2) = (\nabla_X Y_1) \otimes Y_2 + Y_1 \otimes (\nabla_X Y_2)$.

Up to a constant multiple, the norm $\|\omega_1(z) \wedge \omega_2(z)\|$ is given by $\|\omega_1(z)\|\|\omega_2(z)\|$ for any $\omega_1, \omega_2 \in \mathrm{T}^*\mathbb{H}$. In the following we assume that $\|\omega_1(z) \wedge \omega_2(z)\| = \|\omega_1(z)\|\|\omega_2(z)\|$. This does not cause any problem.

Since $\|dr(z)\| = 1$ and $\|d\theta(z)\| = 1/\sinh r$ ($z = \tanh(r/2)\, e^{\sqrt{-1}\theta}$ as before), it follows from (7.1) that $\|\omega(z)\| = |Br|/\sinh r$,

$$\begin{aligned}
\|\nabla\omega(z)\|^2 &= \frac{\left|(\partial_r - \frac{1}{\tanh r})(Br)\right|^2}{\sinh^2 r} + \frac{|\partial_\theta(rB)|^2}{\sinh^4 r}, \\
\|\nabla^2\omega(z)\|^2 &= \frac{\left|(\partial_r - \frac{1}{\tanh r})^2(Br)\right|^2}{\sinh^2 r} + \frac{\left|(\partial_r - \frac{2}{\tanh r})\partial_\theta(Br)\right|^2}{\sinh^4 r} \\
&\quad + \left|\sinh r\cosh r\left(\partial_r - \frac{1}{\tanh r}\right)^2(Br) + \partial_\theta^2(Br)\right|^2 \frac{1}{\sinh^6 r}.
\end{aligned}$$

Hence, we find that $R_{00} = |rB(r,\theta)| = \|\omega(z)\|$,

$$\begin{aligned}
R_{10} &= \frac{|\partial_r(rB)|}{\sinh r} \le \frac{\left|(\partial_r - \frac{1}{\tanh r})(Br)\right|}{\sinh r} + \frac{|Br|}{\tanh r \sinh r} \\
&\le \|\nabla\omega(z)\| + C\|\omega(z)\|, \\
R_{01} &= \frac{|\partial_\theta(rB)|}{\sinh^2 r} \le \|\nabla\omega(z)\|, \\
R_{11} &= \frac{|\partial_r\partial_\theta(rB)|}{\sinh^2 r} \le \frac{\left|\left(\partial_r - \frac{2}{\tanh r}\partial_\theta(Br)\right)\right|}{\sinh^2 r} + \frac{2}{\tanh r}\frac{|\partial_\theta(Br)|}{\sinh^2 r} \\
&\le \|\nabla^2\omega(z)\| + C\|\nabla\omega(z)\|
\end{aligned}$$

for all $r \ge 1$, where the constant $C > 0$ is independent of z, ω. Similar calculations show that $R_{20} \le \|\nabla^2\omega(z)\| + C\|\nabla\omega(z)\| + C\|\omega(z)\|$ and $R_{02} \le \|\nabla^2\omega(z)\| + \|\nabla\omega(z)\|$ hold for some $C > 0$. This proves the lemma. ▌

Lemma 7.4 *Let $\omega = Brdr \wedge d\theta \in C^2(\Lambda^2 T^*\mathbb{H})$ and let R be as above. Then there exists $\mathbf{a} \in C^2(\Lambda T^*\mathbb{H})$ such that $d\mathbf{a} = \omega$ holds and the following estimate*

$$\|\mathbf{a}(z)\| + \|\nabla\mathbf{a}(z)\| + \|\nabla^2\mathbf{a}(z)\| \le Cr\int_0^1 R(tr,\theta)dt \tag{7.2}$$

holds for any $z \in \mathbb{H}$, where $C > 0$ is independent of $z = \tanh(r/2)e^{\sqrt{-1}\theta}$.

Proof. Given ω, we take $\mathbf{a} = \left(\int_0^1 B(tr,\theta)r^2tdt\right)d\theta$. Then, by an integration by parts, one can observe that $d\mathbf{a} = \partial_r(\int_0^1 B(tr,\theta)r^2tdt)dr \wedge d\theta = B(r,\theta)rdr \wedge d\theta = \omega$ holds.

We show (7.2). In this proof we denote the function $\int_0^1 B(tr,\theta)trdt$ by b for simplicity.

Direct calculations show that

$$\begin{aligned}
\nabla \mathbf{a} &= \left(r\left(\partial_r - \frac{1}{\tanh r}\right) b + b \right) d\theta \otimes d\theta + r(\partial_\theta b) d\theta \otimes d\theta, \\
\nabla^2 \mathbf{a} &= \left(r\left(\partial_r - \frac{1}{\tanh r}\right)^2 b + 2\left(\partial_r - \frac{1}{\tanh r}\right) b \right) d\theta \otimes dr \otimes dr, \\
&\quad - \left(\frac{r(\partial_r b)}{\tanh r} - \frac{2rb}{\tanh^2 r} + \frac{b}{\tanh r} \right) dr \otimes dr \otimes d\theta \\
&\quad - \left(r\left(\partial_r - \frac{1}{\tanh r}\right)\left(\frac{b}{\tanh r}\right) + \frac{b}{\tanh r} \right) dr \otimes d\theta \otimes dr \\
&\quad - \frac{2r(\partial_\theta b)}{\tanh r} dr \otimes d\theta \otimes d\theta \\
&\quad + \left(r\left(\partial_r - \frac{2}{\tanh r}\right)(\partial_\theta b) + (\partial_\theta b) \right) d\theta \otimes (dr \otimes d\theta + d\theta \otimes dr) \\
&\quad + \sinh r \cosh r \left(r\left(\partial_r - \frac{2}{\tanh r}\right) b + b \right) d\theta \otimes d\theta \otimes d\theta.
\end{aligned}$$

Hence it follows from the fact that $\|dr(z)\| = 1$ and $\|d\theta(z)\| = 1/\sinh r$ that

$$\begin{aligned}
\|\mathbf{a}(z)\| &\leq \frac{r|b|}{\sinh r} \leq r \int_0^1 \frac{|trB(tr,\theta)|}{\sinh r} dt \leq r \int_0^1 R_{00}(tr,\theta) dt \leq r \int_0^1 R(tr,\theta) dt, \\
\|\nabla \mathbf{a}(z)\|^2 &= \left| \left(r\left(\partial_r - \frac{1}{\tanh r}\right) b + b \right) \frac{1}{\sinh r} \right|^2 + \left| \frac{r\partial_\theta b}{\sinh^2 r} \right|^2 \\
&\leq C \left| r \int_0^1 (R_{01}(tr,\theta) + R_{00}(tr,\theta)) dt \right|^2 + C \left| r \int_0^1 R_{01}(tr,\theta) dt \right|^2 \\
&\leq C \left(r \int_0^1 R(tr,\theta) dt \right)^2
\end{aligned}$$

holds for some $C > 0$. Similarly we can show that $\|\nabla^2 \mathbf{a}(z)\| \leq Cr \int_0^1 R(tr,\theta) dt$ holds for some $C > 0$. This proves the lemma. ▮

Lemma 7.5 *Let $\mathcal{G}$ be the class as in Section 1. Assume that $R \in C([0,\infty))$ and there exists $\tilde{a} \in \mathcal{G}$ such that $0 \leq R(r) \leq \tilde{a}(r)$ holds for all $\rho \geq 0$. Set $\tilde{R}(r) = \int_0^1 R(tr) dt$. Then there exists $\tilde{c} \in \mathcal{G}$ such that $|\tilde{R}(r)| \leq \tilde{c}(r)$ holds for all $r \geq 0$.*

Proof. If we set $\tilde{c}(r) = \int_0^1 \tilde{a}(tr) dt$, then we have the estimate $|\tilde{R}(r)| \leq \tilde{c}(r)$ by definition. Moreover, it follows that $\tilde{c} \in \mathcal{G}$ since $\tilde{a} \in \mathcal{G}$. ▮

Now we introduce the following condition (B) and we show the claim at beginning at Appendix.

(B) ω belongs to $C^2(\Lambda^2 \mathrm{T}^*\mathbb{H})$ and moreover, there exists $\tilde{c} \in \mathcal{G}$ such that ω satisfies the estimate

$$d(z, \sqrt{-1}) \left(\|\omega(z)\| + \|\nabla\omega(z)\| + \|\nabla^2\omega(z)\| \right) \leq \tilde{c}(d(z, \sqrt{-1}))$$

for any $z \in \mathbb{H}$.

Proposition 7.6 *Assume that the 2-form ω satisfies the condition (B). Then, there exists a 1-form $\mathbf{a}$ such that $d\mathbf{a} = \omega$ and the condition (A) holds. Moreover, as $\tilde{a} \in \mathcal{G}$ in (A), we may choose a function of the following form:*

$$\tilde{a}(r) = Cr \int_0^1 R(tr)dt + C,$$

where C is a positive constant, $R(r) = \tilde{c}(r)/r$ except in a neighborhood of $r = 0$ and $\tilde{c}$ is as in (B).

Proof. By Lemma 7.3, the function $rR(r, \theta)$ is dominated by some $\tilde{a} \in \mathcal{G}$, from which we can deduce that $R(r, \theta)$ is also dominated by some element of $\mathcal{G}$ if $r \geq 1$ since any continuous function defined on $[0, \infty)$ which coincides with $1/r$ except near $r = 0$ belongs to $\mathcal{G}$, and the class $\mathcal{G}$ is closed under multiplication. Finally the claim obeys from Lemmas 7.1, 7.4 and Lemma 7.5 with $R(r) = R(r, \theta)$. ▮

References

[1] Aida, S., Logarithmic derivatives of heat kernels and logarithmic Sobolev inequalities with unbounded diffusion coefficients on loop spaces, J. Funct. Anal. 174 (2000), no.2, 430–477.

[2] Broderix, K., Hundertmark, D. and Leschke, H., Continuity properties of Schrödinger semigroups with magnetic fields, Rev. Math. Phys. 12 (2000), no. 2, 181-225.

[3] Bingham, N. H., Goldie, C. M. and Teugels, J. L., *Regular variation*, Cambridge Univ. Press, 1987.

[4] Brislawn, C., Kernels of trace class operators, Proc. Amer. Math. Soc. 104 (1988), no. 4, 1181–1190.

[5] Davies, E.B., *Heat kernels and spectral theory*, Cambridge Univ. Press, Cambridge, 1989.

[6] Eberle, A., Absence of spectral gaps on a class of loop spaces, J .Math. Pures Appl. (9), 81 (2002), no. 10, 915-955.

[7] Ikeda, N. and Matsumoto, H., Brownian motion on the hyperbolic plane and Selberg trace formula, J. Funct. Anal. 163 (1999), no. 1, 63–110.

[8] Ikeda, N. and Watanabe, S., *Stochastic Differential Equations and Diffusion Processes* (second edition), North Holland-Kodansha, 1989.

[9] Inahama, Y. and Shirai, S., Eigenvalue asymptotics for the Schrödinger operators on the hyperbolic plane, J. Funct. Anal. 211 (2004) no. 2, 424–456.

[10] Malliavin, P., *Stochastic Analysis,* Springer-Verlag, 1997.

[11] Malliavin, P. and Nualart, D., Quasi sure analysis of stochastic flows and Banach space valued smooth functionals on the Wiener space, J. Funct. Anal. 112 (1993), no. 2, 287–318.

[12] Matsumoto, H., The short time asymptotics of the traces of the heat kernels for the magnetic Schrödinger operators, J. Math. Soc. Japan, 42 (1990), no. 4, 677-689.

[13] Matsumoto, H., The asymptotic distributions of the eigenvalues for the Schrödinger operators with magnetic field, *Asymptotic problems in probability theory: Wiener functionals and asymptotics (Sanda/Kyoto, 1990),* 169–193, Pitman Res. Notes Math. Ser., 284, Longman Sci. Tech., Harlow, 1993.

[14] Reed, M., and Simon, B., *Methods of modern mathematical physics*, Vol. IV, Academic Press, New York, 1978.

[15] Shubin, M., Essential self-adjointness for semi-bounded magnetic Schödinger operators on non-compact manifolds, J. Funct. Anal. 186 (2001), no. 1, 92–116.

[16] Sugita, H., Positive generalized functions and potential theory over abstract Wiener spaces, Osaka J. Math. 25 (1988), no. 3, 665–696.

[17] Terras, A., *Harmonic Analysis on Symmetric Spaces and Applications I,* Springer Verlag, 1985.

Yuzuru INAHAMA

Department of Mathematics

Tokyo Institute of Technology,

2-12-1, Oh-okayama, Meguro-ku, Tokyo, 152-8551, Japan.

Shin-ichi SHIRAI

Department of Mathematics

Osaka Institute of Technology,

Asahi, Osaka, 535-8585, Japan.