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Regularity of local manifolds in dispersing billiards

N. Chernov
Department of Mathematics
University of Alabama at Birmingham
Email: chernov@math.uab.edu

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Abstract

This work is devoted to 2D dispersing billiards with smooth boundary, i.e. periodic Lorentz gases (with and without horizon). We revisit several fundamental properties of these systems and make a number of improvements. The necessity of such improvements became obvious during our recent studies of gases of several particles [CD]. We prove here that local (stable and unstable) manifolds, as well as singularity curves, have uniformly bounded derivatives of all orders. We establish sharp estimates on the size of local manifolds, on distortion bounds, and on the Jacobian of the holonomy map.

1 Introduction

A billiard is a mechanical system in which a point particle moves in a compact container $\mathcal{D}$ and bounces off its boundary $\partial\mathcal{D}$. The dynamical properties of a billiard are determined by the shape of $\partial\mathcal{D}$, and they may vary greatly from completely regular (integrable) to strongly chaotic. The first class of chaotic billiards was introduced by Ya. Sinai in 1970, see [S2], who considered containers defined by

$$\mathcal{D} = \mathrm{Tor}^2 \setminus \cup_{i=1}^{p} \mathbb{B}_i, \tag{1.1}$$

where Tor^2 denotes the unit 2-torus and $\mathbb{B}_i \subset \mathrm{Tor}^2$ disjoint strictly convex domains with C^ℓ ($\ell \geq 3$) smooth boundary whose curvature never vanishes. Sinai proved that the billiard flows and maps in such domains are hyperbolic, ergodic, and K-mixing. He called these systems *dispersing billiards*, now they are known as *Sinai billiards*.

Dispersing billiards have a lot in common with canonical chaotic models, namely Anosov diffeomorphisms and Axiom A attractors: they are hyperbolic with uniform expansion and contraction rates [S2], they possess a physically observable invariant measure (celled Sinai-Ruelle-Bowen measure, or SRB measure), with respect o that measure they are ergodic, mixing [S2], and Bernoulli [GO], they admit Markov partitions [BS2, BSC1], enjoy exponential decay of correlations [Y, C2], and satisfy Central Limit Theorem and other probabilistic limit theorems [BS3, BSC2].

However, on a technical level there is a substantial difference between Sinai billiards and smooth Anosov and Axiom A maps: billiard maps have singularities. Furthermore, the derivatives of the billiard map are unbounded (they blows up) near singularities. These facts cause enormous difficulties in the analysis of billiard dynamics. Images and preimages of singularities make a dense set in phase space. For this reason Markov partitions cannot be finite, they are always countable.

A traditional approach in the studies of chaotic billiards, which goes back to Sinai's school [S2, BS1, SC], is to work *locally* – construct local stable and unstable manifolds, prove 'local ergodicity', etc. In this approach one picks a point in the phase space, at which all the positive and/or negative iterations of the billiard map are smooth, and works it its vicinity trying to 'stay away' from the nearest singularities. For example, stable and unstable manifolds are constructed by successive approximations [S2, C1], which is a variant of the classical Hadamard technique [H] and is local in nature.

A. Katok and J. M. Strelcyn generalized that 'local' approach and developed a theory of *smooth hyperbolic maps with singularities*, presented in their fundamental monograph [KS]. They proved many general facts analogous to those established earlier for Anosov and Axiom A systems. For example, they showed that for a C^k hyperbolic map local stable and unstable manifolds are C^{k-1} smooth. They also proved *absolute continuity*, i.e. showed that the Jacobian of the holonomy map between stable (unstable) was finite. As their work was primarily motivated by billiards, all their general results applied to Sinai billiards as well.

The Katok-Strelcyn theory supplies sufficient tools for the study of ergodic properties of billiards, which were mostly completed in the 1970s. However, recent studies of statistical properties of chaotic billiards required much finer tools and sharper estimates than those furnished by the general Katok-Strelcyn theory. For example, one needs *uniform* bounds on the second derivatives of stable and unstable manifolds, because their curvature must be uniformly bounded in the analysis of statistical properties.

One also needs sharp quantitative estimates on distortion bounds and the Jacobian of the holonomy map. Unlike the curvature, though, the Jacobian of the holonomy map cannot be uniformly bounded (it blows up near singularities). To establish necessary sharp estimates, one has to carefully partition local stable and unstable manifolds into shorter components (called *homogeneous submanifolds*), see [BSC2, Y], which are made arbitrarily short near singularities. Thus a precise analysis of the immediate vicinity of singularities becomes necessary.

Furthermore, in our latest studies [CD] we used billiards to approximate systems of two particles, and a need arose in estimation of the higher order derivatives of stable and unstable manifolds in Sinai billiards. We found that existing technical estimates obtained in 1990s, see [BSC1, BSC2, Y, C2] are inadequate, they need to be improved and sharpened. Thus a revision of the existing arsenal of results became necessary.

In this paper we adopt quite a novel approach to the construction of stable and unstable manifolds, in which the singularities play an instrumental role. Instead of 'staying away' from singularities, we *use* them as 'frames' to carve stable and unstable manifolds. Roughly speaking, we partition the phase space into connected domains bounded by the images/ preimages of singularities taken up to time n. The limit of this partition, as $n \to \infty$, produces all unstable/stable manifolds.

This new approach gives us a much better control the size of local manifolds. We also combine this construction with the traditional 'local' analysis and obtain necessary uniform bounds on all the higher order derivatives of stable and unstable manifolds, as well as establish new, better estimates on distortion bounds and the Jacobian of the holonomy map.

Many results of this work seem rather technical in nature, but they are important for the studies of chaotic billiards and systems of several particles that can be approximated by chaotic billiards, see [CD]. All our results improve and sharpen the existing results and estimates published earlier [BSC1, BSC2, Y, C2].

Our approach is essentially two-dimensional and hardly can be generalized to billiards in spatial domains. In particular, it is known [BCST] that the curvature of singularity manifolds is no longer uniformly bounded even in 3D. But we believe our methods and results can be extended to other classes of planar chaotic billiards (dispersing tables with corners, Bunimovich stadia, chaotic models by Wojtkowski, Markarian, Donnay, etc.).

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2 Preliminaries

Here we recall basic facts about dispersing billiards. All of them are well known, see [S2, BSC1, BSC2, Y, C2].

Let $\mathcal{D} \subset \mathrm{Tor}^2$ be a domain defined by (1.1). Its boundary

$$\partial\mathcal{D} = \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_p \tag{2.1}$$

is a union of p smooth closed curves, where $\Gamma_i = \partial\mathbb{B}_i$.

The billiard particle moves in $\mathcal{D}$ with a unit speed and upon reaching $\partial\mathcal{D}$ it gets reflected according to the law "the angle of incidence is equal to the angle of reflection". The state of the particle is a pair (q, v), where $q \in \mathcal{D}$ is its position and $v \in S^1$ its (unit) velocity vector. At reflection, $q \in \Gamma$, and the velocity changes discontinuously

$$v^+ = v^- - 2\,\langle v, n\rangle\, n \tag{2.2}$$

where v^+ and v^- refer to the postcollisional and precollisional velocities, respectively, and n denotes the unit normal vector to Γ at the point q. We assume n is pointing inside $\mathcal{D}$, so that $\langle v_-, n\rangle \leq 0$ and $\langle v_+, n\rangle \geq 0$. A collision is said to be grazing (or tangential) if $\langle v_-, n\rangle = \langle v_+, n\rangle = 0$, i.e. $v_+ = v_-$.

The phase space is $\Omega = \mathcal{D} \times S^1$, and the billiards dynamics generates a flow $\Phi^t \colon \Omega \to \Omega$. We denote by π_q the natural projection of Ω onto $\mathcal{D}$, i.e. $\pi_q(q, v) = q$. The 3D space Ω can be coordinatized by x, y, ω, where x, y are the rectangular coordinates of the position point $q \in \mathcal{D}$ and ω the angle between v and the positive x axis, so that $v = (\cos\omega, \sin\omega)$. The flow Φ^t is a Hamiltonian (contact) flow, and it preserves Liouville (uniform) measure $dx\, dy\, d\omega$.

Let $X = (x, y, \omega) \in \Omega$. Consider the coordinate system $(d\eta, d\xi, d\omega)$ in the 3D tangent space $\mathcal{T}_X\Omega$ with

$$d\eta = \cos\omega\, dx + \sin\omega\, dy, \qquad d\xi = -\sin\omega\, dx + \cos\omega\, dy. \tag{2.3}$$

Note that $d\eta$ is the component of the vector (dx, dy) along the velocity vector v, and $d\xi$ is its orthogonal component. The coordinates $(d\eta, d\xi, d\omega)$ are convenient to describe the action of the flow Φ^t, see below.

For any tangent vector $dX = (d\eta, d\xi, d\omega)$ we denote by $dX_t = (d\eta_t, d\xi_t, d\omega_t) = D_X\Phi^t(dX)$ its image at time $t \in \mathbb{R}$. Since the flow has constant (unit) speed, we have

$$D_X\Phi^t \colon (d\eta, 0, 0) \mapsto (d\eta_t, 0, 0), \qquad d\eta_t = d\eta.$$

Since Φ^t is a contact flow, we have

$$D_X\Phi^t \colon (0, d\xi, d\omega) \mapsto (0, d\xi_t, d\omega_t), \tag{2.4}$$

i.e. if (dx, dy) is orthogonal to v at time $t = 0$, then the orthogonality will be preserved at all times. Thus the linear map $(d\eta, d\xi, d\omega) \mapsto (d\eta_t, d\xi_t, d\omega_t)$ is given by the 3×3 matrix

$$D_X\Phi^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}$$

whose bottom right 2×2 block contains all the essential information about $D_X\Phi^t$. If there are no collisions between X and $\Phi^t X$, then obviously

$$d\xi_t = d\xi + t\, d\omega, \qquad d\omega_t = d\omega, \tag{2.5}$$

At a moment of collision, the precollisional vector $(d\xi^-, d\omega^-)$ and the postcollisional vector $(d\xi^+, d\omega^+)$ are related by

$$\begin{aligned} d\xi^+ &= -d\xi^- \\ d\omega^+ &= -\mathcal{R}\, d\xi^- - d\omega^-, \qquad \mathcal{R} = \tfrac{2\mathcal{K}}{\langle v^+, n\rangle}, \end{aligned} \tag{2.6}$$

where $\mathcal{K} > 0$ denotes the curvature of the boundary $\partial\mathcal{D}$ at the collision point; since every domain $\mathbb{B}_i$ is strictly convex, the curvature is strictly positive, and it is uniformly bounded:

$$0 < \mathcal{K}_{\min} \leq \mathcal{K} \leq \mathcal{K}_{\max} < \infty.$$

The quantity $\mathcal{R} = 2\mathcal{K}/\langle v^+, n\rangle$ in (2.6) is called the *collision parameter*. It is uniformly bounded below

$$\mathcal{R} \geq \mathcal{R}_{\min} = 2\mathcal{K}_{\min} > 0,$$

but not above, as $\langle v^+, n\rangle$ may be arbitrarily small at nearly grazing collisions.

Let $X = (x, y, \omega) \in \Omega$ and $dX = (0, d\xi, d\omega) \in \mathcal{T}_X\Omega$ a tangent vector (we assume $d\eta = 0$). Its slope

$$\mathcal{B} = d\omega/d\xi \tag{2.7}$$

has the following geometric meaning. Consider an arbitrary smooth curve $\gamma' \subset \Omega$ passing through X and tangent to dX. Its images $\Phi^t(\gamma')$ for all small t make a 2D surface Σ in Ω. Its projection $\pi_q(\Sigma)$ onto $\mathcal{D}$ is a family of rays (directed line segments). Denote by σ the orthogonal cross-section of this family passing through $q = (x, y)$, see Fig. 1. If we equip every point $q' \in \sigma$ with a unit normal vector v' to σ pointing in the direction of motion, we obtain a curve $\gamma \subset \Omega$ that is also tangent to dX at the point X.

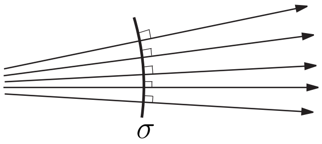

Figure 1: An orthogonal cross-section of a family of directed lines.

A smooth curve $\sigma \subset \mathcal{D}$ equipped with a continuous family of unit normal vectors is called a *wave front* and σ its *support*. The value of $\mathcal{B}$ in (2.7) is the curvature of the support σ at the point $q = (x, y)$. In fact $\mathcal{B} > 0$ if and only if the front is divergent (i.e. the corresponding family of directed line segments 'opens up', as shown on Fig. 1).

Let $\gamma \subset \Omega$ be a wave front with curvature $\mathcal{B}$ at a point $X \in \gamma$; denote by $\mathcal{B}_t$ the curvature of its image $\Phi^t(\gamma)$ at the point $\Phi^t(X)$. If there are no collisions between X and $\Phi^t(X)$, then, according to (2.5)

$$\mathcal{B}_t = \frac{\mathcal{B}}{1+t\mathcal{B}} = \frac{1}{t+\dfrac{1}{\mathcal{B}}}. \tag{2.8}$$

At a moment of collision, the precollisional curvature $\mathcal{B}^-$ and the postcollisional curvature $\mathcal{B}^+$ are related by

$$\mathcal{B}^+ = \mathcal{R} + \mathcal{B}^-, \tag{2.9}$$

as it follows from (2.6). The formula (2.9) is known in geometrical optics as *mirror equation*. Wave fronts play an instrumental role in the analysis of chaotic billiards.

Next, the billiard map (also called the collision map) acts on the 2D collision space

$$\mathcal{M} = \cup_i \mathcal{M}_i, \quad \mathcal{M}_i = \{x = (q,v) \in \Omega\colon\ q \in \Gamma_i,\ \langle v, n\rangle \geq 0\}, \tag{2.10}$$

that consists of the 'postcollisional' velocity vectors attached to $\partial\mathcal{D}$. The billiard map $\mathcal{F}$ takes a point $x = (q,v) \in \mathcal{M}$ to the point $x_1 = (q_1, v_1) \in \mathcal{M}$ of the next collision with $\partial\mathcal{D}$, see Fig. 2. If the boundary $\partial\mathcal{D}$ is C^ℓ smooth, then the map $\mathcal{F}$ is $C^{\ell-1}$ smooth.

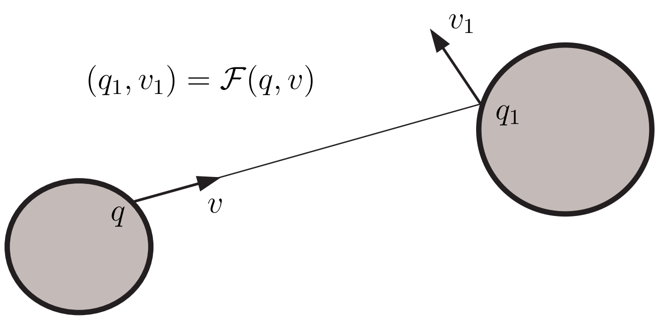

Figure 2: The collision map $\mathcal{F}$.

Standard coordinates on $\mathcal{M}$ are the arc length parameter r and the angle $\varphi \in [-\pi/2, \pi/2]$ between the vectors v and n; the orientation of r and φ is shown on Fig. 3. Note that $\langle v, n \rangle = \cos\varphi$. The space $\mathcal{M}$ is the union of p cylinders on which r is a cyclic ('horizontal') coordinate and φ is a 'vertical' coordinate. The map $\mathcal{F}\colon \mathcal{M} \to \mathcal{M}$ preserves smooth measure

$$d\mu = c_\mu \cos\varphi \; dr\, d\varphi \tag{2.11}$$

where $c_\mu = (2|\Gamma|)^{-1}$ is the normalizing constant; $|\Gamma|$ denotes the length of the boundary $\Gamma = \partial\mathcal{D}$.

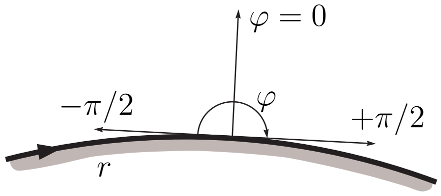

Figure 3: Orientation of r and φ.

For $x \in \mathcal{M}$ denote by $\tau(x)$ the length of the free path between the collision points at x and $\mathcal{F}(x)$. Clearly $\tau(x) \geq \tau_{\min} > 0$, where $\tau_{\min}$ is the minimal distance between the domains $\mathbb{B}_i \subset \mathrm{Tor}^2$. The flow Φ^t can be represented as a suspension flow over the map $\mathcal{F}\colon \mathcal{M} \to \mathcal{M}$ under the ceiling function $\tau(x)$. If $\tau(x)$ is bounded ($\tau(x) \leq \tau_{\max} < \infty$), then the billiard is said to have *finite horizon*. We will consider both types of billiards – with finite horizon and without it (the latter case usually requires a special treatment).

Let $W \subset \mathcal{M}$ be a smooth curve and $x \in W$ an arbitrary point on it. Denote by $\mathcal{V} = d\varphi/dr$ the slope of W at x. The 'outgoing' trajectories $\Phi^t(x')$ of the points $x' \in W$ for all small $t > 0$ make a family of directed line segments in $\mathcal{D}$. Again, let σ^+ be the orthogonal cross-section of that family, passing through x, and $\mathcal{B}^+$ denote its curvature at x. Similarly, the 'incoming' trajectories $\Phi^t(x')$, $x' \in W$, for all small $t < 0$ make another family of directed line segments in $\mathcal{D}$; let σ^- be the orthogonal cross-section of that family, passing through x, and $\mathcal{B}^-$ denote its curvature at x. Then

$$\mathcal{V} = \mathcal{B}^- \cos\varphi + \mathcal{K} = \mathcal{B}^+ \cos\varphi - \mathcal{K} \tag{2.12}$$

where again $\mathcal{K} > 0$ is the curvature of the boundary at the point x.

For every tangent vector $dx = (dr, d\varphi)$ at $x \in \mathcal{M}$ we denote by $\|dx\| = \sqrt{(dr)^2 + (d\varphi)^2}$ its Euclidean norm and by

$$\|(dr, d\varphi)\|_p = \cos\varphi\, |dr|$$

the so-called p-norm (it is technically a pseudo-norm, it corresponds to the pseudo-norm $|d\xi|$ in the tangent space $\mathcal{T}_x\Omega$). Thus we will have two metrics in $\mathcal{M}$: the Euclidean metric and the p-metric (the latter is, technically, a pseudo-metric). Note that the Euclidean norm is related to the p-norm by

$$
\begin{aligned}
\|dx\| &= |dr|\sqrt{1+\mathcal{V}^2} \\
&= \frac{\|dx\|_p}{\cos\varphi}\sqrt{1+(\mathcal{B}^+\cos\varphi-\mathcal{K})^2}.
\end{aligned} \tag{2.13}
$$

3 Stable and unstable curves

Here we use stable and unstable fronts to describe the hyperbolicity of billiards. Mostly we deal with unstable fronts, but due to the time reversibility stable fronts have all similar properties.

In dispersing billiards, a dispersing wave front will always remain dispersing in the future, i.e. if $\mathcal{B} > 0$, then $\mathcal{B}_t > 0$ for all $t > 0$. The curvature $\mathcal{B}_t$ of a dispersing front slowly decreases between collisions due to (2.8), but jumps up at every collision due to (2.9). If the horizon is finite ($\tau(x) \le \tau_{\max}$), then $\mathcal{B}_t$ remains bounded away from zero

$$\mathcal{B}_t \ge \mathcal{B}_{\min}\colon = \frac{1}{\tau_{\max} + 1/\mathcal{R}_{\min}} > 0.$$

Dispersing wave fronts can be used to define an invariant family of unstable cones for the map $\mathcal{F}\colon \mathcal{M} \to \mathcal{M}$: at every point $x \in \mathcal{M}$ the unstable cone

$$\mathcal{C}_x^u = \{(dr, d\varphi) \in \mathcal{T}_x\mathcal{M}\colon\ \mathcal{K} \le d\varphi/dr \le \infty\},$$

is induced by all incoming dispersing (or flat) wave fronts, i.e. such that $\mathcal{B}^- \ge 0$, in the notation of (2.12). The cones $\mathcal{C}_x^u$ are strictly invariant under $D_x\mathcal{F}$. However, we prefer to use more narrow cones

$$\hat{\mathcal{C}}_x^u = \{(dr, d\varphi) \in \mathcal{T}_x\mathcal{M}\colon\ \mathcal{K} \le d\varphi/dr \le \mathcal{K} + \cos\varphi/\tau_{-1}\}. \tag{3.1}$$

(here $\tau_{-1} = \tau(\mathcal{F}^{-1}x)$ denotes the free path since the previous collision), which are also strictly invariant under $D_x\mathcal{F}$. We note that $D_x\mathcal{F}(\mathcal{C}_x^u) \subset \hat{\mathcal{C}}_{\mathcal{F}(x)}^u$.

We say that a C^m smooth curve $W \subset \mathcal{M}$ is *unstable* if at every point $x \in W$ the tangent line $\mathcal{T}_xW$ belongs in the unstable cone $\hat{\mathcal{C}}_x^u$. Similarly, a

C^m smooth curve $W \subset \mathcal{M}$ is said to be *stable* if at every point $x \in W$ the tangent line $\mathcal{T}_x W$ belongs in the stable cone

$$\hat{\mathcal{C}}^s_x = \{(dr, d\varphi) \in \mathcal{T}_x\mathcal{M} \colon \ -\mathcal{K} - \cos\varphi/\tau \le d\varphi/dr \le -\mathcal{K}\}, \tag{3.2}$$

where $\tau = \tau(x)$ (note that the cones (3.1) and (3.2) are symmetric under time reversal). Observe that stable (unstable) curves are monotonically decreasing (resp., increasing) in the coordinates r and φ. Moreover, their slopes are uniformly bounded above and below:

$$0 < \mathcal{V}_{\min} < |d\varphi/dr| < \mathcal{V}_{\max} < \infty \tag{3.3}$$

with $\mathcal{V}_{\min} = \mathcal{K}_{\min}$ and $\mathcal{V}_{\max} = \mathcal{K}_{\max} + 1/\tau_{\min}$. In what follows, we use a shorthand notation $F \asymp G$ meaning

$$F \asymp G \qquad \Longleftrightarrow \qquad C_1 < F/G < C_2 \tag{3.4}$$

for some constants $0 < C_1 < C_2$ depending only on the table $\mathcal{D}$. In this notation, (3.3) can be written as $|d\varphi/dr| \asymp 1$.

We will use the following notation: given a point $x = (r, \varphi) \in \mathcal{M}$, we denote by $x_n = (r_n, \varphi_n) = \mathcal{F}^n(x)$ its images and by $\mathcal{K}_n$, $\mathcal{R}_n$, etc. the corresponding parameters at x_n. Given a wave front moving along the trajectory of x, its image at the nth collision induces a curve $W_n \subset \mathcal{M}$ containing x_n. Let $dx_0 = (dr_0, \varphi_0)$ denote a tangent vector to W_0 at x_0, then its image $dx_n = (dr_n, d\varphi_n) = D_x\mathcal{F}^n(dx)$ will be tangent to W_n at x_n,and we denote by $\mathcal{V}_n$, $\mathcal{B}^{\pm}_n$, etc. the corresponding characteristics of that vector. We also denote by t_n the time of the nth collision and by $\tau_n = \tau(x_n) = t_{n+1} - t_n$ the intervals between collisions.

The next formulas are standard (and easily follow from our previous analysis). First, the precollisional curvature is uniformly bounded above: $\mathcal{B}^-_n \le \mathcal{B}^-_{\max} \colon = 1/\tau_{\min}$. On the other hand, the postcollisional curvature is bounded below:

$$\mathcal{B}^+_n \ge \mathcal{R}_{\min}/\cos\varphi_n \ge \mathcal{R}_{\min} > 0.$$

Using the notation of (3.4) we note that

$$\mathcal{B}^+_n \asymp 1/\cos\varphi_n. \tag{3.5}$$

The Jacobian of $D_{x_n}\mathcal{F}$ along the vector dx_n in the p-norm is

$$\frac{\|dx_{n+1}\|_p}{\|dx_n\|_p} = 1 + \tau_n\mathcal{B}^+_n \ge 1 + \tau_n\mathcal{R}_{\min}. \tag{3.6}$$

Furthermore, at nearly grazing collisions a better estimate holds:

$$\frac{\|dx_{n+1}\|_p}{\|dx_n\|_p} \asymp \frac{\tau_n}{\cos\varphi_n}. \tag{3.7}$$

We denote the minimal expansion factor (in the p-norm) by

$$\Lambda = 1 + \tau_{\min}\mathcal{R}_{\min} > 1 \tag{3.8}$$

hence a uniform hyperbolicity (in the p-norm) holds:

$$\frac{\|dx_n\|_p}{\|dx_0\|_p} \geq \Lambda^n \qquad \forall n \geq 1.$$

In the Euclidean norm $\|dx\| = \sqrt{(dr)^2 + (d\varphi)^2}$, a uniform hyperbolicity also holds, but this requires a bit more work. First, (2.13) gives us

$$\frac{\|dx_n\|}{\|dx_0\|} = \frac{\|dx_n\|_p}{\|dx_0\|_p}\,\frac{\cos\varphi_0}{\cos\varphi_n}\,\frac{\sqrt{1+\mathcal{V}_n^2}}{\sqrt{1+\mathcal{V}_0^2}}. \tag{3.9}$$

The last fraction is uniformly bounded away from zero and infinity due to (3.3). The middle fraction can be arbitrarily small, but it can be handled as we have

$$\frac{\|dx_1\|_p}{\|dx_0\|_p} \geq \frac{\text{const}}{\cos\varphi_0},$$

according to (3.7). Therefore,

$$\frac{\|dx_n\|}{\|dx_0\|} \geq \text{const} \times \frac{\|dx_n\|_p}{\|dx_1\|_p} \geq \hat{c}\,\Lambda^n, \tag{3.10}$$

where $\hat{c} = \hat{c}(\mathcal{D}) > 0$ is a constant, which means uniform hyperbolicity. Also, (3.7) and (2.13) imply another useful relation:

$$\frac{\|dx_{n+1}\|}{\|dx_n\|} \asymp \frac{\tau_n}{\cos\varphi_{n+1}}. \tag{3.11}$$

Next, suppose a curve $W \subset \mathcal{M}$ is defined by a function $\varphi = \varphi_W(r)$ for some $r'_W \leq r \leq r''_W$. We say that W is C^m smooth ($m \geq 1$) if the function $\varphi_W(r)$ is C^m smooth, up to the endpoints r'_W and r''_W. If a stable/unstable curve W is C^m smooth, $m \geq 1$, then its image $\mathcal{F}(W)$ is (locally) $C^{m'}$ smooth, where $m' = \min\{m, \ell - 1\}$, because the map $\mathcal{F}$ is $C^{\ell-1}$ smooth. For this reason, we will only consider $C^{\ell-1}$ curves.

The billiard map $\mathcal{F}$ has discontinuities which are analyzed in the next section. Since $\mathcal{F}$ is discontinuous, the image of an unstable curve W may consist of not just one, but finitely or countably many unstable curves.

Here is our first result:

Theorem 3.1. *Let $\mathcal{D}$ be a dispersing billiard (1.1). Then for each $\nu = 1, \dots, \ell - 1$ there exists a constant $C_\nu = C_\nu(\mathcal{D}) > 0$ such that for every $C^{\ell-1}$ smooth unstable curve $W \subset \mathcal{M}$ there is an $n_W \geq 1$ such that for all $n > n_W$ every smooth curve $W' \subset \mathcal{F}^n(W)$ has its νth derivative bounded by C_ν:*

$$|\varphi_{W'}^{(\nu)}(r)| \leq C_\nu. \tag{3.12}$$

We note that the case $\nu = 1$ is trivial due to (3.3), the case $\nu = 2$ was first fully treated in [C3], and the case $\nu = 3$ was first handled, by different techniques, in our recent manuscript [CD] (which also demonstrates the necessity of bounding higher order derivatives). Our present work is first to cover the general case.

Proof. The first derivative $d\varphi/dr$ is bounded by (3.3). Differentiating (2.12) gives

$$d^2\varphi/dr^2 = d\mathcal{K}/dr - \mathcal{B}^- \sin\varphi \; d\varphi/dr + \cos\varphi \; d\mathcal{B}^-/dr,$$

hence the second derivative would be bounded if we had a uniform bound on $d\mathcal{B}^-/dr$. Further differentiation allows us to reduce Theorem 3.1 to the following

Proposition 3.2. *For each $\nu = 1, \dots, \ell - 2$ there exists a constant $C'_\nu = C'_\nu(\mathcal{D}) > 0$ such that for every $C^{\ell-1}$ smooth unstable curve $W \subset \mathcal{M}$ there is $n_W \geq 1$ such that for all $n > n_W$ every smooth curve $W' \subset \mathcal{F}^n(W)$ satisfies*

$$|d^\nu \mathcal{B}^-_{W'}/dr^\nu| \leq C'_\nu.$$

where $\mathcal{B}^-_{W'}(r) = \big(d\varphi_{W'}(r)/dr - \mathcal{K}(r)\big)/\cos\varphi_{W'}(r)$, according to (2.12).

Now the value $\mathcal{B}^-_{W'}$ is the curvature of the dispersing wave front corresponding to the unstable curve W' immediately before the collision. Since $\mathcal{B} = d\omega/d\xi$ by (2.7), it will be more convenient to differentiate $\mathcal{B}$ with respect to ξ rather than r. This change of variables requires certain analysis, which we do next.

Let γ be a dispersing wave front, in the notation of Section 2, and s a smooth parameter on the curve $\pi_q(\gamma) \subset \mathcal{D}$. For every point $X_s = (x_s, y_s, \omega_s) \in \gamma$ denote by $X_{st} = (x_{st}, y_{st}, \omega_{st}) = \Phi^t(X_s)$ its image at time $t > 0$ (here we use the coordinates (x, y, ω) introduced in Section 2). Now the points $\{X_{st}\}$ fill a 2D surface $\Sigma \subset \Omega$ parameterized by s and t, see Fig. 4.

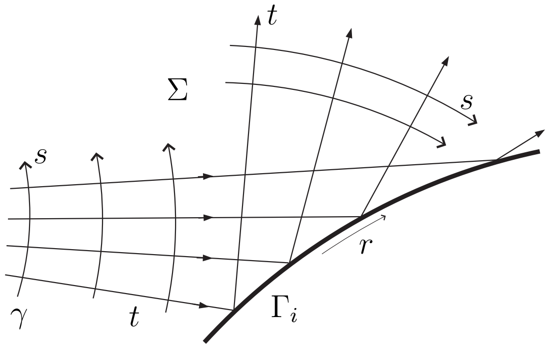

Figure 4: Parametrization of Σ by s and t.

In our calculations below, we denote differentiation with respect to s by primes and that with respect to t by dots. For example,

$$(\dot{x}, \dot{y}, \dot{\omega}) = (\cos\omega, \sin\omega, 0).$$

The vector (x', y') remains orthogonal to the velocity vector $(\cos\omega, \sin\omega)$ at all times, according to (2.4), hence $x' \cos\omega + y' \sin\omega = 0$. Put

$$u = -x' \sin\omega + y' \cos\omega.$$

Note that $u^2 = (x')^2 + (y')^2$, hence $|u|$ equals the distance on the table $\mathcal{D}$ corresponding to the unit increment of the parameter s.

It is easy to check that for any smooth function $F \colon \Sigma \to \mathbb{R}$

$$\frac{dF}{d\xi} = \frac{1}{u} \frac{dF}{ds}. \tag{3.13}$$

In particular,

$$\mathcal{B} = \frac{d\omega}{d\xi} = \frac{\omega'}{u}. \tag{3.14}$$

One also easily checks that

$$\dot{u} = u\mathcal{B}. \tag{3.15}$$

Combining (3.13) with (3.15) gives

$$\frac{d}{dt}\frac{d}{d\xi}F = -\mathcal{B}\frac{d}{d\xi}F + \frac{d}{d\xi}\frac{d}{dt}F. \tag{3.16}$$

In particular, letting $F = \omega$ and using (3.14) gives

$$\frac{d\mathcal{B}}{dt} = -\mathcal{B}^2, \qquad \text{hence} \qquad \frac{d}{dt}\left(\frac{1}{\mathcal{B}}\right) = 1, \tag{3.17}$$

which agrees with (2.8). Denote

$$\mathcal{E}_\nu = \frac{d^\nu \mathcal{B}}{d\xi^\nu}, \qquad \nu = 1, \dots, \ell - 2.$$

Then (3.16) implies

$$\dot{\mathcal{E}}_\nu = -\mathcal{E}_\nu \mathcal{B} + \frac{d}{d\xi}\dot{\mathcal{E}}_{\nu-1}, \qquad \nu = 1, \dots, \ell - 2$$

(assuming $\mathcal{E}_0 = \mathcal{B}$, of course). In particular, one easily checks that

$$\dot{\mathcal{E}}_1 = -3\mathcal{E}_1\mathcal{B}, \qquad \dot{\mathcal{E}}_2 = -4\mathcal{E}_2\mathcal{B} - 3\mathcal{E}_1^2, \qquad \text{etc.}$$

Observe that $\dot{\mathcal{E}}_\nu$ is a homogeneous quadratic polynomial of $\mathcal{B}, \mathcal{E}_1, \dots, \mathcal{E}_\nu$ for every $\nu \geq 1$.

Our previous analysis dealt with the free motion between collisions. Next we consider what happens at a collision with a wall Γ_i, see Fig. 4. For every collision point $\big(\bar{x}(r), \bar{y}(r)\big) \in \Gamma_i$ there is a unique pair (s, t) such that

$$x_{st} = \bar{x}(r) \qquad \text{and} \qquad y_{st} = \bar{y}(r), \tag{3.18}$$

hence s and t, constrained to the collision, become functions of r. Differentiating (3.18) and using some elementary trigonometry yield

$$\frac{dt}{dr} = \sin\varphi \qquad \text{and} \qquad \frac{ds}{dr} = \frac{\cos\varphi}{u^-} = -\frac{\cos\varphi}{u^+} \tag{3.19}$$

(note that $u^+ = -u^-$ because the (s,t) coordinate system changes orientation after the collision, see Fig. 4).

Now for any smooth function $F\colon \Sigma \to \mathbb{R}$ we denote by F^- and F^+ its restrictions to the parts of Σ before and after the collision, respectively. Then, due to (3.19),

$$\begin{aligned} \frac{dF^-(s(r),t(r))}{dr} &= \frac{(F')^- \cos\varphi}{u^-} + (\dot F)^- \sin\varphi \\ &= \left(\frac{dF}{d\xi}\right)^- \cos\varphi + (\dot F)^- \sin\varphi \end{aligned} \tag{3.20}$$

and, similarly,

$$\frac{dF^+(s(r),t(r))}{dr} = -\left(\frac{dF}{d\xi}\right)^+ \cos\varphi + (\dot F)^+ \sin\varphi. \tag{3.21}$$

We now return to the proof of Proposition 3.2. Using (3.20) and (3.17) yields

$$d\mathcal{B}^-/dr = \mathcal{E}_1^- \cos\varphi - (\mathcal{B}^-)^2 \sin\varphi, \tag{3.22}$$

so $d\mathcal{B}^-/dr$ would be uniformly bounded if we had a uniform bound on $\mathcal{E}_1^- \cos\varphi$. We will prove even more – a uniform bound on $\mathcal{E}_1^-$, which will be useful for the second derivative, see next. Differentiating further gives

$$\begin{aligned} d^2\mathcal{B}^-/dr^2 = \mathcal{E}_2^- \cos^2\varphi - 5\,\mathcal{E}_1^- \mathcal{B}^- \cos\varphi \sin\varphi - \mathcal{E}_1^- \mathcal{V} \sin\varphi \\ - (\mathcal{B}^-)^2 \mathcal{V} \cos\varphi + 2(\mathcal{B}^-)^3 \sin^2\varphi, \end{aligned}$$

where $\mathcal{V} = d\varphi/dr = \mathcal{B}^- \cos\varphi + \mathcal{K}$, see (2.12). Hence $d^2\mathcal{B}^-/dr^2$ would be uniformly bounded if we had a uniform bound on $\mathcal{E}_2^-$. Subsequent differentiation (it is straightforward, so we leave it out) reduces Proposition 3.2 to the following

Proposition 3.3. *For each $\nu = 1, \dots, \ell - 2$ there exists a constant $C''_\nu = C''_\nu(\mathcal{D}) > 0$ such that for every $C^{\ell-1}$ smooth dispersing wave front $\gamma \subset \Omega$ there is $n_\gamma \geq 1$ such that for all $n \geq n_\gamma$ the νth derivative $\mathcal{E}_\nu = d^\nu\mathcal{B}/d\xi^\nu$ of the curvature $\mathcal{B}$ of its image $\Phi^t(\gamma)$ before the nth collision satisfies*

$$|\mathcal{E}_\nu^-| \leq C''_\nu.$$

In a sense, Proposition 3.3 is a "flow" version of the "map" statement 3.1.

Proof. First, we do it for $\nu = 1$, and then use induction on ν. Suppose that F is a smooth function on the surface Σ introduced above such that $dF/dt = 0$, i.e. the function F is independent of t between collisions. Then (3.17) and (3.16) imply

$$\frac{d}{dt}\left(\frac{1}{\mathcal{B}}\frac{d}{d\xi}F\right) = \frac{\mathcal{B}^2}{\mathcal{B}^2}\frac{d}{d\xi}F - \frac{\mathcal{B}}{\mathcal{B}}\frac{d}{d\xi}F = 0, \tag{3.23}$$

hence the function $G = \mathcal{B}^{-1}\,dF/d\xi$ is also independent of t between collisions.

Due to (3.17), the function $F_0 = t - 1/\mathcal{B}$ satisfies $\dot{F}_0 = 0$. Therefore, the function

$$F_1 = \frac{1}{\mathcal{B}}\frac{dF_0}{d\xi} = \frac{\mathcal{E}_1}{\mathcal{B}^3} \tag{3.24}$$

is independent of t between collisions, i.e. $\dot{F}_1 = 0$. To compute its increment at collisions, we recall that

$$\mathcal{B}^+ = \mathcal{B}^- + \mathcal{R},$$

where $\mathcal{R} = 2\mathcal{K}(r)/\cos\varphi$. Differentiating this equation with respect to r and using (3.20)–(3.21) gives

$$-\mathcal{E}_1^+ = \mathcal{E}_1^- + \mathcal{R}_1$$

where

$$\begin{aligned}\mathcal{R}_1 &= (\mathcal{B}^+)^2\tan\varphi - (\mathcal{B}^-)^2\tan\varphi + \frac{1}{\cos\varphi}\frac{d\mathcal{R}}{dr}\\ &= \frac{6\mathcal{K}^2\sin\varphi + 6\mathcal{K}\mathcal{B}^-\sin\varphi\cos\varphi + 2\mathcal{K}'\cos\varphi}{\cos^3\varphi},\end{aligned}$$

where $\mathcal{K}' = d\mathcal{K}/dr$. Therefore,

$$-F_1^+ = \left(\frac{\mathcal{B}^-}{\mathcal{B}^+}\right)^3 F_1^- + H_1, \tag{3.25}$$

where $H_1 = \mathcal{R}_1/(\mathcal{B}^+)^3$, i.e.

$$H_1 = \frac{6\mathcal{K}^2\sin\varphi + 6\mathcal{K}\mathcal{B}^-\sin\varphi\cos\varphi + 2\mathcal{K}'\cos\varphi}{(2\mathcal{K} + \mathcal{B}^-\cos\varphi)^3}.$$

Observe that H_1 is represented by a fraction whose all terms are uniformly bounded and the denominator is larger than a positive constant $8\mathcal{K}_{\min}^3$. Thus $|H_1| \le \tilde{C}_1 = \tilde{C}_1(\mathcal{D})$, an absolute constant. Also note that

$$\frac{\mathcal{B}^-}{\mathcal{B}^+} = \frac{\mathcal{B}^-}{\mathcal{B}^- + \mathcal{R}} \le \theta, \qquad \theta \colon = \frac{\mathcal{B}^-_{\max}}{\mathcal{B}^-_{\max} + \mathcal{R}_{\min}} < 1.$$

Since F_1 is constant between collisions, let us denote by $F_1(n)$ its value between the nth and $(n+1)$st collision. Then (3.25) implies

$$|F_1(n)| \le \theta^3 |F_1(n-1)| + \tilde{C}_1$$

and by a simple induction on n we obtain

$$\begin{aligned} |F_1(n)| &\le \theta^{3n}|F_1(0)| + \tilde{C}_1\bigl(1 + \theta^3 + \ldots + \theta^{3(n-1)}\bigr) \\ &\le \theta^{3n}|F_1(0)| + \tilde{C}_1/(1-\theta^3). \end{aligned} \tag{3.26}$$

Therefore, for all

$$n > n_{\gamma,1} \colon = \ln(|\tilde{C}_1|/|F_1(0)|)/\ln\theta^3$$

we have $|F_1(n)| < \tilde{C}_1' \colon = 2\tilde{C}_1/(1-\theta^3)$. Lastly, (3.24) implies $|\mathcal{E}_1| < \tilde{C}_1'\mathcal{B}^3$, hence

$$|\mathcal{E}_1^-| < \tilde{C}_1'(\mathcal{B}^-)^3 \le \tilde{C}_1'(\mathcal{B}^-_{\max})^3 =\colon C_1''$$

for all collisions past the $n_{\gamma,1}$th collision. This proves Proposition 3.3 for $\nu = 1$.

We now turn to the the inductive step. First we make inductive assumptions. Suppose for some $1 \le \nu \le \ell - 3$ we have a function F_ν on Σ with the following properties (the reader is advised to check that they have been proved for $\nu = 1$):

(a) it is given by

$$F_\nu = \frac{Q_\nu(\mathcal{E}_\nu, \mathcal{E}_{\nu-1}, \ldots, \mathcal{E}_1, \mathcal{B})}{\mathcal{B}^{2\nu+1}}$$

where Q_ν is a homogeneous polynomial of degree ν of the variables $\mathcal{E}_\nu, \ldots, \mathcal{E}_1$ and $\mathcal{B}$, in which $\mathcal{E}_\nu$ appears in one term, namely in $\mathcal{E}_\nu\mathcal{B}^{\nu-1}$; moreover, if we replace $\mathcal{E}_i$ by x^i and $\mathcal{B}$ by $x^0 = 1$, then every term of $Q_\nu(x)$ will be proportional to x^ν;

(b) $|F_\nu| < \tilde{C}'_\nu$ for some constant $\tilde{C}'_\nu(\mathcal{D}) > 0$ for all times past the $n_{\gamma,\nu}$-th collision;

(c) $\dot{F}_\nu = 0$, i.e. F_ν is independent of t between collisions;

(d) the increments of F_ν at collisions satisfy the equation

$$(-1)^\nu F_\nu^+ = \left(\frac{\mathcal{B}^-}{\mathcal{B}^+}\right)^{\nu+2} F_\nu^- + H_\nu \tag{3.27}$$

where

$$H_\nu = \frac{G_\nu}{(2\mathcal{K} + \mathcal{B}^- \cos\varphi)^{2\nu+1}}$$

and G_ν is an algebraic expression (a polynomial) whose terms ("variables") are $\cos\varphi$, $\sin\varphi$, $\mathcal{B}^-$, $\mathcal{E}_1^-, \ldots, \mathcal{E}_{\nu-1}^-$ and $\mathcal{K}$, $d\mathcal{K}/dr, \ldots, d^\nu\mathcal{K}/dr^\nu$.

We also assume that it has been proven that $|\mathcal{E}_i| \le \tilde{C}'''_\nu \mathcal{B}^{i+2}$ for some constants $\tilde{C}'''_\nu(\mathcal{D}) > 0$ and all $1 \le i \le \nu$ at all times past the $n_{\gamma,\nu}$th collision.

We now proceed to the inductive step per se. Due to (3.23), the function

$$F_{\nu+1} = \frac{1}{\mathcal{B}} \frac{dF_\nu}{d\xi}$$

is independent of t between collisions, i.e. $\dot{F}_{\nu+1} = 0$. Property (a) of F_ν implies

$$F_{\nu+1} = \frac{\mathcal{B}\, dQ_\nu/d\xi - (2\nu+1)Q_\nu\mathcal{E}_1}{\mathcal{B}^{2\nu+3}} = \frac{Q_{\nu+1}(\mathcal{E}_{\nu+1}, \ldots, \mathcal{E}_1, \mathcal{B})}{\mathcal{B}^{2\nu+3}} \tag{3.28}$$

where $Q_{\nu+1}$ is a homogeneous polynomial of degree $\nu+1$ in which $\mathcal{E}_{\nu+1} = d\mathcal{E}_\nu/d\xi$ appears in one term, namely in $\mathcal{E}_{\nu+1}\mathcal{B}^\nu$. For example, $Q_2 = \mathcal{E}_2\mathcal{B} - 3\mathcal{E}_1^2$ and $Q_3 = \mathcal{E}_3\mathcal{B}^2 - 10\mathcal{E}_2\mathcal{E}_1\mathcal{B} + 15\mathcal{E}_1^3$.

Remark 3.4. *One easily shows, by induction on ν, that if we replace $\mathcal{E}_i$ by x^i and $\mathcal{B}$ by $x^0 = 1$, then every term of $Q_\nu(x)$ will be proportional to x^ν. Moreover, if we replace $\mathcal{E}_i$ by x^{i+2} and $\mathcal{B}$ by x, then every term of $Q_\nu(x)$ will have degree $\ge 2\nu+1$.*

To estimate the change of $F_{\nu+1}$ at collisions, we differentiate (3.27) with respect to r and use (3.20)–(3.21), which gives

$$(-1)^{\nu+1} F_{\nu+1}^+ = \left(\frac{\mathcal{B}^-}{\mathcal{B}^+}\right)^{\nu+3} F_{\nu+1}^- + H_{\nu+1} \tag{3.29}$$

where

$$H_{\nu+1} = \frac{1}{\mathcal{B}^+ \cos\varphi} \frac{dH_\nu}{dr} + (\nu+2)F_\nu^- (\mathcal{B}^-)^{\nu+1} \frac{\mathcal{R}\, d\mathcal{B}^-/dr - \mathcal{B}^-\, d\mathcal{R}/dr}{(\mathcal{B}^+)^{\nu+4} \cos\varphi}.$$

One can verify directly that

$$H_{\nu+1} = \frac{G_{\nu+1}}{(2\mathcal{K} + \mathcal{B}^- \cos\varphi)^{2\nu+3}}$$

where $G_{\nu+1}$ is an algebraic expression (a polynomial) whose terms ("variables") are $\cos\varphi$, $\sin\varphi$, $\mathcal{B}^-$, $\mathcal{E}_1^-, \dots, \mathcal{E}_\nu^-$ and $\mathcal{K}$, $d\mathcal{K}/dr, \dots, d^{\nu+1}\mathcal{K}/dr^{\nu+1}$ (here we use Remark 3.4). Therefore $|H_{\nu+1}| \le \tilde{C}_{\nu+1}$, where $\tilde{C}_{\nu+1}(\mathcal{D}) > 0$ is a constant.

We note that at the last step, when $\nu = \ell - 3$, the function $H_{\nu+1} = H_{\ell-2}$ will involve $d^{\ell-2}\mathcal{K}/dr^{\ell-2}$, which is a continuous but not necessarily differentiable function. Hence $H_{\ell-2}$ may not be differentiable anymore, but it is still uniformly bounded.

The rest of the proof is similar to the case $\nu = 1$. Let $F_{\nu+1}(n)$ denote the value of $F_{\nu+1}$ between the nth and $(n+1)$st collision. Then by (3.29) we have

$$|F_{\nu+1}(n)| \le \theta^{\nu+3} |F_{\nu+1}(n-1)| + \tilde{C}_{\nu+1}$$

and by induction on n

$$|F_{\nu+1}(n)| \le \theta^{(\nu+3)n} |F_{\nu+1}(n_{\gamma,\nu})| + \tilde{C}_{\nu+1}/(1-\theta^{\nu+3}),$$

hence for all sufficiently large n we have $|F_{\nu+1}(n)| < \tilde{C}'_{\nu+1} :\ = 2\tilde{C}_{\nu+1}/(1 - \theta^{\nu+3})$.

Lastly, we estimate $\mathcal{E}_{\nu+1}$. Due to (3.28) and the subsequent comments

$$|\mathcal{E}_{\nu+1}| \le \tilde{C}'_{\nu+1} |\mathcal{B}|^{\nu+3} + \hat{Q}_{\nu+1}/\mathcal{B}^\nu$$

where $\hat{Q}_{\nu+1}$ is a homogeneous polynomial of degree $\nu+1$ of variables $\mathcal{E}_\nu, \dots, \mathcal{E}_1$ and $\mathcal{B}$. Since $|\mathcal{E}_i| \le \tilde{C}''_i \mathcal{B}^{i+2}$ for all $1 \le i \le \nu$, Remark 3.4 implies that $|\hat{Q}_{\nu+1}/\mathcal{B}^\nu| \le$ const$\cdot\mathcal{B}^{\nu+3}$, thus $|\mathcal{E}_{\nu+1}| \le \tilde{C}''_{\nu+1}\mathcal{B}^{\nu+3}$ for some constant $\tilde{C}''_{\nu+1}(\mathcal{D}) > 0$, and

$$|\mathcal{E}_{\nu+1}^-| \le \tilde{C}''_{\nu+1}(\mathcal{B}^-)^{\nu+3} \le \tilde{C}''_{\nu+1}(\mathcal{B}^-_{\max})^{\nu+3} =: C''_{\nu+1}.$$

This proves Proposition 3.3 by induction on ν. Theorem 3.1 is proved. $\square$

4 Singularities

We denote by $\mathcal{S}_0 = \partial\mathcal{M} = \{\cos\varphi = 0\}$ the boundary of the collision space (it consists of all grazing collisions). Then the map $\mathcal{F}$ lacks smoothness on the set $\mathcal{S}_1 = \mathcal{S}_0 \cup \mathcal{F}^{-1}(\mathcal{S}_0)$ (we call it the *singularity set* for $\mathcal{F}$). In fact, $\mathcal{F}$ is discontinuous on $\mathcal{S}_1 \setminus \mathcal{S}_0$. More generally, the singularity sets for the maps $\mathcal{F}^n$ and $\mathcal{F}^{-n}$ are

$$\mathcal{S}_n = \cup_{i=0}^{n} \mathcal{F}^{-i}(\mathcal{S}_0) \qquad \text{and} \qquad \mathcal{S}_{-n} = \cup_{i=0}^{n} \mathcal{F}^{i}(\mathcal{S}_0).$$

We note that the time reversibility of the billiard dynamics implies that if $x = (r, \varphi) \in \mathcal{S}_n$, then $(r, -\varphi) \in \mathcal{S}_{-n}$.

For each $n \geq 1$, the set $\mathcal{S}_{-n} \setminus \mathcal{S}_0$ is a finite or countable union of compact smooth unstable curves (in fact, it is finite for billiards with finite horizon and countable otherwise). Similarly, the set $\mathcal{S}_n \setminus \mathcal{S}_0$ is a finite or countable union of compact smooth stable curves. Here is our second result:

Theorem 4.1. *Every curve $S \subset \mathcal{S}_{-n} \backslash \mathcal{S}_0$, $n \geq 1$, is $C^{\ell-1}$ smooth with bounded derivatives (up to its endpoints). Moreover, for each $\nu = 1, \ldots, \ell - 1$, the ν-th derivative is uniformly bounded by a constant $C_\nu(\mathcal{D}) > 0$ independent of n and the curve $S \subset \mathcal{S}_{-n} \setminus \mathcal{S}_0$.*

We note that the smoothness of singularity curves follows from the general theory [KS], but uniform bounds on derivatives constitute a new result. By the way, such uniform bounds appear to be specific to 2D dispersing billiards, as in general billiards, and even in dispersing billiards in 3D, singularity manifolds may have unbounded curvature [BCST].

Proof. It is enough to prove this for $n = 1$ and then use the results of Section 3. Every curve $S \subset \mathcal{S}_{-1} \setminus \mathcal{S}_0$ is formed by a dispersing wave front consisting of trajectories coming from grazing collisions. This front actually focuses (has infinite curvature) at its origin (i.e., at the grazing collisions). Therefore, when the above wave front approaches the collision at a point $x = (r, \varphi) \in S \subset \mathcal{S}_{-1} \setminus \mathcal{S}_0$, it has curvature $\mathcal{B} = 1/\tau_{-1}$, so the slope of the curve S is, due to (2.12)

$$\mathcal{V} = d\varphi/dr = \mathcal{K} + \cos\varphi/\tau_{-1}, \tag{4.1}$$

where, as before, $\mathcal{K} = \mathcal{K}(x)$ and $\tau_{-1} = \tau(\mathcal{F}^{-1}x)$.

In (4.1), the function $\mathcal{K}(x)$ depends on r only and is $C^{\ell-2}$ smooth with uniformly bounded derivatives, hence it is enough to prove that $\tau_{-1} = \tau_{-1}(r)$ is

$C^{\ell-2}$ smooth with uniformly bounded derivatives; here $\tau_{-1}(r) = \tau_{-1}(r, \varphi_S(r))$ is the function τ_{-1} restricted to the curve S (we use the fact that $\tau \geq \tau_{\min} > 0$). In fact, we prove a little more:

Lemma 4.2. *The function $\tau_{-1}(r)$ along every curve $S \subset \mathcal{S}_{-1} \setminus \mathcal{S}_0$ is $C^{\ell-1}$ smooth, with uniformly bounded derivatives, up to the endpoints of S.*

Proof. For every point $(r, \varphi) \in S$ we denote by $(\bar{r}, \bar{\varphi}) = \mathcal{F}^{-1}(r, \varphi)$ its preimage, where, of course, $\bar{\varphi} = \pm\pi/2$. This makes $\bar{r}$ a function of r. Let (x, y) denote the Cartesian coordinates of the point $\pi_q(r, \varphi)$ and $(\bar{x}, \bar{y})$ those of the point $\pi_q(\bar{r}, \bar{\varphi})$ Observe that

$$\tau_{-1}^2 = (x - \bar{x})^2 + (y - \bar{y})^2 \tag{4.2}$$

and note that x and y are C^ℓ functions of r, and likewise, $\bar{x}$ and $\bar{y}$ are C^ℓ functions of $\bar{r}$. Thus it is enough to prove that $\bar{r}$ is a $C^{\ell-1}$ function of r. Let $(\dot{\bar{x}}, \dot{\bar{y}}) = (d\bar{x}/d\bar{r}, d\bar{y}/d\bar{r})$ denote the unit tangent vector to $\partial\mathcal{D}$ at the point $(\bar{x}, \bar{y})$ (our use of dots here differs from that in the previous section). We remind the reader that r (and $\bar{r}$) is an arc length, hence that vector is unit. Observe that the vector $(x - \bar{x}, y - \bar{y})$ is parallel to $(\dot{\bar{x}}, \dot{\bar{y}})$, hence

$$(x - \bar{x})\dot{\bar{y}} = (y - \bar{y})\dot{\bar{x}}.$$

Direct differentiation gives

$$\frac{d\bar{r}}{dr} = \frac{\dot{\bar{x}}\dot{y} - \dot{x}\dot{\bar{y}}}{(x - \bar{x})\,\ddot{\bar{y}} - (y - \bar{y})\,\ddot{\bar{x}}}, \tag{4.3}$$

where $\dot{x} = dx/dr$ and $\dot{y} = dy/dr$) Observe that $\bar{n} = (\ddot{\bar{x}}, \ddot{\bar{y}})$ is a normal vector at the point $(\bar{x}, \bar{y})$, and $\|\bar{n}\| = \bar{\mathcal{K}}$ is the curvature of $\partial\mathcal{D}$ at that same point. Therefore the absolute value of the denominator in (4.3) equals $\bar{\mathcal{K}}\tau_{-1} \geq \mathcal{K}_{\min}\tau_{\min} > 0$, so it is bounded away from zero. Now Lemma 4.2 follows from Implicit Function Theorem. We note that our argument works even without a finite horizon assumption. $\square$

This completes the proof of Lemma 4.2 and Theorem 4.1. $\square$

Below we list some other properties of the singularity set $\mathcal{S}_{-n}$, which are proven in earlier works [S2, BSC1, BSC2, Y, C2].

Each smooth curve $S \subset \mathcal{S}_{-n} \setminus \mathcal{S}_0$ terminates on $\mathcal{S}_0 = \partial\mathcal{M}$ or on another smooth curve $\tilde{S} \subset \mathcal{S}_{-n} \setminus \mathcal{S}_0$. Furthermore, every curve $S \subset \mathcal{S}_{-n} \setminus \mathcal{S}_0$ is a part of some monotonically increasing continuous (and piece-wise smooth) curve $\tilde{S} \subset \mathcal{S}_{-n} \setminus \mathcal{S}_0$ which stretches all the way from $\varphi = -\pi/2$ to $\varphi = \pi/2$ (i.e., it terminates on $S_0 = \partial\mathcal{M}$). This property is often referred to as *continuation of singularity lines*.

Next, for each $n', n'' \geq 0$ the set $\mathcal{M} \setminus (\mathcal{S}_{-n'} \cup \mathcal{S}_{n''})$ is a finite or countable union of open domains with piecewise smooth boundaries (curvilinear polygons), such that the interior angles made by their boundary components do not exceed π (i.e. those polygons are 'convex', as far as the interior angles are concerned). Some interior angles may be equal to zero, see below.

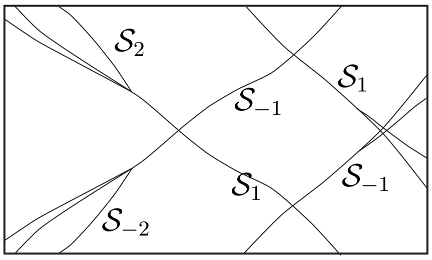

Figure 5: Singularity curves.

A typical structure of singularity curves is shown on Fig. 5.

In billiards without horizon the singularity set $\mathcal{S}_{-1}$ consists of an infinite number of smooth curves, which we describe briefly referring to [BSC1, BSC2] for more detail. The smooth components of $\mathcal{S}_{-1}$ accumulate at points $x \in \mathcal{M}$ near which the function $\tau_{-1}(x)$ is unbounded. There are only finitely many such points in $\mathcal{M}$, we denote them by $x_k = (r_k, \varphi_k) \in \mathcal{M}$, $1 \leq k \leq k_{\max}$; these are points whose trajectories only experience grazing collisions, so that they are periodic (closed) geodesic lines on the torus, see an example on Fig. 6. Of course, $\varphi_k = \pm\pi/2$ at every such point, i.e. they all lie on $\mathcal{S}_0$.

In the vicinity of every point $x_x \in \mathcal{M}$, the singularity curves $\mathcal{S}_{-1}$ make a structure shown on Fig. 7 (right). There $\mathcal{S}_{-1}$ consists of (i) a long curve $S^-_{k,0}$ running from x_k down into $\mathcal{M}$ and (ii) infinitely many short curves $S^-_{k,n}$, $n \geq 1$, running roughly parallel to $S^-_{k,0}$ and terminating on $S^-_{k,0}$ and $\mathcal{S}_0$. The origin of these curves can be traced to grazing collisions on Fig. 6.

The length of $S^-_{k,n}$ is $|S^-_{k,n}| \sim 1/\sqrt{n}$, the distance between the upper endpoint of $S^-_{k,n}$ and the limit point x_k is $\sim 1/n$, and the distance between $S^-_{k,n}$ and $S^-_{k,n+1}$ is $\sim 1/n^2$, as indicated on Fig. 7 (right). Here the notation

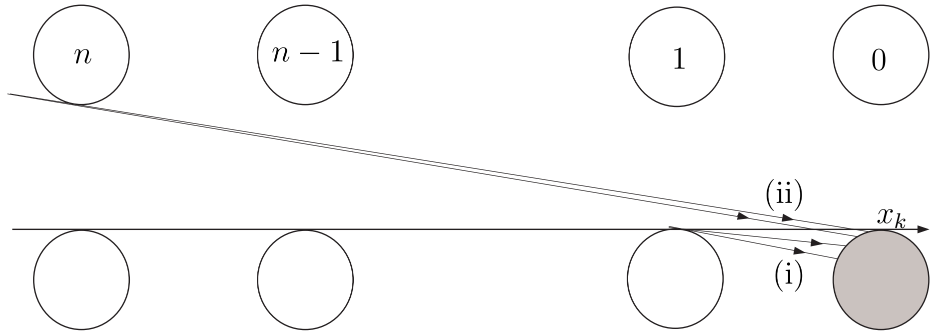

Figure 6: A point x_k where singularity curves accumulate.

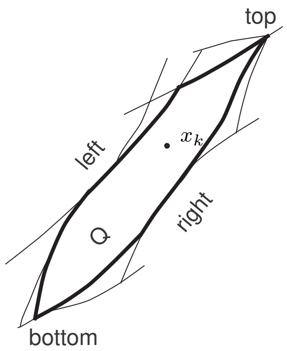

Figure 7: Singularity curves (unbounded horizon).

$\sim 1/n^a$, $a > 0$, means $= c/n^a + o(1/n^a)$ as $n \to \infty$ for some constant $c = c(k, \mathcal{D}) > 0$.

The set $\mathcal{S}_1$ has a similar (in fact, symmetric) structure, see Fig. 7 (left); note that the left and right parts of Fig. 7 should be superimposed for a proper view. We denote the components of $\mathcal{S}_1$ near x_k by $S^+_{k,0}$ and $S^+_{k,n}$, respectively.

Next, let $D^-_{k,n}$ denote the region bounded by the curves $S^-_{k,n}$, $S^-_{k,n-1}$, $S^-_{k,0}$, and $\mathcal{S}_0$ (this region is sometimes called *n-cell* in the literature). Similarly, the n-cell $D^+_{k,n}$ is defined to be the region bounded by the curves $S^+_{k,n}$, $S^+_{k,n-1}$, $S^+_{k,0}$, and $\mathcal{S}_0$. Note that $\mathcal{F}^{-1}(D^-_{k,n}) = D^+_{k',n}$ for some $1 \leq k' \leq k_{\max}$, i.e. "positive" n-cells are transformed into "negative" n-cells by $\mathcal{F}$. The following technical estimate will be useful:

Lemma 4.3. *Let $dx \in \mathcal{T}_x\mathcal{M}$ be an unstable tangent vector at a point $x \in D_{k,n}^+$. Then $\|D_x\mathcal{F}(dx)\|_p/\|dx\|_p \geq \text{const}\, n^{3/2}$, as well as $\|D_x\mathcal{F}(dx)\|/\|dx\| \geq \text{const}\, n^{3/2}$.*

Proof. Observe that $\tau(x) \geq cn$ for $x \in D_{k,n}^+$, where $c = c(\mathcal{D}) > 0$ is a constant. Note also that $\cos\varphi \leq \text{const}/\sqrt{n}$ both at x and $\mathcal{F}(x)$. Now the lemma follows from (3.7) and (3.11). ☐

Lemma 4.3 gives only a lower bound, the actual expansion factor may be much higher and approach infinity.

5 Stable and unstable manifolds

In this section we present a novel construction of stable/unstable manifolds, as was mentioned in Introduction, that will give us a much better control on their parameters, compared to those provided by traditional methods.

First we recall certain related facts. Given a point $X \in \Omega$, we denote by $t^+(X)$ the time of its first collision with $\partial\mathcal{D}$ in the future and by $t^-(X)$ the time elapsed since the last collision in the past. We put $x = \Phi^{t^+(X)}(X) \in \mathcal{M}$, and then use the respective notation $x_n, \mathcal{R}_n, \tau_n$, related to the point x, as introduced in Section 3.

If $X \in \Omega$ is a hyperbolic point for the flow Φ^t, then there are stable and unstable subspaces $E_X^s, E_X^u \subset \mathcal{T}_X\Omega$ in its tangent space. For any tangent vector $(d\eta^u, d\xi^u, d\omega^u) \subset E_X^u$ we have $d\eta^u = 0$ and the slope $\mathcal{B}^u(X) = d\omega^u/d\xi^u$ is known to be given by a remarkable formula discovered by Sinai which involves an infinite continued fraction

$$\mathcal{B}^u(X) = \cfrac{1}{t^-(X) + \cfrac{1}{\mathcal{R}_{-1} + \cfrac{1}{\tau_{-2} + \cfrac{1}{\mathcal{R}_{-2} + \cfrac{1}{\tau_{-3} + \cfrac{1}{\ddots}}}}}}. \tag{5.1}$$

The alternative structure of the τ's and $\mathcal{R}$'s in the fraction corresponds to the alternation of free paths and collisions along the past orbit of X. The continued fraction (5.1) converges for every point whose past orbit is well defined (i.e. Φ^t is smooth at X for all $t < 0$).

Similarly, for any tangent vector $(d\eta^s, d\xi^s, d\omega^s) \subset E^s_X$ we have $d\eta^s = 0$ and the slope $\mathcal{B}^s(X) = d\omega^s/d\xi^s$ is given by a continued fraction

$$\mathcal{B}^s(X) = \cfrac{1}{t^+(X) + \cfrac{1}{\mathcal{R}_0 + \cfrac{1}{\tau_0 + \cfrac{1}{\mathcal{R}_1 + \cfrac{1}{\tau_1 + \cfrac{1}{\ddots}}}}}} \tag{5.2}$$

At every collision point $X \in \mathcal{M}$ the function $\mathcal{B}^u(X)$ takes different values immediately before and immediately after collision, and we denote its values, respectively, by $\mathcal{B}^{u-}(X)$ and $\mathcal{B}^{u+}(X)$. Similarly, we define $\mathcal{B}^{s\pm}(X)$.

Now if $X \in \Omega$ is hyperbolic, then the point $x = \Phi^{t^+(X)}(X) \in \mathcal{M}$ is hyperbolic for the map $\mathcal{F}$, and we denote by $E^s_x, E^u_x \subset \mathcal{T}_x\mathcal{M}$ its stable and unstable subspaces, respectively. We denote by $\mathcal{V}^s(x)$ and $\mathcal{V}^u(x)$ their slopes in the r, φ coordinates, and they satisfy

$$\mathcal{V}^u(x) = \mathcal{B}^{u-}(x)\cos\varphi + \mathcal{K} = \mathcal{B}^{u+}(x)\cos\varphi - \mathcal{K} \tag{5.3}$$

and

$$\mathcal{V}^s(x) = \mathcal{B}^{s-}(x)\cos\varphi + \mathcal{K} = \mathcal{B}^{s+}(x)\cos\varphi - \mathcal{K}. \tag{5.4}$$

The function $\mathcal{V}^u(x)$ is defined by (5.3) on the set of all points where all the past iterations of $\mathcal{F}$ are smooth, i.e. on the set

$$\hat{\mathcal{M}}^-: = \mathcal{M} \setminus \cup_{n=-\infty}^{0} \mathcal{S}_n. \tag{5.5}$$

Furthermore, $\mathcal{V}^u(x)$ is continuous on $\hat{\mathcal{M}}^-$. Similarly, $\mathcal{V}^s(x)$ is defined and continuous on the set

$$\hat{\mathcal{M}}^+: = \mathcal{M} \setminus \cup_{n=0}^{\infty} \mathcal{S}_n \tag{5.6}$$

of points where all the forward iterations of $\mathcal{F}$ are smooth.

It is known that the singularity curves $S \subset \mathcal{S}_{-n} \setminus \mathcal{S}_{-n+1}$ align with the unstable subspaces E^u, as n increases (this property is called the *alignment of singularity curves*). More precisely, there is a sequence $\beta_n \to 0$ as $n \to \infty$ such that for any curve $S \subset \mathcal{S}_{-n} \setminus \mathcal{S}_{-(n-1)}$ and any point $y \in S$ there is an open neighborhood $U_y \subset \mathcal{M}$ such that the slope $\mathcal{V}_S(y)$ of the curve S at y satisfies

$$\sup_{x \in U_y \cap \hat{\mathcal{M}}^-} |\mathcal{V}^u(x) - \mathcal{V}_S(y)| < \beta_n. \tag{5.7}$$

The proof of (5.7) uses continued fraction (5.1).

We now turn to the novel construction of stable and unstable manifolds, where singularities play an instrumental role. Consider the sets

$$\mathcal{S}_\infty = \cup_{n=0}^\infty \mathcal{S}_n \qquad \text{and} \qquad \mathcal{S}_{-\infty} = \cup_{n=0}^\infty \mathcal{S}_{-n}, \tag{5.8}$$

of points where *some future* and, respectively, *some past* iterate of $\mathcal{F}$ is singular. It easily follows from the uniform hyperbolicity that both sets are dense in $\mathcal{M}$. The set $\mathcal{S}_\infty \setminus \mathcal{S}_0$ is a countable union of smooth stable curves, while $\mathcal{S}_{-\infty} \setminus \mathcal{S}_0$ is a countable union of smooth unstable curves. For each $\mathcal{M}_i \subset \mathcal{M}$, cf. (2.10), the sets $\mathcal{M}_i \cap \mathcal{S}_\infty$ and $\mathcal{M}_i \cap \mathcal{S}_{-\infty}$ are pathwise connected due to the continuation property.

On the other hand, the set $\mathcal{M} \setminus \mathcal{S}_{-\infty}$ has full μ measure, but it is badly disconnected. We will show that its connected components are exactly unstable manifolds. Let $x \in \mathcal{M} \setminus \mathcal{S}_{-\infty}$ and for any $n \geq 1$ denote by $\mathcal{Q}_{-n}(x)$ the connected component of the open set $\mathcal{M} \setminus \mathcal{S}_{-n}$ that contains x. Recall that $\mathcal{Q}_{-n}(x)$ is a curvilinear polygon with interior angles $\leq \pi$.

One easily checks that $\partial \mathcal{Q}_{-n}(x)$ consists of two monotonically increasing (and piecewise smooth) curves, whose endpoints are the top and bottom vertices of $\mathcal{Q}_{-n}(x)$, see Fig. 8. We call those curves left and right sides and denote them by $\partial^L \mathcal{Q}_{-n}(x)$ and $\partial^R \mathcal{Q}_{-n}(x)$, respectively.

Obviously, $\mathcal{Q}_{-n}(x) \supseteq \mathcal{Q}_{-(n+1)}(x)$ for all $n \geq 1$ and the intersection of their closures

$$\tilde{W}^u(x) \colon = \cap_{n=1}^\infty \overline{\mathcal{Q}_{-n}}(x) \tag{5.9}$$

is a closed continuous monotonically increasing curve (which follows from the fact that $\mathcal{S}_{-\infty}$ is dense in $\mathcal{M}$). We denote by $W^u(x)$ the curve $\tilde{W}^u(x)$ without its endpoints.

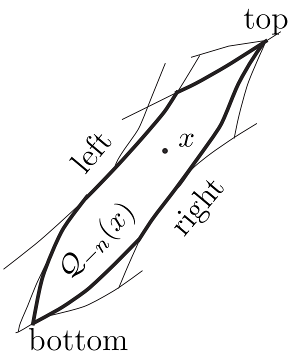

Figure 8: The connected component $\mathcal{Q}_{-n}(x)$.

We remark that the endpoints of $W^u(x)$ need not coincide with the top and bottom vertices of the figure $\mathcal{Q}_{-n}(x)$ for any n; in fact $W^u(x)$ typically terminates on interior points of singularity curves.

Lemma 5.1. *We have* $W^u(x) \subset \cap_{n\geq 1}\mathcal{Q}_{-n}(x)$.

Proof. Suppose that some point $y \in W^u(x)$ belongs to a singularity curve $S \subset \partial\mathcal{Q}_{-n}(x)$ for a finite $n \geq 1$. Without loss of generality, we assume $S \subset \partial^L \mathcal{Q}_{-n}(x)$. Then y is also a limit point for a sequence of curves $S'_{-m} \subset \partial^R \mathcal{Q}_{-m}(x)$. Since $S \subset \mathcal{S}_{-n}$ for a finite n, the curve S has a slope at y different from the limit slope of the curves S'_{-m}, according to the alignment property (5.7) and the 'convexity' of the curvilinear polygons. This leads to a contradiction with the continuation property. $\square$

Next, one easily checks that at every point $y \in W^u(x)$ the slope of the curve $W^u(x)$ equals $\mathcal{V}^u(y)$ (this in fact follows from the continuity of $\mathcal{V}^u(y)$).

Observe that $\mathcal{F}^{-n}\big(W^u(x)\big) \subset \mathcal{M} \setminus \mathcal{S}_{-\infty}$ is an unstable curve for every $n \geq 1$. Let $\mathcal{L}_n \subset \mathcal{M}$ be the linear segment (in the r, φ coordinates) joining the endpoints of $\mathcal{F}^{-n}\big(W^u(x)\big)$. We claim that $\mathcal{L}_n \cap \mathcal{S}_n = \emptyset$. Indeed, observe that $\mathcal{F}^{-n}\big(W^u(x)\big) \cap \mathcal{S}_n = \emptyset$ and recall that $\mathcal{S}_n$ consists of monotonically decreasing curves satisfying the continuation property; as a result, the unstable curves $\mathcal{F}^n(\mathcal{L}_n)$ converge, as $n \to \infty$, to $W^u(x)$ in the C^0 metric.

Observe that the unstable curves $\mathcal{F}^n(\mathcal{L}_n)$, $n \geq 1$, are $C^{\ell-1}$ smooth and have all their $\ell - 1$ derivatives uniformly bounded, see Theorem 3.1. We need the following elementary lemma:

Lemma 5.2. *Let $f_n(t)$, $n \ge 1$, be C^q smooth functions on an interval (a, b) such that $|f_n^{(\nu)}(t)| \le C =$ const for all $n \ge 1$, $1 \le \nu \le q$, and $t \in (a, b)$. Suppose $\lim_{n\to\infty} f_n(t) = f(t)$ for all $a < t < b$. Then $f(t)$ is at least C^{q-1} smooth, $f_n^{(\nu)}(t) \to f^{(\nu)}(t)$ and $|f^{(\nu)}(t)| \le C$ for all $1 \le \nu \le q-1$ and $t \in (a, b)$. Moreover, $f^{(q-1)}(t)$ is a Lipschitz continuous function with constant C, i.e. $|f^{(q-1)}(t) - f^{(q-1)}(t')| \le C|t - t'|$.*

This gives us our next result:

Theorem 5.3. *The curve $W^u(x)$ is $C^{\ell-2}$ smooth, all its $\ell - 2$ derivatives are bounded by a constant independent of x, and its $(\ell - 2)$nd derivative is Lipschitz continuous with a Lipschitz constant independent of x.*

Remark 5.4. *It follows from a general theory [KS] that for almost every $x \in \mathcal{M}$ the curve $W^u(x)$ is actually $C^{\ell-1}$ smooth. Then the last, $(\ell - 1)$st derivative, will be also uniformly bounded, according to Theorem 5.3.*

Recall that $\mathcal{F}^{-n}\big(W^u(x)\big) \subset \mathcal{M}\setminus\mathcal{S}_{-\infty}$ is an unstable curve for every $n \ge 1$. The uniform hyperbolicity of $\mathcal{F}$, see (3.10), implies

$$\big|\mathcal{F}^{-n}\big(W^u(x)\big)\big| \le C\Lambda^{-n}\big|W^u(x)\big|, \qquad \forall n \ge 1,$$

hence the preimages of $W^u(x)$ contract exponentially fast, so $W^u(x)$ is an unstable manifold; in fact it is a *maximal* unstable manifold (it cannot be continued any further).

We remark that our construction produces the maximal unstable manifold at every point $x \in \mathcal{M} \setminus \mathcal{S}_{-\infty}$, but it does not guarantee that $W^u(x) \neq \emptyset$. It may happen that the intersection (5.9) consists of the single point x, then $W^u(x) = \emptyset$, we return to this possibility below.

All our constructions have their time-reversals, in particular, for every point $x \in \mathcal{M} \setminus S_\infty$ we obtain a (maximal) stable manifold $W^s(x)$.

Our construction of stable and unstable manifolds allows us to establish sharp estimates on their size, see below, which do not follow from traditional methods. We first consider the simpler case of finite horizon, and then extend our results to tables without horizon.

Let $W \subset \mathcal{M}$ be a smooth unstable (or stable) curve. Any point $x \in W$ divides W into two segments, and we denote by $r_W(x)$ the length (in the Euclidean metric) of the shorter one. For brevity, we put $r^u(x) :=$ $r_{W^u(x)}(x)$. If it happens that $W^u(x) = \emptyset$, then we set $r^u(x) = 0$. Clearly, $r^u(x)$ characterizes the size of $W^u(x)$.

Theorem 5.5. *There is a constant $C = C(\mathcal{D}) > 0$ such that for all $\varepsilon > 0$*

$$\mu\{x\colon r^u(x) < \varepsilon\} \le C\varepsilon. \tag{5.10}$$

We note that the general theory of hyperbolic maps with singularities [KS, Theorem 6.1] only guarantees that $\mu\{x\colon r^u(x) < \varepsilon\} \le C\varepsilon^a$ for some $a > 0$. We actually show that $a = 1$.

Proof. For every point $x \in \mathcal{M}$ denote by $d^u(x, \mathcal{S}_1)$ the length of the shortest unstable curve that connects x with the set $\mathcal{S}_1$.

Lemma 5.6. *For any $x \in \mathcal{M}$*

$$r^u(x) \ge \min_{n\ge 1} \hat{c}\,\Lambda^n\, d^u(\mathcal{F}^{-n}x, \mathcal{S}_1), \tag{5.11}$$

where $\hat{c} = \hat{c}(\mathcal{D}) > 0$ is a constant from (3.10).

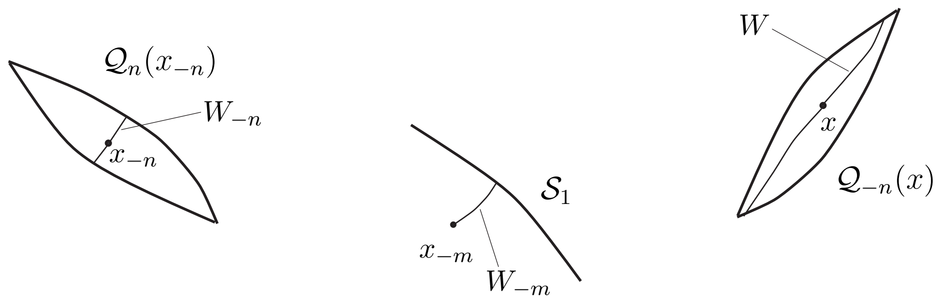

Figure 9: Proof of Lemma 5.6.

Proof. We may assume that $x \in \mathcal{M} \setminus \mathcal{S}_{-\infty}$. Denote $x_{-n} = \mathcal{F}^{-n}(x)$ for $n \ge 1$ and again let $\mathcal{Q}_{-n}(x)$ be the connected component of $\mathcal{M} \setminus \mathcal{S}_{-n}$ containing the point x. Obviously, $\mathcal{Q}_n(x_{-n})\colon = \mathcal{F}^{-n}(\mathcal{Q}_{-n}(x))$ is the connected component of $\mathcal{M} \setminus \mathcal{S}_n$ containing the point x_{-n}.

Let W'_{-n} be an arbitrary unstable curve passing through x_{-n} and terminating on the opposite sides of $\mathcal{Q}_n(x_{-n})$, see Fig. 9. Then $W' = \mathcal{F}^n(W'_{-n})$ is an unstable curve passing through x and terminating on $\partial\mathcal{Q}_{-n}(x)$. It is divided by the point x into two segments, and we denote by W the shorter one. Put $W_{-m} = \mathcal{F}^{-m}(W)$ for all $m \ge 1$.

Since W_{-n} terminates on $\mathcal{S}_n$, there is an $m \in [1, n]$ such that W_{-m} joins x_{-m} with $\mathcal{S}_1$. Due to the uniform hyperbolicity of $\mathcal{F}$, see (3.10), we have

$$r_{W'}(x) = |W| \geq \hat{c}\,\Lambda^m |W_{-m}| \geq \hat{c}\,\Lambda^m d^u(\mathcal{F}^{-m}x, \mathcal{S}_1).$$

Since this holds for all $n \geq 1$, the lemma follows. □

Next, we pick $\hat{\Lambda} \in (1, \Lambda)$ and observe that

$$r^u(x) \geq r^u_*(x) \colon = \min_{n \geq 1} \hat{c}\,\hat{\Lambda}^n\, d^u\big(\mathcal{F}^{-n}x, \mathcal{S}_1\big)$$

(we need to assume $\hat{\Lambda} < \Lambda$ for a later use in Propositions 6.3 and 5.9 and Theorem 6.4).

So if $r^u(x) = 0$, then $r^u_*(x) = 0$, i.e. the past semitrajectory of x approaches the singularity set $\mathcal{S}_1$ faster than the exponential function $\hat{\Lambda}^{-n}$. For any $\varepsilon > 0$ denote by $\mathcal{U}_\varepsilon(\mathcal{S}_1)$ the ε-neighborhood of the set $\mathcal{S}_1$ and let

$$\mathcal{U}^u_\varepsilon(\mathcal{S}_1) = \{x \colon d^u(x, \mathcal{S}_1) < \varepsilon\}.$$

Due to (3.3), $\mathcal{U}^u_\varepsilon(\mathcal{S}_1) \subset \mathcal{U}_{D\varepsilon}(\mathcal{S}_1)$ for some constant $D > 0$ (because the set $\mathcal{S}_1$ consists of monotonically decreasing curves and horizontal lines), hence

$$\mu\big(\mathcal{U}^u_\varepsilon(\mathcal{S}_1)\big) < C\varepsilon \tag{5.12}$$

for some $C > 0$ and all $\varepsilon > 0$ (here we use the assumption of finite horizon). Next we show that $r^u_*(x) > 0$ for μ-almost every point $x \in \mathcal{M}$. Let

$$B_n \colon = \mathcal{F}^n\big(\mathcal{U}^u_{\hat{\Lambda}^{-n}}(\mathcal{S}_1)\big) = \big\{x \colon d^u\big(\mathcal{F}^{-n}(x), \mathcal{S}_1\big) \leq \hat{\Lambda}^{-n}\big\}.$$

By the invariance of the measure μ we have

$$\sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \mu\big(\mathcal{U}^u_{\hat{\Lambda}^{-n}}(\mathcal{S}_1)\big) < \infty.$$

Denote by $B = \cap_{m \geq 1} \cup_{n \geq m} B_n$ the set of points that belong to infinitely many B_n's. Borel-Cantelli lemma implies that $\mu(B) = 0$, and it is easy to check that $r^u_*(x) > 0$ for every $x \in \mathcal{M} \setminus B$. □

We remark that Theorem 5.5 implies a well known fact that almost every point $x \in \mathcal{M}$ has a nontrivial unstable manifold.

Lemma 5.7. *We have $\mu\{x\colon r_*^u(x) < \varepsilon\} \le C\varepsilon$ for some constant $C = C(\mathcal{D}) > 0$ and all $\varepsilon > 0$.*

Proof. Observe that $r_*^u(x) < \varepsilon$ iff $x \in \mathcal{F}^n\big(\mathcal{U}^u_{\varepsilon\hat{c}^{-1}\hat{\Lambda}^{-n}}(\mathcal{S}_1)\big)$ for some $n \ge 1$. Now (5.12) implies the result. □

This completes the proof of Theorem 5.5. □

Next we show that an estimate similar to (5.10) holds in the p-metric as well. Any smooth curve $W \subset \mathcal{M}$ is divided by any point $x \in W$ into two segments, and we denote by $p_W(x)$ the p-length of the shorter one. We put $p^u(x)\colon = p_{W^u(x)}(x)$.

Theorem 5.8. *We have $\mu\{x\colon p^u(x) < \varepsilon\} \le C\varepsilon$ for some constant $C = C(\mathcal{D}) > 0$ and all $\varepsilon > 0$.*

Proof. Denote by $d_p^u(y, \mathcal{S}_1)$ the p-length of the shortest unstable curve that connects a point y with the set $\mathcal{S}_1$. An argument similar to the proof of Lemma 5.6 yields

$$p^u(x) \ge \min_{n\ge 1} \Lambda^n d_p^u(\mathcal{F}^{-n}x, \mathcal{S}_1). \tag{5.13}$$

We pick $\hat{\Lambda} \in (1, \Lambda)$ and observe that

$$p^u(x) \ge p_*^u(x)\colon = \min_{n\ge 1} \hat{\Lambda}^n\, d^u\big(\mathcal{F}^{-n}x, \mathcal{S}_1\big).$$

For any $\varepsilon > 0$ let

$$\mathcal{U}^u_{p,\varepsilon}(\mathcal{S}_1) = \{x\colon d_p^u(x, \mathcal{S}_1) < \varepsilon\}.$$

Next we have an analogue of (5.12):

$$\mu\big(\mathcal{U}^u_{p,\varepsilon}(\mathcal{S}_1)\big) < C\varepsilon, \tag{5.14}$$

where $C = C(\mathcal{D}) > 0$ is a constant. This is not so immediate as (5.12), because the set $\mathcal{U}^u_{p,\varepsilon}(\mathcal{S}_1)$ is getting thicker as it approaches $\mathcal{S}_0$, and has width $\sim \sqrt{\varepsilon}$ near $\mathcal{S}_0$, see Fig. 10. However, the density of the μ measure is proportional to $\cos\varphi$, which vanishes on $\mathcal{S}_0$, and the smallness of the density perfectly compensates for large size of the set $\mathcal{U}^u_{p,\varepsilon}(\mathcal{S}_1)$.

We now observe that $p_*^u(x) < \varepsilon$ iff $x \in \mathcal{F}^n\big(\mathcal{U}^u_{p,\varepsilon\hat{\Lambda}^{-n}}(\mathcal{S}_1)\big)$ for some $n \ge 1$. Now (5.14) implies that $\mu\{x\colon p_*^u(x) < \varepsilon\} \le C\varepsilon$ for some constant $C = C(\mathcal{D}) > 0$ and all $\varepsilon > 0$. This completes the proof of Theorem 5.8. □

We emphasize that points with arbitrarily short unstable manifolds are dense in $\mathcal{M}$; in fact, for any $\varepsilon > 0$ and any open set $\mathcal{U} \subset \mathcal{M}$ we have $\mu\{x \in \mathcal{U}\colon r^u(x) < \varepsilon\} > 0$; this follows from the fact that $\mathcal{S}_{-\infty}$ is dense in $\mathcal{M}$.

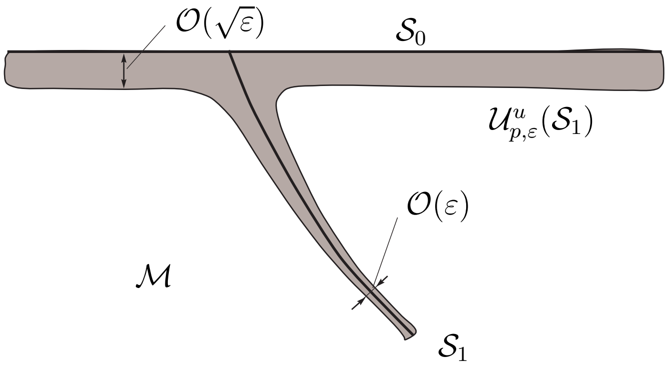

Figure 10: The set $\mathcal{U}^u_{p,\varepsilon}(\mathcal{S}_1)$.

Proposition 5.9. *If $r^u_*(x) > 0$ (which happens for almost every point $x \in \mathcal{M}$), then both endpoints of $W^u(x)$ belong to the singularity set $\mathcal{S}_{-\infty}$.*

Proof. First, $r^u_*(x) > 0$ implies that $r^u_*(y) > 0$ for all $y \in W^u(x)$. Indeed, it is enough to observe that $|\mathcal{F}^{-n}(W^u(x))| \leq \hat{c}\Lambda^{-n}$ and use the assumption $\hat{\Lambda} < \Lambda$. It now follows that if y is an endpoint of $W^u(x)$, the either $r^u_*(y) > 0$ or $y \in \mathcal{S}_{-\infty}$. However, $r^u_*(y) > 0$ is impossible, since in that case $W^u(x)$ could be extended beyond the point y. $\square$

Observe that no two unstable manifolds can intersect each other, but they can have a common endpoint (which lies on a singularity curve).

Lastly, we extend our results to billiards without horizon:

Theorem 5.10. *For dispersing billiards without horizon, we have*

$$\mu\{x\colon r^u(x) < \varepsilon\} \leq C\varepsilon \tag{5.15}$$

$$\mu\{x\colon p^u(x) < \varepsilon\} \leq C\varepsilon \tag{5.16}$$

for some constant $C = C(\mathcal{D}) > 0$ and all $\varepsilon > 0$.

Proof. We need to refine the arguments of the previous section, since the estimates (5.12) and (5.14), as stated, are no longer valid, because the total length of the set $\mathcal{S}_1$ is now infinite and the measure of its ε-neighborhood is no longer $\mathcal{O}(\varepsilon)$ (in fact, $\mu\big(\mathcal{U}^u_\varepsilon(\mathcal{S}_1)\big) \sim \varepsilon \ln(1/\varepsilon)$ and $\mu\big(\mathcal{U}^u_{p,\varepsilon}(\mathcal{S}_1)\big) \sim \varepsilon^{4/5}$, but we will not need these facts). ‘

We will use the notation of the previous section. Let $E_u(x)$ be a piecewise constant function on $\mathcal{M}$ defined as follows: $E_u(x) = n^{3/2}$ for all $x \in D^+_{k,n}$ with $1 \le k \le k_{\max}$ and $n \ge 1$, and $E_u(x) = 1$ for all other points in $\mathcal{M}$. It follows from (3.10) and Lemma 4.3 that there is a small constant $\tilde{c} = \tilde{c}(\mathcal{D}) > 0$ such that $\tilde{c}E_u(x)$ is the lower bound on the factor of expansion of unstable vectors $dx \in \mathcal{T}_x\mathcal{M}$ for all $x \in \mathcal{M}$, i.e. $\|D_x\mathcal{F}(dx)\| \ge \tilde{c}E_u(x)\,\|dx\|$.

Now (5.11) can be modified as follows:

$$r^u(x) \ge \min_{n\ge1} \tilde{c}E_u(\mathcal{F}^{-n}x)\,\hat{c}\,\Lambda^{n-1}\,d^u(\mathcal{F}^{-n}x,\mathcal{S}_1). \tag{5.17}$$

Indeed, one can repeat the proof of Lemma 5.6 with the following modification:

$$|W| = |W_{-m}|\,\frac{|W_{-m+1}|}{|W_{-m}|}\,\frac{|W|}{|W_{-m+1}|} \ge \tilde{c}E_u(x_{-m})\,\hat{c}\,\Lambda^{m-1}|W_{-m}|$$

(we note that whenever $x_{-m} \in D^+_{k,n}$, we also have $W_{-m} \subset D^+_{k,n}$), then one gets (5.17). As before, we pick $\hat{\Lambda} \in (1,\Lambda)$ and observe that

$$r^u(x) \ge \tilde{r}^u_*(x) \colon = \min_{n\ge1} \tilde{c}\,\hat{c}\,E_u(\mathcal{F}^{-n}x)\,\hat{\Lambda}^{n-1}\,d^u\big(\mathcal{F}^{-n}x,\mathcal{S}_1\big).$$

For any $\varepsilon > 0$ let

$$\tilde{\mathcal{U}}^u_\varepsilon(\mathcal{S}_1) = \{x\colon E_u(x)\,d^u(x,\mathcal{S}_1) < \varepsilon\}.$$

Observe that $\tilde{r}^u_*(x) < \varepsilon$ iff $x \in \mathcal{F}^n\big(\tilde{\mathcal{U}}^u_{\varepsilon\tilde{c}^{-1}\hat{c}^{-1}\hat{\Lambda}^{-n+1}}(\mathcal{S}_1)\big)$ for some $n \ge 1$. Thus, to prove (5.15) it is enough to derive the following analogue of (5.12):

$$\mu\big(\tilde{\mathcal{U}}^u_\varepsilon(\mathcal{S}_1)\big) < C\varepsilon \tag{5.18}$$

for some constant $C > 0$ and all $\varepsilon > 0$. For every $n \ge 1$ and $1 \le k \le k_{\max}$, the set $D^+_{k,n} \cap \tilde{\mathcal{U}}^u_\varepsilon(\mathcal{S}_1)$ lies within the $\mathcal{O}\big(n^{-3/2}\varepsilon\big)$-neighborhood of the boundary $\partial D^+_{n,k}$. Since the length of the latter is $\mathcal{O}\big(n^{-1/2}\big)$, we get

$$\mu\big(D^+_{k,n} \cap \tilde{\mathcal{U}}^u_\varepsilon(\mathcal{S}_1)\big) < \text{ const}\,n^{-5/2}\varepsilon \tag{5.19}$$

(the additional factor of $n^{-1/2}$ results from the density $\cos\varphi = \mathcal{O}\big(n^{-1/2}\big)$ of the measure μ on $D^+_{k,n}$). Denote $\mathcal{M}_0 = \mathcal{M} \setminus \cup_{k,n} D^+_{k,n}$. Since the total length of the singularity lines $\mathcal{S}_1$ within the region $\mathcal{M}_0$ is finite, it follows that

$$\mu\big(\mathcal{M}_0 \cap \tilde{\mathcal{U}}^u_\varepsilon(\mathcal{S}_1)\big) < \text{ const}\,\varepsilon.$$

Adding these estimates up proves (5.18) and hence (5.15).

The proof of (5.16) is similar, but certain changes are noteworthy. The function $E_u(x)$ is defined in the same way as above, and $\tilde{c}E_u(x)$ will be again a lower bound on the factor of expansion of unstable vectors due to Lemma 4.3. Now (5.13) can be modified as follows:

$$p^u(x) \geq \min_{n\geq 1} \tilde{c}E_u(\mathcal{F}^{-n}x)\,\Lambda^{n-1}\,d_p^u(\mathcal{F}^{-n}x,\mathcal{S}_1),$$

we leave the verification to the reader. As before, we pick $\hat{\Lambda} \in (1,\Lambda)$ and observe that

$$p^u(x) \geq \tilde{p}_*^u(x)\colon = \min_{n\geq 1} \tilde{c}E_u(\mathcal{F}^{-n}x)\,\hat{\Lambda}^{n-1}\,d_p^u\big(\mathcal{F}^{-n}x,\mathcal{S}_1\big).$$

For any $\varepsilon > 0$ let

$$\tilde{\mathcal{U}}^u_{p,\varepsilon}(\mathcal{S}_1) = \{x\colon E_u(x)\,d_p^u(x,\mathcal{S}_1) < \varepsilon\}.$$

Observe that $\tilde{p}_*^u(x) < \varepsilon$ iff $x \in \mathcal{F}^n\big(\tilde{\mathcal{U}}^u_{p,\varepsilon\tilde{c}^{-1}\hat{\Lambda}^{-n+1}}(\mathcal{S}_1)\big)$ for some $n \geq 1$. Thus, to prove (5.16) it is enough to derive the following analogue of (5.14):

$$\mu\big(\tilde{\mathcal{U}}^u_{p,\varepsilon}(\mathcal{S}_1)\big) < C\varepsilon \tag{5.20}$$

for some constant $C > 0$ and all $\varepsilon > 0$. One can verify directly that

$$\mu\big(\mathcal{M}_0 \cap \tilde{\mathcal{U}}^u_{p,\varepsilon}(\mathcal{S}_1)\big) < \text{ const}\,\varepsilon$$

and

$$\mu\big(D^+_{k,n} \cap \tilde{\mathcal{U}}^u_{p,\varepsilon}(\mathcal{S}_1)\big) < \text{ const}\,n^{-2}\varepsilon$$

for all $1 \leq k \leq k_{\max}$ and $n \geq 1$ (remember that $E_u(x) = n^{3/2}$ for $x \in D^+_{k,n}$ and the length of $D^+_{k,n}$ is $\mathcal{O}(1/\sqrt{n})$). Adding these estimates up yields (5.20) and hence (5.16). $\square$

6 Homogeneous stable and unstable manifolds

In the previous section we constructed the maximal unstable manifold $W^u(x)$ for almost every point $x \in \mathcal{M}$. We note that $W^u(x)$ does not include its end points, hence for any $x, y \in \mathcal{M}$ the manifolds $W^u(x)$ and $W^u(y)$ either coincide or are disjoint.

Let ξ^u denote the partition of $\mathcal{M}$ into maximal unstable manifolds. Precisely, for every point $x \in \mathcal{M}$ we put $\xi^u_x = W^u(x)$, if the latter is not empty, and $\xi^u_x = \{x\}$ otherwise. Note that if y is an endpoint of $W^u(x)$, then $W^u(y) = \emptyset$, hence $\xi^u_y = \{y\}$.

Theorem 6.1. *The partition ξ^u is measurable.*

Proof. We recall that a partition ξ of a Lebesgue space is said to be measurable if there exists a countable collection of measurable ξ-sets[1] $\{B_n\}$, $n \geq 1$, such that for any distinct elements $C_1, C_2 \in \xi$ there is a B_n, for which $C_1 \subset B_n$ and $C_2 \subset X \setminus B_n$, or vice versa (see, for example, Appendix 1 in [CFS] or [Bou, pp. 57–72]). Now one can easily construct a countable collection of sets B_n by using the domains $\mathcal{Q}_{-n}(x)$, their closures, and the singularity curves, according to (5.9). $\square$

Since the partition ξ^u is measurable, the invariant measure μ induces conditional (probability) measure, $\nu_{W^u(x)}$, on a.e. unstable manifold $W^u(x)$. Our next goal is to describe those measures. For any point $x \in W$ on any unstable (or stable) curve $W \subset \mathcal{M}$ we denote by

$$\mathcal{J}_W \mathcal{F}^n(x) = \frac{\|D_x \mathcal{F}^n(dx)\|}{\|dx\|} \tag{6.1}$$

the Jacobian of the restriction of the map $\mathcal{F}^n$ to W, at the point x, in the Euclidean metric (here dx denotes a nonzero tangent vector to W at x).

The following fact is given without proof.

Theorem 6.2. *For almost every $x \in \mathcal{M}$, the conditional measure $\nu_{W^u(x)}$ on the unstable manifold $W = W^u(x)$ is absolutely continuous with respect to the Lebesgue measure on W, and its density $\rho_W(y)$ is a $C^{\ell-2}$ smooth function satisfying*

$$\frac{\rho_W(y)}{\rho_W(z)} = \lim_{n\to\infty} \frac{\mathcal{J}_W \mathcal{F}^{-n}(y)}{\mathcal{J}_W \mathcal{F}^{-n}(z)} \tag{6.2}$$

for every $y, z \in W$ (since ν_W has to be a probability measure, ρ_W is specified completely by the above relation).

[1] A ξ-set is a union of some elements of the partition ξ.

Smooth conditional measures on unstable manifolds with the above properties were first constructed by Sinai in 1968 [S1] (for Anosov diffeomorphisms); and now all the above facts have been proven for large classes of hyperbolic maps and became standard in ergodic theory. We refer the reader to [S1] and [PS, Theorem 3] for full proofs of these facts (a similar argument is also given in [L, Proposition 3.1]).

For any unstable manifold $W \subset \mathcal{M}$, the unique probability density ρ_W satisfying (6.2) is often called the *u-SRB density*, and the corresponding probability measure ν_W is called the *u-SRB* measure (or, sometimes, the *Gibbs u-measure*).

General theory only guarantees that the u-SRB density $\rho_{W^u(x)}$ exists for almost every point $x \in \mathcal{M}$. We make this claim more precise:

Proposition 6.3. *Let $r_*^u(x) > 0$ (in the notation of the previous section). Then the limit (6.2) is finite for all $y, z \in W^u(x)$, hence $\rho_{W^u(x)}$ exists.*

Proof. Taking logarithm and using the chain rule reduces this problem to the convergence of the series

$$\sum_{n=0}^{\infty}\Big(\ln \mathcal{J}_{W_n}\mathcal{F}^{-1}(y_n) - \ln \mathcal{J}_{W_n}\mathcal{F}^{-1}(z_n)\Big), \tag{6.3}$$

where $W_n = \mathcal{F}^{-n}(W)$ and $y_n = \mathcal{F}^{-n}(y)$ (the same for z_n). To describe the contraction of unstable manifolds under the inverse map $\mathcal{F}^{-1}$, we pick $x \in W$ and denote $x_n = \mathcal{F}^{-n}(x)$. Then, using the results and notation of (3.9) and (3.6), we have

$$\begin{aligned} \mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n) &= \frac{1}{1+\tau_{n+1}\,\mathcal{B}^+_{n+1}}\,\frac{\cos\varphi_n}{\cos\varphi_{n+1}}\,\frac{\sqrt{1+\mathcal{V}^2_{n+1}}}{\sqrt{1+\mathcal{V}^2_n}} \\ &= \frac{\cos\varphi_n}{2\mathcal{K}_{n+1}\tau_{n+1}+\cos\varphi_{n+1}\big(1+\tau_{n+1}\mathcal{B}^-_{n+1}\big)}\,\frac{\sqrt{1+\mathcal{V}^2_{n+1}}}{\sqrt{1+\mathcal{V}^2_n}}, \end{aligned} \tag{6.4}$$

where $\tau_n = \tau(x_n)$, and $\mathcal{B}_n$, $\mathcal{K}_n$, etc. are taken at the point x_n; note that we used the mirror equation (2.9). Therefore

$$\begin{aligned} \ln \mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n) = \ln\cos\varphi_n &+ \tfrac12\ln(1+\mathcal{V}^2_{n+1}) - \tfrac12\ln(1+\mathcal{V}^2_n) \\ &- \ln\big[2\mathcal{K}_{n+1}\tau_{n+1}+\cos\varphi_{n+1}\big(1+\tau_{n+1}\mathcal{B}^-_{n+1}\big)\big]. \end{aligned} \tag{6.5}$$

Due to the uniform hyperbolicity of $\mathcal{F}$, cf. (3.10), $\mathrm{dist}(y_n, z_n) \leq \mathrm{diam}(W_n) \leq \hat{C}\Lambda^{-n}$, where $\hat{C} = 1/\hat{c}$. Thus,

$$\big|\ln \mathcal{J}_{W_n}\mathcal{F}^{-1}(y_n) - \ln \mathcal{J}_{W_n}\mathcal{F}^{-1}(z_n)\big| \leq \hat{C}\Lambda^{-n} \max\Big|\frac{d}{dx_n} \ln \mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n)\Big|$$

where the maximum is taken over all $x_n \in W_n$, and d/dx_n denotes the derivative with respect to the Euclidean length on W_n (we do not fix an orientation of W_n, since we only need the absolute value of this derivative).

Next we differentiate (6.5) with respect to x_n. Observe, however, that (6.5) contains functions that depend on x_{n+1}, but those can be handled by the chain rule

$$\frac{d}{dx_n} = \frac{dx_{n+1}}{dx_n}\,\frac{d}{dx_{n+1}} = \mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n)\,\frac{d}{dx_{n+1}}$$

(note also that the factor $\mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n)$ is uniformly bounded).

Now, we recall that the quantities φ, $\mathcal{K}$, $\mathcal{B}^-$, and $\mathcal{V}$ are $C^{\ell-2}$ smooth functions on unstable manifolds, with uniformly bounded derivatives, cf. Theorem 5.3, Remark 5.4, and Proposition 3.2, while τ_{n+1} is a C^ℓ smooth function of x_{n+1} and x_n, which easily follows from (4.2). Also, all the expressions on the right hand side of (6.5), except the first one, are uniformly bounded both below and above. Thus,

$$\frac{d}{dx_n} \ln \mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n) = \frac{Q_n(x_n)}{\cos\varphi_n} \tag{6.6}$$

where Q_n is a $C^{\ell-3}$ smooth uniformly bounded function of x_n with uniformly bounded derivatives. Due to (6.6)

$$\big|\ln \mathcal{J}_{W_n}\mathcal{F}^{-1}(y_n) - \ln \mathcal{J}_{W_n}\mathcal{F}^{-1}(z_n)\big| \leq \frac{\mathrm{const}\cdot\Lambda^{-n}}{\min_x \cos\varphi_n},$$

where the minimum is taken over all $x \in W$. To deal with the potentially small denominator here, we use our assumption that $r_*^u(x) > 0$ for some point $x \in W$ (and hence, for all points $x \in W$, see the proof of Proposition 5.9). We also restrict our analysis to the segment $W[y,z] \subset W$ of the curve W with endpoints y and z. Then we have $d^u(\mathcal{F}^{-n}x, \mathcal{S}_1) \geq c\hat{\Lambda}^{-n}$, hence $\cos\varphi_n > c\hat{\Lambda}^{-n}$, for all $n \geq 1$ and all $x \in W[y,z]$ with some $c = c(y,z) > 0$. We also recall that $\hat{\Lambda} < \Lambda$, hence $\hat{\Lambda}^{-n} \gg \Lambda^{-n}$, and so

$$\min_x \cos\varphi_n > \tfrac{1}{2}\, c\hat{\Lambda}^{-n},$$

thus the series (6.3) converges exponentially, and the u-SRB density ρ_W exists. $\square$

Thus, the u-SRB densities ρ_W exist on almost all unstable manifolds $W^u \subset \mathcal{M}$, but the ratio (6.2) may strongly depend on the points y and z, hence the densities may be very nonuniform. The fluctuations of ρ_W on W result from uneven contraction of the preimages $\mathcal{F}^{-n}(W)$ under the inverse map $\mathcal{F}^{-1}$, and the greatest differences in the contraction factor (strongest distortions of unstable manifolds by $\mathcal{F}^{-1}$) occur near the singularities $\mathcal{S}_0 = \{\cos\varphi = 0\}$.

To control distortions of unstable manifolds, Ya. Sinai proposed [BSC2] to partition unstable manifolds by countably many lines that are parallel to $\mathcal{S}_0$ and accumulate on $\mathcal{S}_0$. Let $k_0 \geq 1$ be a large constant. For each $k \geq k_0$ we define two *homogeneity strips* $\mathbb{H}_{\pm k} \subset \mathcal{M}$ by

$$\mathbb{H}_k = \{(r,\varphi)\colon \pi/2 - k^{-2} < \varphi < \pi/2 - (k+1)^{-2}\}$$

and

$$\mathbb{H}_{-k} = \{(r,\varphi)\colon -\pi/2 + (k+1)^{-2} < \varphi < -\pi/2 + k^{-2}\}.$$

We also put

$$\mathbb{H}_0 = \{(r,\varphi)\colon -\pi/2 + k_0^{-2} < \varphi < \pi/2 - k_0^{-2}\}. \tag{6.7}$$

Now $\mathcal{M}$ is divided into homogeneity strips $\mathbb{H}_k$. Denote by

$$\mathbb{S}_k = \{(r,\varphi)\colon |\varphi| = \pi/2 - k^{-2}\}$$

for $|k| \geq k_0$ the boundaries of the homogeneity strips and put

$$\mathbb{S} = \cup_{|k|\geq k_0} \mathbb{S}_k. \tag{6.8}$$

A stable or unstable curve $W \subset \mathcal{M}$ is said to be *weakly homogeneous* if W belongs to one strip $\mathbb{H}_k$. For any point $x = (r,\varphi)$ on a weakly homogeneous unstable or stable curve $W \subset \mathbb{H}_k$ we have

$$|W| \leq \text{ const } (|k|+1)^{-3} \leq \text{ const } \cos^{3/2}\varphi, \tag{6.9}$$

which easily follows from (3.3) and the definition of $\mathbb{H}_k$.

DEFINITION. An unstable manifold $W \subset \mathcal{M}$ is said to be *homogeneous* if $\mathcal{F}^{-n}(W)$ is weakly homogeneous for every $n \geq 0$. Similarly, a stable manifold

$W \subset \mathcal{M}$ is said to be homogeneous if $\mathcal{F}^n(W)$ is weakly homogeneous for every $n \geq 0$.

For brevity, we call homogeneous manifolds *H-manifolds*. Observe that given an unstable manifold $W \subset \mathcal{M}$, the connected components of $W \setminus \cup_{n\geq 0}\mathcal{F}^n(\mathbb{S})$ will be H-manifolds. Similarly, for any stable manifold $W \subset \mathcal{M}$ the connected components of $W \setminus \cup_{n\geq 0}\mathcal{F}^{-n}(\mathbb{S})$ will be stable H-manifolds. Thus, the lines $\mathbb{S}_k \subset \mathbb{S}$ and their images act as singularities. It is natural then to combine them with the existing singularity sets $\mathcal{S}_n$, $-\infty < n < \infty$, and treat the union of the two as new ('extended') singularities. To make this formal, it is proposed in [C2] to redefine the collision space $\mathcal{M}$ as follows.

We divide each connected component of the the old collision space $\mathcal{M}_i \subset \mathcal{M}$, recall (2.10) into countably many connected homogeneity strips $\mathbb{H}_{i,k} \colon = \mathcal{M}_i \cap \mathbb{H}_k$. The new collision space $\mathcal{M}_{\mathbb{H}}$ is defined to be a disjoint union of the closures of the $\mathbb{H}_{i,k}$'s. Observe that the new space $\mathcal{M}_{\mathbb{H}}$ is closed but no longer compact (it has countably many connected components). Also note that $\partial\mathcal{M}_{\mathbb{H}} = \mathbb{S}$.

The map $\mathcal{F}\colon \mathcal{M} \to \mathcal{M}$ naturally acts on the new collision space $\mathcal{M}_{\mathbb{H}}$, and we denote it by $\mathcal{F}_{\mathbb{H}}\colon \mathcal{M}_{\mathbb{H}} \to \mathcal{M}_{\mathbb{H}}$. The map $\mathcal{F}_{\mathbb{H}}$ lacks smoothness on the 'extended' singularity set $\mathcal{S}_1 \cup \mathbb{S} \cup \mathcal{F}^{-1}(\mathbb{S})$. Similarly, the map $\mathcal{F}_{\mathbb{H}}^n$ is not smooth on the 'extended' singularity set

$$\mathcal{S}_n^{\mathbb{H}} \colon = \mathcal{S}_n \cup \big(\cup_{m=0}^{n} \mathcal{F}^{-m}(\mathbb{S})\big). \tag{6.10}$$

The inverse map $\mathcal{F}_{\mathbb{H}}^{-n}$ is not smooth on the 'extended' singularity set

$$\mathcal{S}_{-n}^{\mathbb{H}} \colon = \mathcal{S}_{-n} \cup \big(\cup_{m=0}^{n} \mathcal{F}^{m}(\mathbb{S})\big). \tag{6.11}$$

Observe that $\mathcal{F}^{-1}(\mathbb{S})$ is a countable union of stable curves that are almost parallel to each other and accumulate on the old singular curves $\mathcal{S}_1 \setminus \mathcal{S}_0$. Thus, for every $n \geq 1$ the set $\mathcal{F}^{-n}(\mathbb{S})$ is a countable union of compact $C^{\ell-1}$ smooth stable curves. Similarly, $\mathcal{F}^n(\mathbb{S})$ is a countable union of compact $C^{\ell-1}$ smooth unstable curves. The results of Section 3 imply that all $\ell - 1$ derivatives of those curves are uniformly bounded.

Let $\mathcal{Q}_{-n}^{\mathbb{H}}(x)$ denote the connected component of the set $\mathcal{M} \setminus \mathcal{S}_{-n}^{\mathbb{H}}$ containing x. These new domains $\mathcal{Q}_{-n}^{\mathbb{H}}(x)$ are nested in the old domains $\mathcal{Q}_{-n}(x)$ used in Section 5 and have a similar shape. The arguments of that section carry over to these new domains without change and imply that the intersection $\cap_{n=1}^{\infty} \overline{\mathcal{Q}_{-n}^{\mathbb{H}}}(x)$ is a closed unstable curve. Let $W_{\mathbb{H}}^u(x)$ denote that curve without its endpoints. Then $W^u_{\mathbb{H}}(x)$ is the maximal unstable H-manifold passing through x. All the facts mentioned in the beginning of this section hold for unstable H-manifolds: the partition $\xi^u_{\mathbb{H}}$ of $\mathcal{M}$ into (maximal) unstable H-manifolds is measurable, and the corresponding conditional measures on the curves $W^u_{\mathbb{H}}$ are u-SRB measures whose density is determined by (6.2).

Due to the time reversibility, the definition and construction of stable H-manifolds $W^s_{\mathbb{H}}(x)$ are very similar. The partition $\xi^s_{\mathbb{H}}$ of $\mathcal{M}$ into (maximal) stable H-manifolds is also measurable.

Next we use the methods of Section 5 to estimate the length of the H-manifold $W^u_{\mathbb{H}}(x)$. Denote by $r^u_{\mathbb{H}}(x) = r_{W^u_{\mathbb{H}}(x)}(x)$ the distance, measured along $W^u_{\mathbb{H}}(x)$, from x to the nearest endpoint of $W^u_{\mathbb{H}}(x)$.

Theorem 6.4. *There is a constant $C = C(\mathcal{D}) > 0$ such that for all $\varepsilon > 0$.*

$$\mu\{x\colon r^u_{\mathbb{H}}(x) < \varepsilon\} \leq C\varepsilon. \tag{6.12}$$

Thus, the unstable H-manifolds, despite being much shorter than ordinary unstable manifolds, satisfy the same linear 'tail bound' (compare (5.10) and (6.12)!). We note, however, that there is no linear tail bound on the length of H-manifolds in the p-metric; in fact one can check directly that $\mu\{x\colon p^u_{\mathbb{H}}(x) < \varepsilon\} \sim \varepsilon^{4/5}$.

Proof. Our argument closely follows the proof of (5.15). First we consider the easier case of finite horizon. Let $E_u(x)$ be a piecewise constant function on $\mathcal{M}$ defined by $E_u(x) = (|k|+1)^2$ for all $x \in \mathcal{F}^{-1}(\mathbb{H}_k)$. There is a small constant $\tilde{c} = \tilde{c}(\mathcal{D}) > 0$ such that the factor of expansion of unstable vectors $dx \in \mathcal{T}_x\mathcal{M}$ for all $x \in \mathcal{M}$ will be between $\tilde{c}E_u(x)$ and $\tilde{c}^{-1}E_u(x)$, i.e.

$$\tilde{c}E_u(x) \leq \frac{\|D_x\mathcal{F}(dx)\|}{\|dx\|} \leq \tilde{c}^{-1}E_u(x).$$

Indeed, if $x \in \mathcal{F}^{-1}(\mathbb{H}_k)$, then $\cos\varphi \approx (|k|+1)^{-2}$ at the point $\mathcal{F}(x) \in \mathbb{H}_k$ and we use the relation (3.11) to conclude that $\|D_x\mathcal{F}(dx)\|/\|dx\| \asymp (|k|+1)^2$.

For every point $x \in \mathcal{M}$ denote by $d^u(x, \mathcal{S}_1^{\mathbb{H}})$ the length of the shortest unstable curve that connects x with the set $\mathcal{S}_1^{\mathbb{H}}$. Similarly to (5.17) we have

$$r^u_{\mathbb{H}}(x) \geq \min_{n\geq 1} \tilde{c}E_u(\mathcal{F}^{-n}x)\, \hat{c}\, \Lambda^{n-1}\, d^u(\mathcal{F}^{-n}x, \mathcal{S}_1^{\mathbb{H}}).$$

We pick $\hat{\Lambda} \in (1, \Lambda)$ and observe that

$$r^u_{\mathbb{H}}(x) \geq r^u_{\mathbb{H}*}(x)\colon = \min_{n\geq 1} \tilde{c}\, \hat{c}\, E_u(\mathcal{F}^{-n}x)\hat{\Lambda}^{n-1}\, d^u\big(\mathcal{F}^{-n}x, \mathcal{S}_1^{\mathbb{H}}\big).$$

For any $\varepsilon > 0$ let

$$\mathcal{U}^u_\varepsilon(\mathcal{S}^{\mathbb{H}}_1) = \{x\colon E_u(x)\, d^u(x, \mathcal{S}^{\mathbb{H}}_1) < \varepsilon\}.$$

Observe that $r^u_{\mathbb{H}*}(x) < \varepsilon$ iff $x \in \mathcal{F}^n\bigl(\mathcal{U}^u_{\varepsilon\tilde{c}^{-1}\hat{c}^{-1}\hat{\Lambda}^{-n+1}}(\mathcal{S}^{\mathbb{H}}_1)\bigr)$ for some $n \geq 1$. The proof of (5.18) can be easily adapted to show that $\mu\bigl(\mathcal{U}^u_\varepsilon(\mathcal{S}^{\mathbb{H}}_1)\bigr) < C\varepsilon$ for some constant $C > 0$ and all $\varepsilon > 0$; one should only observe that for every k the set $\mathcal{F}^{-1}(\mathbb{H}_k) \cap \mathcal{U}^u_\varepsilon(\mathcal{S}^{\mathbb{H}}_1)$ lies within the $\mathcal{O}\bigl((|k|+1)^{-2}\varepsilon\bigr)$-neighborhood of the boundary $\partial\mathcal{F}^{-1}(\mathbb{H}_k)$. It is also easy to check directly that the ε-neighborhood of the set $\mathbb{S}$ (which is a part of $\mathcal{S}^{\mathbb{H}}_1$) has μ-measure $\mathcal{O}(\varepsilon)$. Hence

$$\mu\bigl(\mathcal{F}^{-1}(\mathbb{H}_k) \cap \mathcal{U}^u_\varepsilon(\mathcal{S}^{\mathbb{H}}_1)\bigr) < \text{ const}\,(|k|+1)^{-2}\varepsilon.$$

It now follows that $\mu\{x\colon r^u_{\mathbb{H}*}(x) < \varepsilon\} \leq C\varepsilon$, hence we obtain (6.12) in the finite horizon case.

Now we indicate the changes in the proof for billiards without horizon. First, the function $E_u(x)$ must be redefined so that for each $x \in \mathcal{F}^{-1}(\mathbb{H}_k \cap D^-_{l,n})$ (equivalently, for each $x \in \mathcal{F}^{-1}(\mathbb{H}_k) \cap D^+_{l,n}$), see the definition of the n-cells $D^\pm_{l,n}$ in Section 4, we have $E_u(x) = n(|k|+1)^2$. Again, there is a small constant $\tilde{c} = \tilde{c}(\mathcal{D}) > 0$ such that the factor of expansion of unstable vectors $dx \in \mathcal{T}_x\mathcal{M}$ for all $x \in \mathcal{M}$ will be between $\tilde{c}E_u(x)$ and $\tilde{c}^{-1}E_u(x)$, i.e.

$$\tilde{c}E_u(x) \leq \frac{\|D_x\mathcal{F}(dx)\|}{\|dx\|} \leq \tilde{c}^{-1}E_u(x)$$

(we note that that $\tau(x) \asymp n$ for $x \in D^+_{l,n}$).

One more modification is required in the proof of the crucial estimate $\mu\bigl(\mathcal{U}^u_\varepsilon(\mathcal{S}^{\mathbb{H}}_1)\bigr) < C\varepsilon$. One can check directly that for each n-cell $D^+_{l,n}$ the set $\mathcal{F}^{-1}(\mathbb{H}_k) \cap D^+_{l,n}$ is a narrow strip stretching from top to bottom of this cell, see Fig. 11. The length of this cell is $\mathcal{O}\bigl(n^{-1/2}\bigr)$, see Section 4, hence

$$\mu\bigl(\mathcal{F}^{-1}(\mathbb{H}_k) \cap D^+_{l,n} \cap \mathcal{U}^u_\varepsilon(\mathcal{S}^{\mathbb{H}}_1)\bigr) < \text{ const}\, n^{-2}(|k|+1)^{-2}\varepsilon$$

(the additional factor of $n^{-1/2}$ results from the density $\cos\varphi = \mathcal{O}\bigl(n^{-1/2}\bigr)$ of the measure μ on $D^+_{l,n}$, just as in our early estimate (5.19)). Now adding these estimates over all k and n completes the proof of Theorem 6.4. $\square$

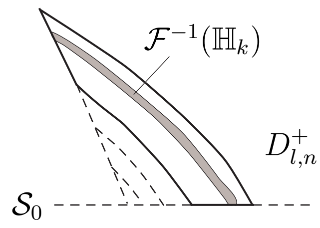

Figure 11: The set $\mathcal{F}^{-1}(\mathbb{H}_k) \cap D^+_{l,n}$ within the cell $D^+_{l,n}$ for billiards without horizon.

7 Distortion bounds

Here we derive bounds on distortions and on the derivatives of u-SRB densities on unstable H-manifolds. Such bounds are absolutely necessary in the studies of ergodic and statistical properties of billiards. Our bounds here are sharper than those obtained previously [S2, BSC2, CD].

Our first theorem will apply to rather generic unstable curves (provided they are homogeneous in the appropriate sense), but further theorems will be restricted to unstable H-manifolds.

Let W be an unstable curve such that $W_n = \mathcal{F}^{-n}(W)$ is a weakly homogeneous unstable curve for all $0 \leq n \leq N-1$. We assume that the curves W_n are $C^{\ell-1}$ smooth and all their $\ell - 1$ derivatives are uniformly bounded by constants C_ν, $1 \leq \nu \leq \ell - 1$ involved in Theorem 3.1.

We use the notation of the previous section, in particular for every point $x \in W$ we put $x_n = \mathcal{F}^{-n}(x)$ and denote by $\mathcal{J}_W\mathcal{F}^{-n}(x)$ the factor of contraction of W under the map $\mathcal{F}^{-n}$ at x.

Theorem 7.1 (Distortion bounds). *For every $y, z \in W$ and every $1 \leq n \leq N$ we have*

$$C_{\mathrm{d}} \leq e^{-C|W|^{1/3}} \leq \frac{\mathcal{J}_W\mathcal{F}^{-n}(y)}{\mathcal{J}_W\mathcal{F}^{-n}(z)} \leq e^{C|W|^{1/3}} \leq C_{\mathrm{d}},$$

where $C = C(\mathcal{D}) > 0$ and $C_{\mathrm{d}} = C_{\mathrm{d}}(\mathcal{D}) > 0$ are constants (the subscript d *in C_{d} stands for 'distortions').*

Note that, obviously, $|W|$ can be replaced here by the distance between the points y and z.

Proof. Taking the logarithm and using the chain rule gives

$$\begin{aligned}\left|\ln \mathcal{J}_W \mathcal{F}^{-n}(y) - \ln \mathcal{J}_W \mathcal{F}^{-n}(z)\right| &\leq \sum_{m=0}^{n-1} \left|\ln \mathcal{J}_{W_m} \mathcal{F}^{-1}(y_m) - \ln \mathcal{J}_{W_m} \mathcal{F}^{-1}(z_m)\right| \\ &\leq \sum_{m=0}^{n-1} |W_m| \, \max \left| \frac{d}{dx_m} \ln \mathcal{J}_{W_m} \mathcal{F}^{-1}(x_m)\right| \\ &\leq \text{ const} \sum_{m=0}^{n-1} |W_m| / \cos\varphi_m \\ &\leq \text{ const} \sum_{m=0}^{n-1} |W_m|^{1/3},\end{aligned}$$

in the last two steps we used (6.6) and (6.9). Note that we need uniform bounds on the derivatives of W_n's to use (6.6). Now, due to the uniform hyperbolicity of $\mathcal{F}$, we have $\sum_{m=0}^{n-1} |W_n|^{1/3} \leq \text{ const} \, |W|^{1/3}$. □

Corollary 7.2. *For every $x \in W$ and every $1 \leq n \leq N$*

$$C_{\mathrm{d}}^{-1} \leq \frac{\mathcal{J}_W \mathcal{F}^{-n}(x)}{|W_n|/|W|} \leq C_{\mathrm{d}}.$$

The following two theorems are devoted to the regularity of u-SRB densities.

Theorem 7.3. *There is a constant $C = C(\mathcal{D}) > 0$ such that for every unstable H-manifold $W \subset \mathcal{M}$*

$$\left|\frac{d}{dx} \ln \rho_W(x)\right| \leq \frac{C}{|W|^{2/3}}.$$

Proof. Fix a point $\bar{x} \in W$, then due to (6.2) and the chain rule

$$\ln \rho_W(x) = \ln \rho_W(\bar{x}) + \sum_{n=0}^{\infty} \big(\ln \mathcal{J}_{W_n} \mathcal{F}^{-1}(x_n) - \ln \mathcal{J}_{W_n} \mathcal{F}^{-1}(\bar{x}_n)\big),$$

(the exponential convergence of this series was established in the previous section). Hence

$$\frac{d}{dx}\ln\rho_W(x) = \sum_{n=0}^{\infty}\frac{d}{dx}\,\ln\mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n)$$
$$= \sum_{n=0}^{\infty}\frac{dx_n}{dx}\,\frac{d}{dx_n}\,\ln\mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n)$$
$$= \sum_{n=0}^{\infty}\mathcal{J}_W\mathcal{F}^{-n}(x)\,\frac{Q_n(x_n)}{\cos\varphi_n} \tag{7.1}$$

(provided this series converges uniformly in x, which will follow from the subsequent analysis); here Q_n is a $C^{\ell-3}$ smooth uniformly bounded function of x_n with uniformly bounded derivatives, cf. (6.6). Therefore,

$$\left|\frac{d}{dx}\ln\rho_W(x)\right| \leq \text{ const}\sum_{n=0}^{\infty}\frac{\mathcal{J}_W\mathcal{F}^{-n}(x)}{\cos\varphi_n}$$
$$\leq \text{ const}\sum_{n=0}^{\infty}\frac{|W_n|/|W|}{|W_n|^{2/3}}$$
$$\leq \text{ const}\sum_{n=0}^{\infty}\frac{\Lambda^{-n/3}}{|W|^{2/3}} \tag{7.2}$$

where we used (6.9), Corollary 7.2, and the uniform hyperbolicity of $\mathcal{F}$, cf. (3.10). Theorem 7.3 is proved. □

Corollary 7.4. *For every $x, y \in W$*

$$C_{\mathrm{d}} \leq e^{-C|W|^{1/3}} \leq \frac{\rho_W(x)}{\rho_W(y)} \leq e^{C|W|^{1/3}} \leq C_{\mathrm{d}},$$

where $C = C(\mathcal{D}) > 0$ is a constant.

Theorem 7.3 can be generalized to higher order derivatives:

Theorem 7.5. *For every $\nu = 1, \ldots, \ell-2$ there is a constant $C_\nu = C_\nu(\mathcal{D}) > 0$ such that*

$$\left|\frac{d^\nu}{dx^\nu}\ln\rho_W(x)\right| \leq \frac{C_\nu}{|W|^{2\nu/3}}.$$

Proof. For $\nu = 1$ see the previous theorem. For $\nu \geq 2$ we need to subsequently differentiate (7.1). First observe that

$$\begin{aligned} \frac{d}{dx_n}\,\mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n) &= \mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n)\,\frac{d}{dx_n}\,\ln \mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n) \\ &= \mathcal{J}_{W_n}\mathcal{F}^{-1}(x_n)\,\frac{Q_n(x_n)}{\cos\varphi_n}, \end{aligned}$$

hence

$$\begin{aligned} \frac{d}{dx}\,\mathcal{J}_W\mathcal{F}^{-n}(x) &= \frac{d}{dx}\prod_{m=0}^{n-1}\mathcal{J}_{W_m}\mathcal{F}^{-1}(x_m) \\ &= \sum_{m=0}^{n-1}\frac{\mathcal{J}_W\mathcal{F}^{-n}(x)}{\mathcal{J}_{W_m}\mathcal{F}^{-1}(x_m)}\,\frac{dx_m}{dx}\,\frac{d}{dx_m}\,\mathcal{J}_{W_m}\mathcal{F}^{-1}(x_m) \\ &= \mathcal{J}_W\mathcal{F}^{-n}(x)\sum_{m=0}^{n-1}\mathcal{J}_W\mathcal{F}^{-m}(x)\,\frac{Q_m(x_m)}{\cos\varphi_m}. \end{aligned}$$

Also,

$$\begin{aligned} \frac{d}{dx}\,\frac{Q_n(x_n)}{\cos\varphi_n} &= \frac{dx_n}{dx}\,\frac{d}{dx_n}\,\frac{Q_n(x_n)}{\cos\varphi_n} \\ &= \mathcal{J}_W\mathcal{F}^{-n}(x)\left[\frac{Q_n'(x_n)}{\cos\varphi_n} + \frac{Q_n(x_n)\sin\varphi_n}{\cos^2\varphi_n}\,\frac{d\varphi_n}{dx_n}\right], \end{aligned}$$

where

$$\frac{d\varphi_n}{dx_n} = \frac{d\varphi_n}{\sqrt{dr_n^2 + d\varphi_n^2}} = \frac{\mathcal{V}_n}{\sqrt{1+\mathcal{V}_n^2}}$$

is a $C^{\ell-2}$ smooth uniformly bounded function with uniformly bounded derivatives along unstable manifolds, see the notation of (6.4) and (6.5). Hence

$$\frac{d}{dx}\,\frac{Q_n(x_n)}{\cos\varphi_n} = \mathcal{J}_W\mathcal{F}^{-n}(x)\,\frac{Q_n^{(1)}(x_n)}{\cos^2\varphi_n},$$

where $Q_n^{(1)}$ is a $C^{\ell-4}$ smooth uniformly bounded function of x_n with uniformly bounded derivatives along unstable manifolds. We note that the smoothness of the functions involved in our analysis decreases as we estimate the derivatives of $\ln\rho_W(x)$. Thus, bounds on higher derivatives require higher smoothness of the billiard table $\mathcal{D}$.

Combining the above estimate gives

$$\frac{d^2}{dx^2}\ln\rho_W(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n-1}\mathcal{J}_W\mathcal{F}^{-n}(x)\,\mathcal{J}_W\mathcal{F}^{-m}(x)\,\frac{Q_n(x_n)\,Q_m(x_m)}{\cos\varphi_n\,\cos\varphi_m}$$

$$+\sum_{n=0}^{\infty}\bigl[\mathcal{J}_W\mathcal{F}^{-n}(x)\bigr]^2\,\frac{Q_n^{(1)}(x_n)}{\cos^2\varphi_n}. \tag{7.3}$$

Repeating the estimates in (7.2) we obtain

$$\left|\frac{d^2}{dx^2}\ln\rho_W(x)\right| \leq \text{const}\left(\sum_{n=0}^{\infty}\frac{\Lambda^{-n/3}}{|W|^{2/3}}\right)^2 \leq \frac{\text{const}}{|W|^{4/3}},$$

thus proving the theorem for $\nu = 2$. The proof for $\nu \geq 3$ is a straightforward extension of the above arguments. For example, in the case $\nu = 3$ the third derivative can be expressed in a way similar to (7.3), where the most important term of the respective formula will look like

$$\sum_{n=0}^{\infty}\bigl[\mathcal{J}_W\mathcal{F}^{-n}(x)\bigr]^3\,\frac{Q_n^{(2)}(x_n)}{\cos^3\varphi_n},$$

where $Q_n^{(2)}$ will be a $C^{\ell-5}$ smooth uniformly bounded function of x_n with uniformly bounded derivatives along unstable manifolds. Further analysis is straightforward. □

We note that a limited version of Theorem 7.5 (only for $\nu = 1$ and $\nu = 2$) was proved in [CD, Appendix B], by somewhat different techniques.

8 Absolute continuity

Let $W^1, W^2 \subset \mathcal{M}$ be two unstable curves. Denote $W^i_* = \{x \in W^i \colon W^s(x) \cap W^{3-i} \neq \emptyset\}$ for $i = 1, 2$. The map $\mathbf{h}\colon W^1_* \to W^2_*$ taking every point $x \in W^1_*$ to $\bar{x} = W^s(x) \cap W^2$ is called the *holonomy map*. This map is often described as sliding along stable manifolds (see Fig. 12). We note that W^1 and W^2 are arbitrary unstable curves, but $W^s(x)$ has to be a stable H-manifold.

The sets W^i_* are nowhere dense on W^i, but if W^1 and W^2 are close enough, the sets W^i_* have positive measure on W^i. It is well known (see, e.g., [AS, Equation (5.3)], or [S2], or a modern version in [BP, Theorem 4.4.1]) that

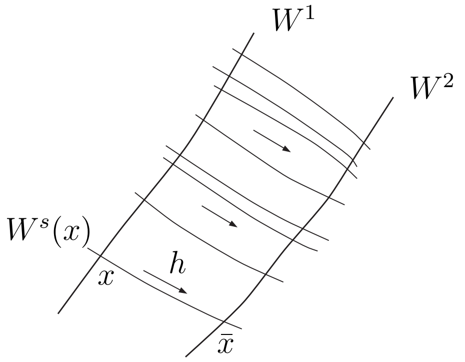

Figure 12: Holonomy map.

the holonomy map $\mathbf{h}$ is absolutely continuous with respect to the Lebesgue measures m_1 and m_2 on the curves W^1 and W^2, respectively, and its Jacobian is given by

$$J\mathbf{h}(x) = \frac{d\mathbf{h}^{-1}(m_2)}{dm_1}(x) = \lim_{n\to\infty} \frac{\mathcal{J}_{W^1}\mathcal{F}^n(x)}{\mathcal{J}_{W^2}\mathcal{F}^n\big(\mathbf{h}(x)\big)}. \tag{8.1}$$

The general theory only guarantees the finiteness of $J\mathbf{h}(x)$. Certain 'crude' estimates on $J\mathbf{h}(x)$ were obtained early in [S2, Section 5] and [Ga, Section 4] in the proofs of ergodicity. In later studies of statistical properties of billiards, uniform bounds on $J\mathbf{h}(x)$ were required [C2], and sometimes a much finer control on $J\mathbf{h}(x)$ was necessary [CD], and to achieve it, the holonomy map $\mathbf{h}$ had to be restricted to stable H-manifolds:

Definition. Let $W^1, W^2 \subset \mathcal{M}$ be two unstable curves. Denote

$$W^i_* = \{x \in W^i\colon W^s_{\mathbb{H}}(x) \cap W^{3-i} \neq \emptyset\}$$

for $i = 1, 2$, where $W^s_{\mathbb{H}}(x)$ is the stable H-manifold passing through x. The map $\mathbf{h}\colon W^1_* \to W^2_*$ taking every point $x \in W^1_*$ to $\bar{x} = W^s_{\mathbb{H}}(x) \cap W^2$ is called the *(modified) holonomy map.*

We derive sharp bounds on the Jacobian $J\mathbf{h}(x)$ of the (modified) holonomy map. We assume that W^1 and W^2 are $C^{\ell-1}$ smooth curves, and at least their first and second derivatives are uniformly bounded (then their future images will have uniformly bounded derivatives as well, due to Theorem 3.1).

Let $x \in W^1_*$ and $\bar{x} = \mathbf{h}(x) \in W^2_*$. We put $\delta = \operatorname{dist}(x, \bar{x})$ and denote by γ the angle between the tangent vectors to the curves W^1 and W^2 at the points x and $\bar{x}$, respectively.

Theorem 8.1. *The Jacobian of the holonomy map* $\mathbf{h}$ *is uniformly bounded*

$$C^{-1} \le J\mathbf{h}(x) \le C,$$

for all $x \in W^1_*$*, here* $C = C(\mathcal{D}) > 1$ *is a constant. Moreover,*

$$A^{-\gamma-\delta^{1/3}} \le J\mathbf{h}(x) \le A^{\gamma+\delta^{1/3}}, \tag{8.2}$$

where $A = A(\mathcal{D}) > 1$ *is a constant.*

Proof. We denote $W^i_n = \mathcal{F}^n(W^i)$ for $i = 1, 2$ and $n \ge 1$. We also denote $x_n = \mathcal{F}^n(x)$ and $\bar{x}_n = \mathcal{F}^n(\bar{x})$ and put $\delta_n = \operatorname{dist}(x_n, \bar{x}_n)$. Due to the uniform hyperbolicity of $\mathcal{F}$, cf. (3.10), $\delta_n \le \hat{C}\delta\Lambda^{-n}$, where $\hat{C} = 1/\hat{c}$.

Taking the logarithm of (8.1) and using the chain rule gives

$$\ln J\mathbf{h}(x) = \sum_{n=0}^{\infty}\Big(\ln \mathcal{J}_{W^1_n}\mathcal{F}(x_n) - \ln \mathcal{J}_{W^2_n}\mathcal{F}(\bar{x}_n)\Big). \tag{8.3}$$

Due to (3.9), (3.6), and (2.12)

$$\begin{aligned}\mathcal{J}_{W^1_n}\mathcal{F}(x_n) &= (1 + \tau_n\mathcal{B}^+_n)\,\frac{\cos\varphi_n}{\cos\varphi_{n+1}}\,\frac{\sqrt{1+\mathcal{V}^2_{n+1}}}{\sqrt{1+\mathcal{V}^2_n}}\\ &= \frac{\cos\varphi_n + \tau_n(\mathcal{K}_n + \mathcal{V}_n)}{\cos\varphi_{n+1}}\,\frac{\sqrt{1+\mathcal{V}^2_{n+1}}}{\sqrt{1+\mathcal{V}^2_n}},\end{aligned}$$

where, as usual, $\tau_n = \tau(x_n)$, and $\mathcal{B}_n$, $\mathcal{K}_n$, etc. are taken at the point x_n. Therefore

$$\begin{aligned}\ln \mathcal{J}_{W^1_n}\mathcal{F}(x_n) &= -\ln\cos\varphi_{n+1} + \tfrac{1}{2}\ln(1+\mathcal{V}^2_{n+1}) - \tfrac{1}{2}\ln(1+\mathcal{V}^2_n)\\ &\quad + \ln\big[\cos\varphi_n + \tau_n(\mathcal{K}_n + \mathcal{V}_n)\big].\end{aligned} \tag{8.4}$$

Using similar notation at $\bar{x}_n$ we get

$$\begin{aligned}\ln \mathcal{J}_{W^2_n}\mathcal{F}(\bar{x}_n) &= -\ln\cos\bar{\varphi}_{n+1} + \tfrac{1}{2}\ln(1+\bar{\mathcal{V}}^2_{n+1}) - \tfrac{1}{2}\ln(1+\bar{\mathcal{V}}^2_n)\\ &\quad + \ln\big[\cos\bar{\varphi}_n + \bar{\tau}_n(\bar{\mathcal{K}}_n + \bar{\mathcal{V}}_n)\big].\end{aligned} \tag{8.5}$$

Comparing the first terms of the above expressions gives

$$\begin{aligned}\big|\ln\cos\varphi_{n+1} - \ln\cos\bar{\varphi}_{n+1}\big| &\le \frac{\text{const}\,|\varphi_{n+1} - \bar{\varphi}_{n+1}|}{\cos\varphi_{n+1}}\\ &\le \frac{\text{const}\,\delta_{n+1}}{\delta^{2/3}_{n+1}} = \text{const}\,\delta^{1/3}_{n+1}\end{aligned}$$

where we applied (6.9) to our stable manifolds. All the other terms in (8.4) and (8.5) are uniformly bounded and differentiable (with respect to their arguments $\mathcal{V}_{n+1}$, φ_n, τ_n, etc.), thus it easily follows that

$$\big|\ln \mathcal{J}_{W_n^1}\mathcal{F}(x_n) - \ln \mathcal{J}_{W_n^2}\mathcal{F}(\bar{x}_n)\big| \leq C\big(\delta_{n+1}^{1/3} + \gamma_n + \gamma_{n+1} + \delta_n\big). \tag{8.6}$$

where $C > 0$ is a constant and $\gamma_n = |\Delta\mathcal{V}_n| = |\mathcal{V}_n - \bar{\mathcal{V}}_n|$ is equivalent to the angle between the tangent vectors to the curves W_n^1 and W_n^2 at the points x_n and $\bar{x}_n$, respectively (note that $\gamma_0 = \gamma$).

It is easy to check that for any stable curve W^s we have $|\mathcal{F}^{-1}(W^s)| \leq C\sqrt{|W^s|}$, where $C = C(\mathcal{D}) > 0$ is a constant (we can apply (6.9) to our stable curves). Thus $\delta_n \leq C\delta_{n+1}^{1/2}$, hence the term δ_n in (8.6) may be dropped. It remains to estimate γ_n for all $n \geq 0$. For brevity, we denote $\Delta\mathcal{B}_n^- = \mathcal{B}_n^- - \bar{\mathcal{B}}_n^-$, $\Delta\tau_n = \tau_n - \bar{\tau}_n$, etc. First we estimate $\Delta\mathcal{B}_{n+1}^-$ by using (2.8) and the obvious relation $A^{-1} - B^{-1} = A^{-1}B^{-1}(B - A)$:

$$\begin{aligned}
\Delta\mathcal{B}_{n+1}^- &= \frac{1}{\tau_n + 1/\mathcal{B}_n^+} - \frac{1}{\bar{\tau}_n + 1/\bar{\mathcal{B}}_n^+} \\
&= -\frac{1}{\tau_n + 1/\mathcal{B}_n^+}\,\frac{1}{\bar{\tau}_n + 1/\bar{\mathcal{B}}_n^+}\left(\Delta\tau_n + \frac{1}{\mathcal{B}_n^+} - \frac{1}{\bar{\mathcal{B}}_n^+}\right) \\
&= -\frac{\Delta\tau_n}{\big(\tau_n + 1/\mathcal{B}_n^+\big)\big(\bar{\tau}_n + 1/\bar{\mathcal{B}}_n^+\big)} + \frac{\Delta\mathcal{R}_n + \Delta\mathcal{B}_n^-}{\big(1 + \tau_n\mathcal{B}_n^+\big)\big(1 + \bar{\tau}_n\bar{\mathcal{B}}_n^+\big)} \qquad (8.7)
\end{aligned}$$

(recall that $\mathcal{B}_n^+ = \mathcal{R}_n + \mathcal{B}_n^-$). The first term in (8.7) is bounded by

$$\text{const}\,\Delta\tau_n \leq \text{const}(\delta_n + \delta_{n+1}) \leq \text{const}\,\delta_n. \tag{8.8}$$

Next,

$$\frac{\Delta\mathcal{R}_n}{\big(1 + \tau_n\mathcal{B}_n^+\big)\big(1 + \bar{\tau}_n\bar{\mathcal{B}}_n^+\big)} \leq \frac{2\mathcal{K}_n\cos\bar{\varphi}_n - 2\bar{\mathcal{K}}_n\cos\varphi_n}{\big(\cos\varphi_n + \tau_n(\mathcal{K}_n + \mathcal{V}_n)\big)\big(\cos\bar{\varphi}_n + \bar{\tau}_n(\bar{\mathcal{K}}_n + \bar{\mathcal{V}}_n)\big)}$$

Since the denominator is bounded away from zero, the fraction does not exceed $\text{const}\,\delta_n$, thus (8.7) reduces to

$$\Delta\mathcal{B}_{n+1}^- = \tilde{Q}_n^{(1)} + \frac{\Delta\mathcal{B}_n^-}{\big(1 + \tau_n\mathcal{B}_n^+\big)\big(1 + \bar{\tau}_n\bar{\mathcal{B}}_n^+\big)},$$

where $|\tilde{Q}_n^{(1)}| \leq \text{const}\,\delta_n$. Next, due to (2.12)

$$\begin{aligned}
\Delta\mathcal{V}_{n+1} &= \Delta\mathcal{K}_{n+1} + \mathcal{B}_{n+1}^-\,\cos\varphi_{n+1} - \bar{\mathcal{B}}_{n+1}^-\,\cos\bar{\varphi}_{n+1} \\
&= \tilde{Q}_{n+1}^{(2)} + \cos\varphi_{n+1}\,\Delta\mathcal{B}_{n+1}^-,
\end{aligned}$$

where $|\tilde{Q}^{(2)}_{n+1}| \leq \text{ const}\,\delta_{n+1}$. Combining the last two estimates gives

$$\Delta\mathcal{V}_{n+1} = \tilde{Q}^{(3)}_n + \frac{\cos\varphi_{n+1}}{\cos\varphi_n}\,\frac{\Delta\mathcal{V}_n}{\bigl(1+\tau_n\mathcal{B}^+_n\bigr)\bigl(1+\bar{\tau}_n\bar{\mathcal{B}}^+_n\bigr)},$$

where $|\tilde{Q}^{(3)}_n| \leq \text{ const}\,\delta_n$. Now consider the fraction

$$u_n\colon = \frac{\cos\varphi_{n+1}}{\cos\varphi_n\bigl(1+\tau_n\mathcal{B}^+_n\bigr)\bigl(1+\bar{\tau}_n\bar{\mathcal{B}}^+_n\bigr)}.$$

One easily verifies that for all $0 \leq k \leq n$

$$u_n u_{n-1}\cdots u_k \leq C/\Lambda^{n-k}$$

where $C = C(\mathcal{D}) > 0$ is a constant (to this end we recall that

$$\mathcal{B}^+_n\cos\varphi_n = 2\mathcal{K}_n + \mathcal{B}^-_n\cos\varphi_n \geq 2\mathcal{K}_{\min} > 0$$

and $1+\bar{\tau}_n\bar{\mathcal{B}}^+_n \geq \Lambda$). Therefore

$$\begin{aligned}\gamma_n = |\Delta\mathcal{V}_n| &\leq \text{ const}\biggl(\sum_{k=0}^{n}\delta_k/\Lambda^{n-k} + \gamma/\Lambda^n\biggr)\\ &\leq \text{ const}\Bigl(\delta n/\Lambda^n + \gamma/\Lambda^n\Bigr) \qquad (8.9)\end{aligned}$$

(we also recall that $\delta_k \leq \hat{C}\delta/\Lambda^k$ due to the uniform hyperbolicity).

Combining (8.9) with (8.6) yields

$$\Bigl|\ln\mathcal{J}_{W^1_n}\mathcal{F}(x_n) - \ln\mathcal{J}_{W^2_n}\mathcal{F}(\bar{x}_n)\Bigr| \leq C\biggl(\frac{\delta^{1/3}}{\Lambda^{n/3}} + \frac{\delta n}{\Lambda^n} + \frac{\gamma}{\Lambda^n}\biggr), \qquad (8.10)$$

where $C = C(\mathcal{D}) > 0$ is a constant. Summing up over n gives

$$\bigl|\ln J\mathbf{h}(x)\bigr| \leq \text{ const}\bigl(\gamma + \delta^{1/3}\bigr),$$

which complete the proof of the theorem. □

The main estimate (8.2) involves two parameters, δ and γ. In most applications, though, the curves W^1 and W^2 are disjoint, and then γ is unnecessary, as we show next. Let

$$W^1_\diamond = \{x \in W^1_*\colon r_{W^1}(x) > D\delta^{1/2},\ r_{W^2}(\bar{x}) > D\delta^{1/2}\},$$

where $D > 0$ is a sufficiently large constant. This is the set of points of W^1_* that are at least $D\delta^{1/2}$-away from the endpoints of both curves.

Lemma 8.2. *If $W^1 \cap W^2 \neq \emptyset$, then for every $x \in W^1_\diamond$ we have $\gamma \leq C\delta^{1/2}$, where $C = C(\mathcal{D}) > 0$ is a constant.*

Proof. It follows from the uniform bounds on the first and second derivatives of the curves W^1 and W^2 that if the claim failed, then W^1 and W^2 would have to cross each other, thus contradicting our assumption. $\square$

So for all the points $x \in W^1_\diamond$ we have $\gamma \ll \delta^{1/3}$, hence the estimate (8.2) takes a simpler form: $A^{-\delta^{1/3}} \leq J\mathbf{h}(x) \leq A^{\delta^{1/3}}$.

It is easy to prove that the Jacobian $J\mathbf{h}(x)$ is a continuous function on W^1_*. Indeed, the series (8.3) converges uniformly in x due to (8.10) and every term depends on x continuously. It is more important to investigate the regularity of the Jacobian $J\mathbf{h}(x)$.

The Jacobian $J\mathbf{h}(x)$ is not necessarily Hölder continuous (unlike its counterpart in Anosov and Axiom A systems), but it has a similar property sometimes called 'dynamically defined Hölder continuity' [Y, p. 597], which we describe below.

Let $\mathcal{Q}^{\mathbb{H}}_n(x)$ again denote the open connected component of the set $\mathcal{M} \setminus \mathcal{S}^{\mathbb{H}}_n$ containing the point x. For two points $x, y \in \mathcal{M}$ denote by

$$\mathbf{s}_+(x,y) = \min\{n \geq 0 \colon y \notin \mathcal{Q}^{\mathbb{H}}_n(x)\} \tag{8.11}$$

the 'separation time' (the images $\mathcal{F}^n(x)$ and $\mathcal{F}^n(y)$ get separated at time $n = s_+(x,y)$ as they lie in different connected components of the new collision space $\mathcal{M}_{\mathbb{H}}$). Clearly, this function is symmetric: $\mathbf{s}_+(x,y) = \mathbf{s}_+(y,x)$. If $y \in \mathcal{Q}^{\mathbb{H}}_n(x)$ for all $n \geq 0$, then $y \in W^s_{\mathbb{H}}(x)$; in that case we set $\mathbf{s}_+(x,y) = \infty$. Observe that if x and y lie on one unstable curve $W \subset \mathcal{M}$, then

$$\operatorname{dist}(x,y) \leq C\Lambda^{-\mathbf{s}_+(x,y)} \tag{8.12}$$

for some constant $C = C(\mathcal{D}) > 0$. In fact, $\operatorname{dist}(x,y)$ can be much smaller than $\Lambda^{-\mathbf{s}_+(x,y)}$, due to unbounded expansion.

Proposition 8.3. *There are constants $C > 0$ and $\boldsymbol{\theta} \in (0,1)$ such that*

$$|\ln J\mathbf{h}(x) - \ln J\mathbf{h}(y)| \leq C\boldsymbol{\theta}^{\mathbf{s}_+(x,y)}.$$

Proof. Denote $\bar{x} = \mathbf{h}(x)$ and $\bar{y} = \mathbf{h}(y)$. Observe that $\mathbf{s}_+(\bar{x},\bar{y}) = \mathbf{s}_+(x,y)$; this follows from the continuation of singularity lines (Section 4). Using the notation of (8.3) and the triangle inequality gives

$$\begin{aligned}\Delta\colon &= |\ln J\mathbf{h}(x) - \ln J\mathbf{h}(y)| \\ &\le \sum_{n=0}^{\infty} |\ln \mathcal{J}_{W_n^1}\mathcal{F}(x_n) - \ln \mathcal{J}_{W_n^2}\mathcal{F}(\bar{x}_n) - \ln \mathcal{J}_{W_n^1}\mathcal{F}(y_n) + \ln \mathcal{J}_{W_n^2}\mathcal{F}(\bar{x}_n)|\end{aligned}$$

Let $m = \mathbf{s}_+(x,y)/2$. Then we apply the estimate (8.10) to all $n > m$ and Theorem 7.1 (on distortion bounds) to all $n \le m$. Note that we group terms in two different manners for $n > m$ and for $n \le m$. This proves our claim with $\boldsymbol{\theta} = \Lambda^{-1/6}$. $\square$

We emphasize that the regularity of the Jacobin $J\mathbf{h}$ can only be expressed in terms of the 'dynamical distance' between the points on the unstable curve W^1, this is dictated by the application of distortion bounds (Theorem 7.1).

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