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Gauge Transformations and Inverse Quantum Scattering with Medium-Range Magnetic Fields

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Abstract

The time-dependent, geometric method for high-energy limits and inverse scattering is applied to nonrelativistic quantum particles in external electromagnetic fields. Both the Schrödinger- and the Pauli equations in $\mathbb{R}^2$ and $\mathbb{R}^3$ are considered. The electrostatic potential A_0 shall be short-range, and the magnetic field $\mathbf{B}$ shall decay faster than $|\mathbf{x}|^{-3/2}$. A natural class of corresponding vector potentials $\mathbf{A}$ of medium range is introduced, and the decay and regularity properties of various gauges are discussed, including the transversal gauge, the Coulomb gauge, and the Griesinger vector potentials. By a suitable combination of these gauges, $\mathbf{B}$ need not be differentiable. The scattering operator S is not invariant under the corresponding gauge transformations, but experiences an explicit transformation. Both $\mathbf{B}$ and A_0 are reconstructed from an X-ray transform, which is obtained from the high-energy limit of S. Here previous results by Arians and Nicoleau are generalized to the medium-range situation. In a sequel paper, medium-range vector potentials are applied to relativistic scattering.

Keywords: inverse scattering, gauge transformation, vector potential.
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1 Introduction

The scattering theory of a nonrelativistic quantum particle in an electromagnetic field will be discussed under weak decay- and regularity assumptions on the magnetic field. Consider first the corresponding classical dynamics, i.e., the Lorentz force:

$$m\ddot{\mathbf{x}} = e(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \;, \tag{1}$$

where m is the mass and e is the charge of the particle, $\mathbf{E}(\mathbf{x})$ is electrostatic field strength, and $\mathbf{B}(\mathbf{x})$ is the magnetostatic field. More precisely, $\mathbf{B} = \mu_0 \mathbf{H}$ is the magnetic flux density, and $\mathbf{H}$ is the magnetic field strength. The field is described in terms of a scalar potential $A_0(\mathbf{x})$ and a vector potential $\mathbf{A}(\mathbf{x})$ according to $\mathbf{E} = -\,\mathbf{grad}\, A_0$ and $\mathbf{B} = \mathbf{curl}\,\mathbf{A}$. (Note that there is an alternative system of units of measure in use, such that $\mathbf{B} = \mathbf{H} = c^{-1}\,\mathbf{curl}\,\mathbf{A}$, where c is the speed of light.) Now (1) is equivalent to a Hamiltonian dynamic system with the Hamilton function

$$H(\mathbf{x},\, \mathbf{p}) := \frac{1}{2m}\Big(\mathbf{p} - e\mathbf{A}(\mathbf{x})\Big)^2 + eA_0(\mathbf{x}) \;, \tag{2}$$

where $\mathbf{p} = m\dot{\mathbf{x}} + e\mathbf{A}(\mathbf{x})$ is the canonical momentum. A nonrelativistic quantum particle is described by a wave function $\psi(\mathbf{x}) \in L^2(\mathbb{R}^\nu,\, \mathbb{C})$. Its time evolution is determined by the Schrödinger equation $\mathrm{i}\hbar\dot{\psi} = H\psi$. The self-adjoint Hamiltonian H is given by (2), with the canonical momentum operator $\mathbf{p} = -\mathrm{i}\hbar\nabla_{\mathbf{x}}$. We set $\hbar = 1$ and $e = 1$. The Schrödinger operator is describing a spin-0 particle, and the similar Pauli operator (53) is describing a particle of spin 1/2, e.g., an electron. If $A_0(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ decay integrably as $|\mathbf{x}| \to \infty$, i.e., faster than $|\mathbf{x}|^{-1}$, then H is a short-range perturbation of $H_0 := \mathbf{p}^2/2m$. Its time evolution is approximated by the free time evolution (generated by H_0) as $t \to \pm\infty$, and the wave operators exist:

$$\Omega_\pm\,\psi := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}Ht}\,\mathrm{e}^{-\mathrm{i}H_0 t}\,\psi \;. \tag{3}$$

Here the free state ψ is an asymptotic state corresponding to the scattering state $\Omega_\pm\,\psi$ as $t \to +\infty$ or $t \to -\infty$, respectively. The scattering operator $S := \Omega_+^*\Omega_-$ is mapping incoming asymptotics to outgoing asymptotics. The vector potential $\mathbf{A}$ is determined by the magnetic field $\mathbf{B}$ only up to a gradient. Under the gauge transformation $\mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\lambda$, the Hamiltonian H is modified, but the scattering operator S is invariant if $\lambda(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$.

If $\mathbf{A}$ does not decay integrably, the short-range wave operators (3) need not exist. The time evolution generated by H may be described asymptotically in terms of long-range scattering theory, i.e., by modifying the free time evolution. Loss and Thaller [21, 36] have shown that the unmodified wave operators (3) still exist, if $\mathbf{A}(\mathbf{x})$ is transversal, i.e., $\mathbf{x}\cdot\mathbf{A}(\mathbf{x}) = 0$, and $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-(1/2+\delta)})$. In (2), $(\mathbf{A}(\mathbf{x}))^2$ is short-range, but $\mathbf{A}(\mathbf{x})\cdot\mathbf{p}$ is formally long-range. It is effectively short-range, since $\mathbf{A}(\mathbf{x}) = -\mathbf{x} \times \mathbf{G}(\mathbf{x})$ with $\mathbf{G}(\mathbf{x})$ short-range, and $\mathbf{A}\cdot\mathbf{p} = \mathbf{G}\cdot\mathbf{L}$ with the angular momentum $\mathbf{L} = \mathbf{x}\times\mathbf{p}$. This approach generalizes to vector potentials $\mathbf{A}$ with the following property, which will be called "medium-range": the transversal component of $\mathbf{A}(\mathbf{x})$, i.e., orthogonal to $\mathbf{x}$, is $\mathcal{O}(|\mathbf{x}|^{-(1/2+\delta)})$, and the longitudinal component, i.e., parallel to $\mathbf{x}$, is decaying integrably. — The aims of the present paper are:

- A class of medium-decay magnetic fields $\mathbf{B}$ is considered, such that there is a medium-range vector potential $\mathbf{A}$ with $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ in $\mathcal{S}'$, i.e., as a tempered distribution. The construction of $\mathbf{A}$ requires a decay $\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-(3/2+\delta)})$ as in the case of the transversal gauge, but the local regularity required of $\mathbf{B}$ can be weakened. The wave operators are obtained analogously to [21, 7].

- The decay- and regularity properties of various gauges are discussed, and the role of gauge transformations is emphasized: since the scattering operator is not invariant in general under the substitution $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\lambda$ when $\mathbf{A}$ and $\mathbf{A}'$ are medium-range, we must extract gauge-invariant quantities from S (only these may be observed in a physical experiment).

- The corresponding inverse scattering problem can be solved by obtaining the X-ray transform of $\mathbf{A}$ from the high-energy limit of S. This was done by Arians [2] for short-range $\mathbf{A}$ under low regularity assumptions, and by Nicoleau [24] for smooth $\mathbf{A}$ of medium-range. Here these results are extended to low-regularity $\mathbf{A}$ of medium range. The inverse problem of relativistic scattering with medium-range $\mathbf{A}$ is addressed in [19], combining the techniques of [22, 36, 18, 37], including obstacle scattering and the Aharanov–Bohm effect as well.

Only fields and particles in $\mathbb{R}^2$ and $\mathbb{R}^3$ are considered here. A measurable function $A_0 : \mathbb{R}^\nu \to \mathbb{R}$ is a scalar potential of *short range*, if the multiplication operator $A_0(\mathbf{x})$ is Kato-small with respect to $H_0 = \frac{1}{2m}\mathbf{p}^2$, and if it satisfies the Enss condition

$$\int_0^\infty \Big\| A_0(\mathbf{x})\,(H_0 + \mathrm{i})^{-1}\,F(|\mathbf{x}| \geq r)\Big\|\,dr \;<\; \infty\;, \tag{4}$$

where $F(\ldots)$ denotes multiplication with the characteristic function of the indicated region. This condition is satisfied, e.g., if $A_0 \in L^2_{\mathrm{loc}}$, and if it decays as $|\mathbf{x}|^{-\mu}$ with $\mu > 1$. The magnetic field $\mathbf{B} : \mathbb{R}^\nu \to \mathbb{R}^{\nu'}$, $\nu' := \nu(\nu-1)/2$, corresponds to a 2-form.

Definition 1.1 (Decay Conditions)
1. *Consider a magnetic field* $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ *or* $\mathbf{B} : \mathbb{R}^2 \to \mathbb{R}$, *which is in* $L^p(\mathbb{R}^\nu)$ *for a* $p > \nu$. *It is of* medium decay, *if it satisfies a decay condition* $|\mathbf{B}(\mathbf{x})| \leq C|\mathbf{x}|^{-\mu}$ *for a* $\mu > 3/2$ *and large* $|\mathbf{x}|$. *In the case of* $\mathbb{R}^3$, *we also require that* $\operatorname{div}\mathbf{B} = 0$ *in* $\mathcal{S}'$.

2. *A vector potential* $\mathbf{A} : \mathbb{R}^\nu \to \mathbb{R}^\nu$ *is of* medium range, *if it is continuous and satisfies* $|\mathbf{A}(\mathbf{x})| \leq C|\mathbf{x}|^{-\mu}$ *for some* $\mu > 1/2$. *In addition, the longitudinal part* $\mathbf{A}(\mathbf{x})\cdot\mathbf{x}/|\mathbf{x}|$ *shall decay integrably, i.e.,*

$$\int_0^\infty \sup\Big\{\,|\mathbf{A}(\mathbf{x})\cdot\mathbf{x}|/|\mathbf{x}|\;\Big|\;|\mathbf{x}| \geq r\Big\}\,dr \;<\; \infty\;. \tag{5}$$

Note that $\mathbf{A}(\mathbf{x}-\mathbf{x}_0)$ is of medium range as well. The magnetic fields of medium decay form a family of Banach spaces, cf. Cor. 2.5. By the decay at ∞, we have $\mathbf{B} \in L^p$ for a $p < 2$ in addition. The local regularity condition $p > \nu$ on $\mathbf{B}$ enables $\mathbf{A}$ to be continuous. Since $\mathbf{B}$ need not be continuous, the case of an infinitely long solenoid is included, where $\mathbf{B} : \mathbb{R}^2 \to \mathbb{R}$ is the characteristic function of the solenoid's cross section. The *Coulomb gauge* vector potential satisfies $\operatorname{div} \mathbf{A} = 0$, it is determined uniquely by $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ and $\mathbf{A}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. (It is called transversal in QED, because $\mathbf{p} \cdot \widehat{\mathbf{A}}(\mathbf{p}) = 0$.) The *transversal gauge* vector potential satisfies $\mathbf{x} \cdot \mathbf{A}(\mathbf{x}) = 0$, it is determined uniquely by $\mathbf{B}$ if it is continuous at $\mathbf{x} = 0$. The *Griesinger gauge* is introduced in Sec. 2.3, motivated by [13].

Theorem 1.2 (Vector Potentials)
1. *If $\mathbf{A}$, $\mathbf{A}'$ are medium-range vector potentials with $\mathbf{curl}\,\mathbf{A}' = \mathbf{curl}\,\mathbf{A}$ in $\mathcal{S}'$, then there is a C^1-function $\lambda : \mathbb{R}^\nu \to \mathbb{R}$ with $\mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\lambda$. Moreover, the homogeneous function $\Lambda(\mathbf{x}) := \lim_{r\to\infty} \lambda(r\mathbf{x})$ exists and is continuous for $\mathbf{x} \neq \mathbf{0}$.*

2. *If $\mathbf{B}$ is a magnetic field of medium decay, consider the vector potential $\mathbf{A}$ with $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ in $\mathcal{S}'$, given in the Griesinger gauge. It is of medium range, moreover it decays as $\mathcal{O}(|\mathbf{x}|^{-(\mu-1)})$, if $\mathbf{B}$ decays as $\mathcal{O}(|\mathbf{x}|^{-\mu})$ and $3/2 < \mu < 2$. If $\mathbf{B}$ is continuous, then the transversal gauge vector potential has the same properties.*

3. *Suppose that $\mathbf{B}$ satisfies a stronger decay condition with $\mu > 2$. Then the Coulomb gauge vector potential is of medium range, too. Moreover, $\mathbf{A}$ is bounded by $C|\mathbf{x}|^{-1}$ in all of these gauges. In $\mathbb{R}^3$, the flux of $\mathbf{B}$ through almost every plane vanishes, and the Coulomb vector potential is short-range. In $\mathbb{R}^2$, the flux Φ of $\mathbf{B}$ is finite, and the Coulomb vector potential is short-range, iff $\Phi = 0$.*

4. *If $\mathbf{B}$ is a magnetic field of medium decay, there is a special choice of a medium-range vector potential $\mathbf{A} = \mathbf{A}^s + \mathbf{A}^r$ with $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ in $\mathcal{S}'$, where $\mathbf{A}^s$ is short-range and continuous, and $\mathbf{A}^r$ is transversal and C^∞, with $|\,\partial_i \mathbf{A}^r_k(\mathbf{x})| \le C|\mathbf{x}|^{-\mu}$, $\mu > 1$. In addition, $\operatorname{div} \mathbf{A}$ is continuous and decays integrably.*

Moreover, the Griesinger gauge $\mathbf{A}$ and the Coulomb gauge $\mathbf{A}$ are regularizing, i.e., all $\partial_i A_k$ have the same local regularity as $\mathbf{B}$. For the transversal gauge $\mathbf{A}$, $\partial_i A_k$ exists only as a distribution in general if $\mathbf{B}$ is continuous. In [36], it is remarked that the transversal gauge vector potential is better adapted to scattering theory than the Coulomb gauge vector potential, if $\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-(3/2+\delta)})$ and $\mathbf{B}$ is sufficiently regular. The Coulomb gauge is superior in other cases:

Remark 1.3 (Advantages of Different Gauges)
Suppose $\mathbf{B} : \mathbb{R}^\nu \to \mathbb{R}^{\nu'}$ is a magnetic field of medium decay with $\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\mu})$:

1. If $3/2 < \mu \le 2$, the Coulomb gauge vector potential is not of medium range in general, and the wave operators (3) need not exist. The transversal gauge vector potential may be used if $\mathbf{B}$ is continuous. The Griesinger gauge works in any case.

2. If $\mu > 2$, $\nu = 2$, and $\int_{\mathbb{R}^2} \mathbf{B}\, dx \neq 0$, then the Coulomb gauge vector potential is of medium range as well, i.e., its longitudinal component is decaying integrably. It is

preferable to the transversal gauge because of its better local regularity properties, and because the Hamiltonian (2) is simplified due to $\mathbf{p}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{p}$.

3. If $\mu > 2$ and $\nu = 3$, or $\nu = 2$ and $\int_{\mathbb{R}^2} \mathbf{B}\,dx = 0$, the Coulomb gauge vector potential is short-range, but the transversal gauge or the Griesinger gauge is short-range only in exceptional cases.

Under the assumptions of item 3, it is natural to use only short-range $\mathbf{A}$, and the scattering operator S is gauge-invariant. In the medium-range case, we have a family of scattering operators, which are related by the transformation formula (6):

Theorem 1.4 (Gauge Transformation, Asymptotics, Inverse Scattering) *Suppose that $\mathbf{B}$ is a magnetic field of medium decay in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\mathbf{A}$ is any medium-range vector potential with $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ in $\mathcal{S}'$, and $\mathbf{A}_0$ is a short-range electrostatic potential according to* (4)*.*

1. *For the Schrödinger- or Pauli operator H, the wave operators $\Omega_\pm$ exist. Consider a gauge transformation $\mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\lambda$ and $\Lambda(\mathbf{x}) = \lim_{r\to\infty}\lambda(r\mathbf{x})$ according to Thm.* 1.2*, and denote the operators corresponding to $\mathbf{A}'$ by H', $\Omega'_\pm$, S'. The wave operators and scattering operators transform under a change of gauge as*

$$\Omega'_\pm \,=\, \mathrm{e}^{\mathrm{i}\lambda(\mathbf{x})}\,\Omega_\pm\,\mathrm{e}^{-\mathrm{i}\Lambda(\pm\mathbf{p})} \qquad S' \,=\, \mathrm{e}^{\mathrm{i}\Lambda(\mathbf{p})}\,S\,\mathrm{e}^{-\mathrm{i}\Lambda(-\mathbf{p})}\,. \tag{6}$$

2. *Consider translations in momentum space by $\mathbf{u} = u\boldsymbol{\omega}$, $\boldsymbol{\omega}\in S^{\nu-1}$. The scattering operator S for the corresponding Schrödinger- or Pauli equations in $\mathbb{R}^2$ and $\mathbb{R}^3$ has the asymptotics*

$$\operatorname*{s-lim}_{u\to\infty}\,\mathrm{e}^{-\mathrm{i}\mathbf{u}\mathbf{x}}\,S\,\mathrm{e}^{\mathrm{i}\mathbf{u}\mathbf{x}} \,=\, \exp\Big\{\mathrm{i}\int_{-\infty}^{\infty}\boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t)\,dt\Big\}\,. \tag{7}$$

$\mathbf{B}$ *is reconstructed uniquely from the relative phase of this high-energy limit of S.* (*The absolute phase is not gauge-invariant, thus not observable.*) *Under stronger decay assumptions, error bounds and the reconstruction of A_0 are given in Sec.* 5.3*.*

Item 1 is due to [31] for $\nu = 2$. Item 2 will be proved with the time-dependent geometric method of Enss and Weder [9]. It is due to Arians [2, 3] for $\mathbf{A}$ of short range, and in addition for $\mathbf{B} : \mathbb{R}^2 \to \mathbb{R}$ of compact support. The inverse scattering problem was solved before in [11] for $\mathbf{A}$ of exponential decay, and in [24] for C^∞-$\mathbf{A}$ of medium range in the transversal gauge. Analogous results for the Aharanov–Bohm effect are discussed by Nicoleau [25] and Weder [37]. The asymptotics for C^∞-vector potentials of medium range or long range are obtained in [25, 30, 39] using the Isozaki–Kitada modification (cf. Sec. 4.5).

Remark 1.5 (Gauge Invariance)

1. Gauge freedom has two sides to it: we may choose a convenient gauge to simplify a proof, but if the result depends on the gauge, it may be insignificant from a physical point of view. Cf. Sec. 6.1. Therefore existence and high-energy limits of

the wave operators are proved in two steps: first by employing the nice properties of the special gauge $\mathbf{A} = \mathbf{A}^s + \mathbf{A}^r$ according to item 4 of Thm. 1.2, and then the result is transfered to an arbitrary gauge by the transformation (6). Thus (7) is valid in any medium-range gauge, and the gauge-invariant relative phase is observable in principle. Moreover, this approach shows that only the decay properties of $\mathbf{A}$ are essential here, while the local regularity properties are for technical convenience.

2. If $\nu = 2$, $\mu > 2$, and the flux of $\mathbf{B}$ is not vanishing, it is possible to replace the medium-range techniques with short-range techniques plus an adaptive gauge transformation, such that the vector potential is decaying integrably in the direction of interest. Cf. Cor. 2.8 and [3, 17, 37]. In Sec. 5.3, a different kind of adaptive gauge transformation is used, such that the high-energy asymptotics of H are simplified.

This paper is organized as follows: Vector potentials are discussed in Sec. 2, including the proof of Thm. 1.2. In Sec. 3, $\mathbf{B}$ is reconstructed from the X-ray transform of $\mathbf{A}$. The direct problem of nonrelativistic scattering theory is addressed in Sec. 4. Existence of the wave operators is proved in detail, because the same techniques are needed later for the high-energy limit, but the reader is referred to [21, 7, 1] for asymptotic completeness. Sec. 5 is dedicated to the inverse problem, and concluding remarks on gauge invariance and on inverse scattering are given in Sec. 6.

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2 Fields and Gauges

To construct medium-range vector potentials, we are employing vector analysis on $\mathbb{R}^2$ and $\mathbb{R}^3$ under low regularity assumptions, controlling the decay at infinity. Some references to similar results for Sobolev spaces over domains in $\mathbb{R}^\nu$ are included, and [29] is a standard reference for vector analysis on manifolds using distributional derivatives. The following notation will be employed in the case of $\mathbb{R}^2$. It is motivated by identifying vectors and scalars in $\mathbb{R}^2$ with vectors in $\mathbb{R}^3$, $\mathbf{v} = (v_1\,,\,v_2)^{\mathrm{tr}} \leftrightarrow (v_1\,,\,v_2\,,\,0)^{\mathrm{tr}}$ and $w \leftrightarrow (0\,,\,0\,,\,w)^{\mathrm{tr}}$:

$$\begin{aligned} \mathbf{x} \times \mathbf{v} &:= x_1 v_2 - x_2 v_1 & \qquad \mathbf{curl}\,\mathbf{v} &:= \partial_1 v_2 - \partial_2 v_1 \\ \mathbf{x} \times w &:= (x_2 w\,,\, -x_1 w)^{\mathrm{tr}} & \qquad \mathbf{curl}\, w &:= (\partial_2 w\,,\, -\partial_1 w)^{\mathrm{tr}}\;. \end{aligned}$$

2.1 Gauge Transformation of $\mathbf{A}$

Suppose $\mathbf{A} : \mathbb{R}^\nu \to \mathbb{R}^\nu$ is a medium-range vector potential according to Def. 1.1. Assume $\mathbf{curl}\,\mathbf{A} = \mathbf{0}$ in $\mathcal{S}'$, i.e., $\int \mathbf{A}\cdot\mathbf{curl}\,\phi\,dx = 0$ for $\phi \in \mathcal{S}(\mathbb{R}^2,\,\mathbb{R})$ or $\phi \in \mathcal{S}(\mathbb{R}^3,\,\mathbb{R}^3)$, respectively. To show that $\mathbf{A}$ is a gradient, one can mollify $\mathbf{A}$ and employ a density argument, or apply mollifiers to the line integral over a closed curve. Here we shall use test functions, and define $\lambda : \mathbb{R}^\nu \to \mathbb{R}$ by the Poincaré formula for closed 1-forms,

$$\lambda(\mathbf{x}) := \int_0^1 \mathbf{x}\cdot\mathbf{A}(s\mathbf{x})\,ds\ . \tag{8}$$

For $\phi \in \mathcal{S}(\mathbb{R}^\nu,\,\mathbb{R})$ and a unit vector $\boldsymbol{\omega} \in S^{\nu-1}$ consider

$$\begin{aligned}
& -\int_{\mathbb{R}^\nu} \lambda(\mathbf{x})\,\boldsymbol{\omega}\cdot\nabla\phi(\mathbf{x})\,dx && (9)\\
= \;& -\int_{\mathbb{R}^\nu}\int_0^1 \mathbf{x}\cdot\mathbf{A}(s\mathbf{x})\,\boldsymbol{\omega}\cdot\nabla\phi(\mathbf{x})\,ds\,dx && (10)\\
= \;& \int_{\mathbb{R}^\nu}\int_0^1 \mathbf{A}(s\mathbf{x})\cdot\mathbf{curl}\Big(\boldsymbol{\omega}\times\mathbf{x}\,\phi(\mathbf{x})\Big)\,ds\,dx\ - && (11)\\
& -\int_{\mathbb{R}^\nu}\int_0^1 \boldsymbol{\omega}\cdot\mathbf{A}(s\mathbf{x})\,\Big((\nu-1)\phi(\mathbf{x}) + \mathbf{x}\cdot\nabla\phi(\mathbf{x})\Big)\,ds\,dx\ . && (12)
\end{aligned}$$

This identity is verified with $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. Now (11) is vanishing by $\mathbf{curl}\,\mathbf{A} = \mathbf{0}$ in $\mathcal{S}'$. The substitution $(s,\,\mathbf{x}) \to (s,\,\mathbf{y})$ with $\mathbf{y} = s\mathbf{x}$ in (12) gives

$$\begin{aligned}
& -\int_{\mathbb{R}^\nu}\int_0^1 \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{y})\,\Big((\nu-1)s^{-\nu}\phi(\mathbf{y}/s) + s^{-\nu-1}\mathbf{y}\cdot\nabla_{\mathbf{x}}\phi(\mathbf{y}/s)\Big)\,ds\,dy && (13)\\
= \;& \int_{\mathbb{R}^\nu} \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{y})\,\Big[s^{1-\nu}\phi(\mathbf{y}/s)\Big]_{s=0+}^1\,dy \;=\; \int_{\mathbb{R}^\nu}\boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{y})\,\phi(\mathbf{y})\,dy\ . && (14)
\end{aligned}$$

This shows $\mathbf{A} = \mathbf{grad}\,\lambda$ in $\mathcal{S}'$, and we have $\lambda \in C^1$ since $\mathbf{A}$ is continuous, thus $\mathbf{A} = \mathbf{grad}\,\lambda$ pointwise. Moreover, line integrals of $\mathbf{A}$ are path-independent. Now consider

$$\Lambda(\mathbf{x}) := \lim_{r\to\infty}\lambda(r\mathbf{x}) = \lim_{r\to\infty}\int_0^r \mathbf{x}\cdot\mathbf{A}(t\mathbf{x})\,dt\ . \tag{15}$$

Since $\frac{\mathbf{x}}{|\mathbf{x}|}\cdot\mathbf{A}(\mathbf{x})$ is short-range, convergence is uniform for $|\mathbf{x}| \ge R$, thus Λ is continuous on $\mathbb{R}^\nu \setminus \{\mathbf{0}\}$. ($\Lambda$ is 0-homogeneous, and discontinuous at $\mathbf{x} = \mathbf{0}$ unless it is constant.) If $\mathbf{A}$ is short-range, it is well-known that λ can be redefined such that $\lim_{|\mathbf{x}|\to\infty}\lambda(\mathbf{x}) = 0$. This follows from the estimate $\mathbf{A}(\mathbf{x}) = o(|\mathbf{x}|^{-1})$, see, e.g., [17, Lemma 2.12]. Item 1 of Thm. 1.2 is proved. ■

Remark 2.1 (Poincaré Lemma I)
1. The same proof shows the following version of the Poincaré Lemma: Suppose $\Omega \subset \mathbb{R}^\nu$, $p > \nu \ge 2$, and $\mathbf{A} \in L^p_{\mathrm{loc}}(\Omega,\,\mathbb{R}^\nu)$ with $\int_\Omega A_i\partial_k\phi - A_k\partial_i\phi\,dx = 0$ for all $\phi \in C_0^\infty(\Omega)$. If Ω is starlike around $\mathbf{x} = \mathbf{0}$, define $\lambda : \Omega \to \mathbb{R}$ by (8) a.e., then λ is weakly differentiable with $\mathbf{grad}\,\lambda = \mathbf{A}$ almost everywhere.

2. If Ω is simply connected, then λ is obtained piecewise. For general Ω, λ need not exist globally. Now $\mathbf{A}$ is a gradient, iff $\int_\Omega \mathbf{A}\cdot\mathbf{v}\,dx = 0$ for all $\mathbf{v}\in C_0^\infty(\Omega,\,\mathbb{R}^\nu)$ with $\operatorname{div}\mathbf{v} = 0$. Under this assumption on $\mathbf{A}\in L^1_{\mathrm{loc}}(\Omega,\,\mathbb{R}^\nu)$, λ is constructed by employing mollifiers [32, 12]. For suitable Ω, this result implies the Helmholtz-Weyl decomposition of $L^p(\Omega,\,\mathbb{R}^\nu)$, $p>1$, which is important, e.g., in fluid mechanics.

2.2 The Transversal Gauge

Suppose that $\mathbf{B}:\mathbb{R}^3\to\mathbb{R}^3$ or $\mathbf{B}:\mathbb{R}^2\to\mathbb{R}$ is a magnetic field of medium decay according to Def. 1.1. The transversal gauge vector potential $\mathbf{A}:\mathbb{R}^\nu\to\mathbb{R}^\nu$ is defined a.e. by the Poincaré formula for closed 2-forms,

$$\mathbf{A}(\mathbf{x}) := -\mathbf{x}\times\int_0^1 s\mathbf{B}(s\mathbf{x})\,ds\ . \tag{16}$$

Proposition 2.2 (Transversal Gauge)
Suppose that $\mathbf{B}:\mathbb{R}^\nu\to\mathbb{R}^{\nu'}$ is a magnetic field of medium decay with $\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\mu})$ as $|\mathbf{x}|\to\infty$, $\mu>3/2$. In the transversal gauge, the vector potential $\mathbf{A}$ is defined by (16). Assume in addition that $\mathbf{B}$ is continuous. Then

1. *$\mathbf{A}$ is continuous and satisfies $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1})$ if $\mu>2$, $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1}\log|\mathbf{x}|)$ if $\mu=2$, and $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-(\mu-1)})$ if $\mu<2$. Since $\mathbf{x}\cdot\mathbf{A}(\mathbf{x}) = 0$ and $\mu>3/2$, $\mathbf{A}$ is of medium range.*
2. *We have $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ in $\mathcal{S}'$, but the weak partial derivatives $\partial_i A_k$ and $\operatorname{div}\mathbf{A}$ do not exist in general.*

$\mathbf{B}$ is required to be continuous, to ensure that $\mathbf{A}$ is continuous. (More generally, $\mathbf{B}$ may have local singularities $c|\mathbf{x}-\mathbf{x}_0|^{-(1-\delta)}$, or a jump discontinuity on a strictly convex line or surface, but a jump discontinuity on a line through the origin is not permitted.) The decay properties are given in [21, 22, 36, 7] for $3/2<\mu<2$, and in [25] for $\mu\neq 2$. To achieve that $\mathbf{A}\in C^1$ with all derivatives decaying integrably, we would have to assume $\mathbf{B}\in C^1$ with derivatives decaying faster than $|\mathbf{x}|^{-2}$.

Proof: 1. Define $b(r) := \sup_{|\mathbf{x}|=r}|\mathbf{B}(\mathbf{x})|$ for $r\ge 0$, then $b(r) = \mathcal{O}(r^{-\mu})$ as $r\to\infty$. Now $|\mathbf{A}(\mathbf{x})|\le|\mathbf{x}|^{-1}\int_0^{|\mathbf{x}|} r\,b(r)\,dr$ gives the desired estimates (which are sharp).
2. Consider $\nu=3$ and a test function $\phi\in\mathcal{S}(\mathbb{R}^3,\,\mathbb{R}^3)$. We have

$$\begin{aligned}
& +\int_{\mathbb{R}^\nu}\mathbf{A}(\mathbf{x})\cdot\mathbf{curl}\,\phi(\mathbf{x})\,dx && (17)\\
=\ & -\int_{\mathbb{R}^\nu}\int_0^1\Big(\mathbf{x}\times s\mathbf{B}(s\mathbf{x})\Big)\cdot\mathbf{curl}\,\phi(\mathbf{x})\,ds\,dx && (18)\\
=\ & \int_{\mathbb{R}^3}\int_0^1 s\mathbf{B}(s\mathbf{x})\cdot\nabla\Big(\mathbf{x}\cdot\phi(\mathbf{x})\Big)\,ds\,dx && (19)\\
& -\int_{\mathbb{R}^3}\int_0^1 s\mathbf{B}(s\mathbf{x})\cdot\Big(\phi(\mathbf{x})+(\mathbf{x}\cdot\nabla)\phi(\mathbf{x})\Big)\,ds\,dx\ , && (20)
\end{aligned}$$

since $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$. The integral (19) is vanishing because of $\operatorname{div}\mathbf{B} = \mathbf{0}$ in $\mathcal{S}'$. In (20) we substitute $\mathbf{y} = s\mathbf{x}$ and obtain

$$-\int_{\mathbb{R}^3}\int_0^1 \mathbf{B}(\mathbf{y})\cdot\Big(s^{-2}\phi(\mathbf{y}/s) + s^{-3}(\mathbf{y}\cdot\nabla_{\mathbf{x}})\phi(\mathbf{y}/s)\Big)\,ds\,dy \tag{21}$$

$$= \int_{\mathbb{R}^3} \mathbf{B}(\mathbf{y})\cdot\Big[s^{-1}\phi(\mathbf{y}/s)\Big]_{s=0+}^1\,dy \;=\; \int_{\mathbb{R}^\nu}\mathbf{B}(\mathbf{y})\cdot\phi(\mathbf{y})\,dy\;. \tag{22}$$

In dimension $\nu = 2$, we have $\phi\in\mathcal{S}(\mathbb{R}^2,\,\mathbb{R})$, and (18) equals

$$-\int_{\mathbb{R}^2}\int_0^1 s\mathbf{B}(s\mathbf{x})\,\mathbf{x}\cdot\nabla\phi(\mathbf{x})\,ds\,dx \tag{23}$$

$$= \;-\int_{\mathbb{R}^2}\int_0^1 \mathbf{B}(\mathbf{y})\,s^{-2}\mathbf{y}\cdot\nabla_{\mathbf{x}}\phi(\mathbf{y}/s)\,ds\,dy \;=\; \int_{\mathbb{R}^2}\mathbf{B}(\mathbf{y})\,\Big[\phi(\mathbf{y}/s)\Big]_{s=0+}^1\,dy\;. \tag{24}$$

Thus $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ in $\mathcal{S}'(\mathbb{R}^3,\,\mathbb{R}^3)$ or $\mathcal{S}'(\mathbb{R}^2,\,\mathbb{R})$, respectively. — For $\nu = 2$, suppose that $\mathbf{B}(r\cos\theta,\,r\sin\theta) = (1+r)^{-\mu}f(\theta)$, where f is singular continuous. Then none of the weak derivatives $\partial_i A_k$ or $\operatorname{div}\mathbf{A}$ exists in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$. In the following example, the derivatives exist but they are not short-range: $\mathbf{B}(r\cos\theta,\,r\sin\theta) = (1+r)^{-\mu}\cos(r^\mu\theta)$. Similar examples are constructed in $\mathbb{R}^3$. (The condition $\operatorname{div}\mathbf{B} = 0$ is satisfied, e.g., by $\mathbf{B}(\mathbf{x}) = \mathbf{x}\times\mathbf{grad}\,g(\mathbf{x})$.) ∎

Remark 2.3 (Poincaré Lemma II)
1. Suppose $\Omega\subset\mathbb{R}^3$, $p > 3/2$, and $\mathbf{B}\in L^p_{\mathrm{loc}}(\Omega,\,\mathbb{R}^3)$ with $\int_\Omega \mathbf{B}\cdot\mathbf{grad}\,\phi\,dx = 0$ for all $\phi\in C_0^\infty(\Omega)$. If Ω is starlike around $\mathbf{x} = \mathbf{0}$, define $\mathbf{A}(\mathbf{x})$ by (16) a.e., then the same proof shows $\int_\Omega \mathbf{A}\cdot\mathbf{curl}\,\phi\,dx = \int_\Omega \mathbf{B}\cdot\phi\,dx$ for $\phi\in C_0^\infty(\Omega,\,\mathbb{R}^3)$. This vector potential is not weakly differentiable in general.

2. On an arbitrary domain Ω, a vector potential $\mathbf{A}$ exists if $\int_\Omega \mathbf{B}\cdot\phi\,dx = 0$ for all $\phi\in C_0^\infty(\Omega,\,\mathbb{R}^3)$ with $\mathbf{curl}\,\phi = \mathbf{0}$. If C_0^∞ can be replaced with C^∞, then $\mathbf{A}$ can be chosen such that it vanishes at the (regular) boundary $\partial\Omega$: $\mathbf{A}$ is constructed in [38] by potential theory, and in [13] by a mollified version of (16), see below. These vector potentials are weakly differentiable with $\|\partial_i A_k\|_p \le c_p\|\mathbf{B}\|_p\,,\ 1 < p < \infty$.

2.3 The Griesinger Gauge

For a magnetic field $\mathbf{B}:\mathbb{R}^\nu\to\mathbb{R}^{\nu'}$ of medium decay, the Griesinger gauge vector potential shall be defined by employing a mollifier $h\in C_0^\infty(\mathbb{R}^\nu,\,\mathbb{R})$ with $\int_{\mathbb{R}^\nu} h\,dx = 1$:

$$\mathbf{A}(\mathbf{x}) \;:=\; -\int_{\mathbb{R}^\nu}\int_0^1 h(\mathbf{z})\,(\mathbf{x}-\mathbf{z})\times s\mathbf{B}(s\mathbf{x}+(1-s)\mathbf{z})\,ds\,dz \tag{25}$$

$$= \;-\int_{\mathbb{R}^\nu}\int_1^\infty h(\mathbf{x}-t(\mathbf{x}-\mathbf{y}))\,t^{\nu-2}(t-1)\,(\mathbf{x}-\mathbf{y})\times\mathbf{B}(\mathbf{y})\,dt\,dy\;. \tag{26}$$

(25) looks like a mollified version of the transversal gauge (16), which is recovered formally for $h(\mathbf{z})\to\delta(\mathbf{z})$. Note that $\mathbf{B}$ is averaged over a ball of radius $\simeq (1-s)$,

which is shrinking to a point as $s \to 1$ in (25), or $\mathbf{y} \to \mathbf{x}$ in (26). It turns out that the integral kernel in the latter equation is weakly singular.

This definition is the exterior domain analog to the construction found by Griesinger for interior domains [13], i.e., $\int_1^\infty ds$ was replaced with $-\int_0^1 ds$. (Her original construction is less suitable for magnetic fields of medium decay, because it would require $\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2-\delta})$, and vanishing flux in the case of $\mathbb{R}^2$.) The technique goes back to Bogovskij's solution of $\operatorname{div}\mathbf{v} = f$ [4, 5, 12]. In the case of a bounded domain, the vector field satisfies $\|\partial_i v_k\|_p \le c_p\|f\|_p$ or $\|\partial_i A_k\|_p \le c_p\|\mathbf{B}\|_p$, respectively $(1 < p < \infty)$. — Item 2 of Thm. 1.2 is contained in the following

Proposition 2.4 (Griesinger Gauge)
Fix $h \in C_0^\infty(\mathbb{R}^\nu, \mathbb{R})$ with $\int_{\mathbb{R}^\nu} h\,dx = 1$. Suppose $\mathbf{B}$ is a magnetic field of medium decay with $\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\mu})$, $\mu > 3/2$. The Griesinger gauge vector potential $\mathbf{A}$ is defined by (25).

1. *$\mathbf{A}$ is continuous and satisfies $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1})$ if $\mu > 2$, $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1}\log|\mathbf{x}|)$ if $\mu = 2$, and $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-(\mu-1)})$ if $\mu < 2$. The longitudinal component of $\mathbf{A}$ is short-range, thus $\mathbf{A}$ is of medium range.*

2. *$\mathbf{A}$ has weak partial derivatives in $L^2_{\mathrm{loc}}(\mathbb{R}^\nu)$, and $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ almost everywhere. But the weak derivatives do not decay as $\mathcal{O}(|\mathbf{x}|^{-\mu})$ in general.*

Proof: 1. Choose $p > \nu$ with $\mathbf{B} \in L^p(\mathbb{R}^\nu, \mathbb{R}^{\nu'})$, $q := 1/(1-1/p)$, and fix $R > 0$ such that $h(\mathbf{z}) = 0$ for $|\mathbf{z}| \ge R$ and $|\mathbf{B}(\mathbf{x})| \le c(1+|\mathbf{x}|)^{-\mu}$ for $|\mathbf{x}| \ge R$. By Hölder's inequality we have

$$|\mathbf{A}(\mathbf{x})| \le (|\mathbf{x}| + R)\,\|h\|_q \int_0^1 s\|\mathbf{B}(s\mathbf{x} + (1-s)\mathbf{z})\,\chi(|\mathbf{z}| \le R)\|_p\,ds\,. \tag{27}$$

Note that the norm of $\mathbf{B}$ is considered on the ball of radius $(1-s)R$ around $s\mathbf{x}$. For $|\mathbf{x}| \le 2R$ consider $\|\ldots\|_p \le \|\mathbf{z} \mapsto \mathbf{B}(s\mathbf{x} + (1-s)\mathbf{z})\|_p = (1-s)^{-\nu/p}\|\mathbf{B}\|_p$, thus

$$|\mathbf{A}(\mathbf{x})| \le 3R\,\|h\|_q\,\|\mathbf{B}\|_p \int_0^1 s(1-s)^{-\nu/p}\,ds \tag{28}$$

is bounded. For $|\mathbf{x}| \ge 2R$, the s-interval in (27) is split:

a) For $0 \le s \le \dfrac{2R}{|\mathbf{x}| + R}$ we have $1 - s \ge 1/3$, thus $\|\ldots\|_p \le 3^{\nu/p}\|\mathbf{B}\|_p =: c_1$.

b) For $\dfrac{2R}{|\mathbf{x}| + R} \le s \le 1$ we have $|s\mathbf{x} + (1-s)\mathbf{z}| \ge R$ and $|s\mathbf{x} + (1-s)\mathbf{z}| \ge s|\mathbf{x}|/2$, therefore $\|\ldots\|_p \le c_2(1 + s|\mathbf{x}|)^{-\mu}$.

Now $\|\mathbf{B}(s\mathbf{x} + (1-s)\mathbf{z})\,\chi(|\mathbf{z}| \le R)\|_p \le c_3(1 + s|\mathbf{x}|)^{-\mu}$ for $0 \le s \le 1$, and (27) yields the desired estimate for $|\mathbf{A}(\mathbf{x})|$, which is optimal. The stronger bound for the longitudinal component is obtained by replacing the factor $(|\mathbf{x}| + R) \to R$ in (27).

Write (26) as $\mathbf{A}(\mathbf{x}) = \int_{\mathbb{R}^\nu} G(\mathbf{x},\, \mathbf{x}-\mathbf{y})\, \mathbf{B}(\mathbf{y})\, dy$. The kernel $G(\mathbf{x},\, \mathbf{z})$ is bounded by $c\,(1+|\mathbf{x}|^{\nu-1})\,|\mathbf{z}|^{-(\nu-1)}$, analogously to (38) below. Thus it is weakly singular, and $\mathbf{B} \in L^p$ implies that $\mathbf{A}$ is continuous (adapting Thm. II.9.2 in [12]).

2. We show $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ in $\mathcal{S}'$ by the techniques from Sec. 2.2, and establish the higher regularity of the Griesinger gauge afterwards (which was not possible for the transversal gauge). Consider $\nu = 3$ and $\phi \in \mathcal{S}(\mathbb{R}^3,\, \mathbb{R}^3)$. As in (19) and (20) we have

$$+\int_{\mathbb{R}^3} \mathbf{A}(\mathbf{x}) \cdot \mathbf{curl}\,\phi(\mathbf{x})\, dx \tag{29}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^1 h(\mathbf{z})\, s\mathbf{B}(s\mathbf{x}+(1-s)\mathbf{z}) \cdot \nabla_{\mathbf{x}}\Big((\mathbf{x}-\mathbf{z})\cdot\phi(\mathbf{x})\Big)\, ds\, dz\, dx \tag{30}$$

$$- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^1 h(\mathbf{z})\, s\mathbf{B}(s\mathbf{x}+(1-s)\mathbf{z}) \cdot \Big(\phi(\mathbf{x}) + ((\mathbf{x}-\mathbf{z})\cdot\nabla_{\mathbf{x}})\phi(\mathbf{x})\Big)\, ds\, dz\, dx\ . \tag{31}$$

In (30), the $\mathbf{x}$-integral is vanishing for a.e. s and $\mathbf{z}$, because $\operatorname{div}\mathbf{B} = \mathbf{0}$ in $\mathcal{S}'$. In (31), the substitution $(s,\, \mathbf{z},\, \mathbf{x}) \to (s,\, \mathbf{z},\, \mathbf{y})$ with $\mathbf{y} = s\mathbf{x} + (1-s)\mathbf{z}$ yields

$$\begin{aligned} &-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^1 h(\mathbf{z})\, \mathbf{B}(\mathbf{y}) \cdot \\ &\cdot\Big(s^{-2}\phi(\mathbf{z}+(\mathbf{y}-\mathbf{z})/s) + s^{-3}((\mathbf{y}-\mathbf{z})\cdot\nabla_{\mathbf{x}})\phi(\mathbf{z}+(\mathbf{y}-\mathbf{z})/s)\Big)\, ds\, dz\, dy \end{aligned} \tag{32}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} h(\mathbf{z})\, \mathbf{B}(\mathbf{y}) \cdot \Big[s^{-1}\phi(\mathbf{z}+(\mathbf{y}-\mathbf{z})/s)\Big]_{s=0+}^{1}\, dz\, dy \tag{33}$$

$$= \int_{\mathbb{R}^3} h(\mathbf{z})\, dz \int_{\mathbb{R}^3} \mathbf{B}(\mathbf{y}) \cdot \phi(\mathbf{y})\, dy\ , \tag{34}$$

thus $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ in $\mathcal{S}'(\mathbb{R}^3,\, \mathbb{R}^3)$. For $\nu = 2$ we obtain $\big[\phi(\mathbf{z}+(\mathbf{y}-\mathbf{z})/s)\big]_{s=0+}^{1} = \phi(\mathbf{y})$ analogously to (24). (The same technique works for $\int_1^\infty ds$, but for $\nu = 2$ we have $\big[\phi(\mathbf{z}+(\mathbf{y}-\mathbf{z})/s)\big]_{s=\infty-}^{1} = \phi(\mathbf{y}) - \phi(\mathbf{z})$, thus $\mathbf{curl}\,\mathbf{A}(\mathbf{x}) = \mathbf{B}(\mathbf{x}) - h(\mathbf{x}) \int_{\mathbb{R}^2} \mathbf{B}(\mathbf{y})\, dy$.)

The existence of all weak derivatives $\partial_i A_k$, $1 \le i,\, k \le \nu$, is shown first for $\widetilde{\mathbf{B}} \in C_0^\infty$ (without restriction on $\operatorname{div}\widetilde{\mathbf{B}}$), and then a density argument covers the general case. To simplify the notation, we consider only $\operatorname{div}\mathbf{A}$: (26) implies, as a principal value,

$$\operatorname{div}\widetilde{\mathbf{A}}(\mathbf{x}) \;=\; \int_{\mathbb{R}^\nu} K(\mathbf{x},\, \mathbf{x}-\mathbf{y}) \cdot \widetilde{\mathbf{B}}(\mathbf{y})\, dy \qquad \text{with} \tag{35}$$

$$K(\mathbf{x},\, \mathbf{z}) \;:=\; -\mathbf{z} \times \int_1^\infty \nabla h(\mathbf{x}-t\mathbf{z})\, t^{\nu-2}(t-1)^2\, dt \tag{36}$$

$$=\; -\frac{\mathbf{z}}{|\mathbf{z}|^{\nu+1}} \times \int_{|\mathbf{z}|}^\infty \nabla h\Big(\mathbf{x} - r\frac{\mathbf{z}}{|\mathbf{z}|}\Big)\, r^{\nu-2}(r-|\mathbf{z}|)^2\, dr\ . \tag{37}$$

The most singular contribution is

$$-\frac{\mathbf{z}}{|\mathbf{z}|^{\nu+1}} \times \int_0^\infty \nabla h\Big(\mathbf{x} - r\frac{\mathbf{z}}{|\mathbf{z}|}\Big)\, r^{\nu}\, dr = \mathcal{O}\Big((1+|\mathbf{x}|^\nu)|\mathbf{z}|^{-\nu}\Big)\ . \tag{38}$$

The Calderón–Zygmund Theorem [6], cf. [13, 12], shows that the principal value integral (35) is well-defined, and

$$\|(1+|\mathbf{x}|)^{-\nu}\operatorname{div}\widetilde{\mathbf{A}}\|_p \le c_p \|\widetilde{\mathbf{B}}\|_p\ . \tag{39}$$

Approximating the given $\mathbf{B} \in L^p$ with $\widetilde{\mathbf{B}} \in C_0^\infty$, (39) shows $\operatorname{div} \mathbf{A} \in L^p_{\mathrm{loc}}$.
Now $\partial_i A_k \in L^p_{\mathrm{loc}}(\mathbb{R}^\nu)$ is shown analogously, and $\mathbf{curl}\, \mathbf{A} = \mathbf{B}$ in $\mathcal{S}'$ implies $\mathbf{curl}\, \mathbf{A} = \mathbf{B}$ a.e.. I do not know if $\partial_i A_k \in L^p(\mathbb{R}^\nu)$, or if these weak derivatives decay integrably as $|\mathbf{x}| \to \infty$ (short-range terms). But assuming that all $\partial_i A_k(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\mu})$, i.e., having the same decay as $\mathbf{B}(\mathbf{x})$, would imply $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-(\mu-1)})$, which is not true in general if $\mu > 2$. ∎

Corollary 2.5 (Banach Spaces)
For $R > 0$, $p > \nu$, and $\mu > 3/2$, define a norm

$$\|\mathbf{B}\|_{R,p,\mu} := \|\mathbf{B}(\mathbf{x})\, \chi(|\mathbf{x}| \le R)\|_p + \|\, |\mathbf{x}|^\mu\, \mathbf{B}(\mathbf{x})\, \chi(|\mathbf{x}| \ge R)\|_\infty \;, \tag{40}$$

and denote by $\mathcal{M}_{R,p,\mu}$ the Banach space of magnetic fields with finite norm, and with $\operatorname{div} \mathbf{B} = 0$ *in $\mathcal{S}'$ if $\nu = 3$. The vector space of magnetic fields with medium decay is the union of these Banach spaces. For fixed $h \in C_0^\infty$, the proof of item 1 above shows that the Griesinger gauge is a bounded operator $\mathcal{M}_{R,p,\mu} \to C^0(\mathbb{R}^\nu, \mathbb{R}^\nu)$.*

2.4 The Coulomb Gauge

In the Coulomb gauge, the vector potential $\mathbf{A}$ is defined by

$$\mathbf{A}(\mathbf{x}) := -\frac{1}{\omega_\nu} \int_{\mathbb{R}^\nu} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^{\nu}} \times \mathbf{B}(\mathbf{y})\, dy \tag{41}$$

with $\omega_2 := |S^1| = 2\pi$ and $\omega_3 := |S^2| = 4\pi$.

Proposition 2.6 (Coulomb Gauge)
Suppose $\mathbf{B} : \mathbb{R}^\nu \to \mathbb{R}^{\nu'}$ is a magnetic field of medium decay with $\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\mu})$, $\mu > 3/2$. The Coulomb gauge vector potential $\mathbf{A} : \mathbb{R}^\nu \to \mathbb{R}^\nu$ is defined by (41)*.*

1. *$\mathbf{A}$ is continuous and weakly differentiable with $\|\partial_i A_k\|_2 \le \|\mathbf{B}\|_2$. It satisfies* $\operatorname{div} \mathbf{A} = 0$ *and* $\mathbf{curl}\, \mathbf{A} = \mathbf{B}$ *a.e..*

2. *If $\mu > 2$, then $\mathbf{A}$ is of medium range. For $\nu = 3$ it is short-range. For $\nu = 2$ it is short-range, iff the flux is vanishing: $\int_{\mathbb{R}^2} \mathbf{B}\, dx = 0$.*

3. *If $3/2 < \mu \le 2$, then $\mathbf{A}$ satisfies $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-(\mu-1)})$, or $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1} \log |\mathbf{x}|)$ if $\mu = \nu = 2$. But the longitudinal component of $\mathbf{A}$ does not decay integrably in general, and $\mathbf{A}$ need not be of medium range.*

Thus the Coulomb $\mathbf{A}$ has better differentiability properties than the transversal gauge, even if $\mathbf{B}$ is continuous. The decay properties are better for $\mu > 2$ (and vanishing flux), but not sufficient in general if $\mu \le 2$. — In [23], $|\mathbf{A}(\mathbf{x})|$ is estimated in $\mathbb{R}^3$ and in exterior domains. We shall employ a convolution of Riesz potentials:

Lemma 2.7 (Riesz Potentials)
For $\nu \in \mathbb{N}$ and $0 < \alpha,\, \beta < \nu$ with $\alpha + \beta > \nu$, we have the convolution on $\mathbb{R}^\nu$

$$|\mathbf{x}|^{-\alpha} \, * \, |\mathbf{x}|^{-\beta} \;=\; C_{\alpha,\,\beta;\,\nu}\, |\mathbf{x}|^{-(\alpha+\beta-\nu)} \;. \tag{42}$$

This formula is obtained form an elementary scaling argument and convergence proof. The constant is determined in [14, p. 136].

Proof of Prop. 2.6: 1. $\mathbf{A}$ is continuous since the integral kernel in (41) is weakly singular and $\mathbf{B} \in L^p$ for some $p > \nu$, cf. Thm. II.9.2 in [12] (the condition of a bounded domain is overcome by splitting $\mathbf{B}$). The convolution (41) is obtained by differentiating the fundamental solution of the Laplacian: we have $\mathbf{A} = \mathbf{curl}\,\mathbf{U}$ with $-\Delta\mathbf{U} = \mathbf{B}$. This implies $\operatorname{div}\mathbf{A} = 0$ and $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ in $\mathcal{S}'$. The Fourier transforms satisfy $i\mathbf{p}\cdot\widehat{\mathbf{A}}(\mathbf{p}) = 0$ and $i\mathbf{p}\times\widehat{\mathbf{A}}(\mathbf{p}) = \widehat{\mathbf{B}}(\mathbf{p})$, thus $\mathbf{p}^2\widehat{\mathbf{A}}(\mathbf{p}) = i\mathbf{p}\times\widehat{\mathbf{B}}(\mathbf{p})$. It remains to show that $\mathbf{A}$ is weakly differentiable. Now $|\widehat{\partial_i A_k}(\mathbf{p})| \le |\widehat{\mathbf{B}}(\mathbf{p})|$ a.e., thus $\|\partial_i A_k\|_2 \le \|\mathbf{B}\|_2$. We have $\mathbf{B} \in L^p$ for $p_1 < p \le p_2$, with $p_1 < 2$ and $p_2 > \nu$. By the Calderón–Zygmund Theorem [6, 13, 12], $\|\partial_i A_k\|_p \le c_p\|\mathbf{B}\|_p$ for $p_1 < p \le p_2$.

2. Now $\mu > 2$, and we may assume $2 < \mu < 3$. Split $\mathbf{B} = \mathbf{B}^{(1)} + \mathbf{B}^{(2)}$, such that $\mathbf{B}^{(1)} \in L^p$ has compact support, with $\int_{\mathbb{R}^2}\mathbf{B}^{(1)}\,dx = 0$ in the case of $\mathbb{R}^2$, and such that $|\mathbf{B}^{(2)}(\mathbf{x})| \le c|\mathbf{x}|^{-\mu}$ for $\mathbf{x} \in \mathbb{R}^\nu$ ($\operatorname{div}\mathbf{B}^{(i)} = 0$ is not required). Split $\mathbf{A} = \mathbf{A}^{(1)} + \mathbf{A}^{(2)}$ according to (41), i.e., by applying the convolution to $\mathbf{B}^{(i)}$ individually. If $\nu = 3$, we have $|\mathbf{A}^{(1)}(\mathbf{x})| = \mathcal{O}(|\mathbf{x}|^{-2})$, since $|\mathbf{x}-\mathbf{y}|^{-2} = \mathcal{O}(|\mathbf{x}|^{-2})$ for $\mathbf{y} \in \operatorname{supp}(\mathbf{B}^{(1)})$, and $|\mathbf{A}^{(2)}(\mathbf{x})| = \mathcal{O}(|\mathbf{x}|^{-(\mu-1)})$ by Lemma 2.7, thus $\mathbf{A}$ is short-range. For $\nu = 2$ we claim

$$\mathbf{A}(\mathbf{x}) = \frac{1}{2\pi|\mathbf{x}|^2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\int_{\mathbb{R}^2}\mathbf{B}(\mathbf{y})\,dy \;+\; \mathcal{O}(|\mathbf{x}|^{-(\mu-1)})\;. \tag{43}$$

The integral kernel of (41) is decomposed as follows:

$$-\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} = -\frac{\mathbf{x}}{|\mathbf{x}|^2} + \frac{(\mathbf{x}\times\mathbf{y})\times(\mathbf{x}-\mathbf{y}) - (\mathbf{x}\cdot\mathbf{y})(\mathbf{x}-\mathbf{y})}{|\mathbf{x}|^2|\mathbf{x}-\mathbf{y}|^2} = -\frac{\mathbf{x}}{|\mathbf{x}|^2} + \mathcal{O}\Big(\frac{|\mathbf{y}|}{|\mathbf{x}|\,|\mathbf{x}-\mathbf{y}|}\Big)$$

When applying this kernel to $\mathbf{B}^{(1)}$, the first integral is vanishing and the second is $\mathcal{O}(|\mathbf{x}|^{-2})$. Applying it to $\mathbf{B}^{(2)}$, the first integral gives the leading term in (43), and the second integral is bounded by

$$c\int_{\mathbb{R}^2}\frac{|\mathbf{y}|}{|\mathbf{x}|\,|\mathbf{x}-\mathbf{y}|}\,|\mathbf{y}|^{-\mu}\,dy = \frac{c}{|\mathbf{x}|}\int_{\mathbb{R}^2}|\mathbf{x}-\mathbf{y}|^{-1}\,|\mathbf{y}|^{-(\mu-1)}\,dy = c'|\mathbf{x}|^{-(\mu-1)} \tag{44}$$

by Lemma 2.7. Thus (43) is proved. (For $\mathbf{B}$ of compact support this is due to [35].) If the flux of $\mathbf{B}$ is vanishing, then $\mathbf{A}$ is short-range. If not, then $\mathbf{A}$ is still of medium range, since the leading term is transversal.

3. Now $3/2 < \mu \le 2$. Split $\mathbf{B}$ and $\mathbf{A}$ as in the previous item, then $\mathbf{A}^{(1)}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2})$, $|\mathbf{B}^{(2)}(\mathbf{x})| \le c|\mathbf{x}|^{-\mu}$, and Lemma 2.7 gives $|\mathbf{A}^{(2)}(\mathbf{x})| \le c'|\mathbf{x}|^{-(\mu-1)}$, except for the case of $\mu = \nu = 2$: then compute the bound explicitly for $c(1+|\mathbf{x}|^2)^{-1}$. Consider the vector potential $\mathbf{A}(\mathbf{x})$ given by

$$\frac{1}{(|\mathbf{x}|^2+1)^2}\begin{pmatrix} x_1(x_1^2-x_2^2+1) \\ x_2(x_1^2-x_2^2-1) \end{pmatrix} \quad\text{or}\quad \frac{1}{(|\mathbf{x}|^2+1)^2}\begin{pmatrix} x_1(x_1^2-x_2^2+x_3^2+1) \\ x_2(x_1^2-x_2^2-x_3^2-1) \\ 0 \end{pmatrix}\;,$$

respectively, and define $\mathbf{B} := \mathbf{curl}\,\mathbf{A}$. We have $\mathbf{B} \in C^\infty$ with $\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2})$ and $\operatorname{div}\mathbf{A} = 0$, thus $\mathbf{A}$ is the Coulomb gauge vector potential for the medium-decay $\mathbf{B}$. Now $A_1(x_1,\,0) = \dfrac{x_1}{x_1^2+1}$ or $A_1(x_1,\,0,\,0) = \dfrac{x_1}{x_1^2+1}$, respectively, shows that the longitudinal component is not short-range, and $\mathbf{A}$ is not of medium range. ∎

In the last example, $\Lambda(\mathbf{x}) := \lim_{r\to\infty}\lambda(r\mathbf{x})$ according to item 1 of Thm. 1.2 does not exist for the gauge transformation from transversal gauge to Coulomb gauge. The scattering operator exists for the transversal gauge, but not for the Coulomb gauge. Note that for rotationally symmetric $\mathbf{B} : \mathbb{R}^2 \to \mathbb{R}$, the Coulomb gauge agrees with the transversal gauge. Therefore, this vector potential combines the regularity properties of the Coulomb gauge with the decay properties of the transversal gauge: it is weakly differentiable and of medium range for all $\mu > 3/2$.

If $\mu > 2$, the flux of $\mathbf{B}$ is finite if $\nu = 2$, and for $\nu = 3$, the flux through almost every plane is vanishing. When $\nu = 2$ and $\int_{\mathbb{R}^2}\mathbf{B}\,dx \neq 0$, consider the natural class of medium-range vector potentials satisfying $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1})$. The corresponding gauge transformations $\lambda(\mathbf{x})$ have the property that $\Lambda(\mathbf{x})$ is Lipschitz continuous for $|\mathbf{x}| > \varepsilon$. This class contains the Coulomb gauge, and the transversal gauge as well if $\mathbf{B}$ is continuous.

Corollary 2.8 (Adaptive Gauges)
Suppose $\nu = 2$ and $\mathbf{B}$ is of medium decay with $\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\mu})$ for a $\mu > 2$, and $\Phi := \int_{\mathbb{R}^2}\mathbf{B}\,dx \neq 0$. For any direction $\boldsymbol{\omega} \in S^1$ there is a vector potential $\mathbf{A}^{\boldsymbol{\omega}}$ of medium range, such that $\mathbf{A}^{\boldsymbol{\omega}}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1})$, and $\mathbf{A}^{\boldsymbol{\omega}}(\mathbf{x})$ decays integrably as $\mathcal{O}(|\mathbf{x}|^{-(\mu-1)})$ within sectors around $\pm\boldsymbol{\omega}$. Moreover, $\operatorname{div}\mathbf{A}^{\boldsymbol{\omega}}$ is continuous with $\operatorname{div}\mathbf{A}^{\boldsymbol{\omega}}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2})$.

Proof: Denote the Coulomb gauge vector potential by $\mathbf{A}$ and observe (43). Choose a 2π-periodic function $f \in C^2(\mathbb{R},\,\mathbb{R})$, such that $f'(\theta) \equiv \Phi/(2\pi)$ for θ in intervals around $\arg(\pm\boldsymbol{\omega})$. Choose $\lambda \in C^2(\mathbb{R}^2,\,\mathbb{R})$ with $\lambda(\mathbf{x}) \equiv f(\arg\mathbf{x})$ for large $|\mathbf{x}|$. Then consider $\mathbf{A}^{\boldsymbol{\omega}} := \mathbf{A} - \mathbf{grad}\,\lambda$. ∎

Similar constructions are found, e.g., in [37]. If $\mathbf{B}$ has compact support, $\mathbf{A}^{\boldsymbol{\omega}}$ can be chosen to vanish in these sectors for large $|\mathbf{x}|$: this is achieved by subtracting a gradient, or by a shifted transversal gauge if $\mathbf{B}$ is continuous [3, 17, 37].

2.5 The Hörmander Decomposition of A

The following lemma is a special case of [15, Lemma 3.3], which is a standard tool in long-range scattering theory. (It is used to improve the decay of derivatives of the long-range part A_0^l, where $A_0 = A_0^s + A_0^l$.)

Lemma 2.9 (Hörmander Decomposition)
Suppose that $V \in C^1(\mathbb{R}^\nu)$ and $V(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-m_0})$, $\partial^\gamma V(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-m_1})$ for $|\gamma| = 1$, with $m_0 \geq m_1 - 1 > 0$. For $0 < \Delta < \min(1,\,m_1 - 1)$ there is a decomposition $V = V_1 + V_2$ such that: $V_1 \in C^1$ is a short-range potential with $V_1(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\lambda})$ for

$\lambda = \max(m_0,\, m_1 - \Delta) > 1$, and $V_2 \in C^\infty$ satisfies $\partial^\gamma V_2(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-m'_j})$ for $|\gamma| = j$, $j \in \mathbb{N}_0$. Here $m'_0 = m_0$ and $m'_j = \max(m_0 + j\Delta,\, m_1 + (j-1)\Delta)$ for $j \in \mathbb{N}$.

For a given medium-decay $\mathbf{B}$ we want to obtain a corresponding medium-range $\mathbf{A} = \mathbf{A}^s + \mathbf{A}^r$, such that $\mathbf{A}^s$ is short-range, $\mathbf{A}^r$ is transversal with short-range derivatives, and $\operatorname{div}\mathbf{A}$ is short-range. The transversal gauge does not satisfy our requirements because it need not be differentiable, and the Coulomb gauge is weakly differentiable and satisfies the condition on $\operatorname{div}\mathbf{A}$, but its longitudinal part need not decay integrably. For the Griesinger gauge, I do not know how to control the decay of the derivatives. Lemma 2.9 cannot be applied directly to $\mathbf{B}$ or to a known $\mathbf{A}$, because it requires $\mathbf{B} \in C^1$ or $\mathbf{A} \in C^1$, respectively.

Proof of Thm. 1.2, item 4: Suppose $\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\mu})$ for $|\mathbf{x}| \to \infty$ and consider the Coulomb gauge vector potential $\mathbf{A}^c$ according to (41). If $\mu > 2$, we may take $\mathbf{A}^s := \mathbf{A}^c$ and $\mathbf{A}^r := \mathbf{0}$, except in the case of $\nu = 2$ and $\int \mathbf{B}\, dx \neq 0$: then $\mathbf{A}^r$ equals the first term in (43) for large $|\mathbf{x}|$. Thus we may assume $3/2 < \mu < 2$, or $\mu = 3/2 + 3\delta$ with $0 < \delta < 1/6$. $\mathbf{A}^c$ is continuous, weakly differentiable, and satisfies $\mathbf{A}^c(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1/2-3\delta})$. Choose a function $\eta \in C^\infty(\mathbb{R}^\nu,\, \mathbb{R})$ with $\eta(\mathbf{x}) = 0$ for $|\mathbf{x}| < 1$ and $\eta(\mathbf{x}) = 1$ for $|\mathbf{x}| > 2$, and consider the decomposition $\mathbf{A}^c = \mathbf{A}^{(0)} + \mathbf{A}^\infty$:

$$\begin{aligned} \mathbf{A}^{(0)}(\mathbf{x}) &= -\frac{1}{\omega_\nu} \int_{\mathbb{R}^\nu} (1 - \eta(\mathbf{x}-\mathbf{y}))\, \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\nu}} \times \mathbf{B}(\mathbf{y})\, dy \;, \\ \mathbf{A}^{\infty}(\mathbf{x}) &= -\frac{1}{\omega_\nu} \int_{\mathbb{R}^\nu} \eta(\mathbf{x}-\mathbf{y})\, \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\nu}} \times \mathbf{B}(\mathbf{y})\, dy \;. \end{aligned}$$

Now $\mathbf{A}^{(0)}$ is short-range and $\mathbf{A}^\infty \in C^\infty$ with $\mathbf{A}^\infty(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1/2-3\delta})$. The derivatives are given by convolutions $\partial_i A^\infty_k = \sum K_{ikl} * B_l$, where $K_{ikl}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\nu})$ as $|\mathbf{x}| \to \infty$ and $K_{ikl}(\mathbf{x}) = 0$ for $|\mathbf{x}| < 1$. Thus $|K_{ikl}(\mathbf{x})| \le c|\mathbf{x}|^{-(\nu-\delta)}$ for $\mathbf{x} \in \mathbb{R}^\nu$. Split $\mathbf{B} = \mathbf{B}^{(1)} + \mathbf{B}^{(2)}$, such that $\mathbf{B}^{(1)}$ has compact support and $|\mathbf{B}^{(2)}(\mathbf{x})| \le c'|\mathbf{x}|^{-3/2-3\delta}$. The estimate $\partial_i A^\infty_k(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-3/2-2\delta})$ is obtained from Lemma 2.7. A medium-range $\mathbf{A}$ can be constructed as $\mathbf{A}^{(0)}$ plus the transversal gauge for $\mathbf{curl}\,\mathbf{A}^\infty$, but the derivatives of the latter need not decay integrably.

Lemma 2.9 with $m_0 = 1/2+3\delta$, $m_1 = 3/2+2\delta$ and $\Delta = 1/2+\delta$ yields a decomposition $\mathbf{A}^\infty = \mathbf{A}^{(1)} + \mathbf{A}^{(2)}$, such that $\mathbf{A}^{(1)}$ is short-range and $\partial^\gamma A^{(2)}_k(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-m'_j})$ for $|\gamma| = j$, with $m'_1 = m_1 = 3/2 + 2\delta$ and $m'_2 = 2 + 3\delta$. Now define $\mathbf{A}^s := \mathbf{A}^{(0)} + \mathbf{A}^{(1)} = \mathbf{A}^c - \mathbf{A}^{(2)}$, then $\mathbf{A}^s$ is continuous, weakly differentiable, and short-range. The longitudinal part of $\mathbf{A}^{(2)}$ need not decay integrably. Consider the decomposition $\mathbf{B} = \mathbf{B}^s + \mathbf{B}^r$ with $\mathbf{B}^s = \mathbf{curl}\,\mathbf{A}^s$ in $\mathcal{S}'$ and $\mathbf{B}^r = \mathbf{curl}\,\mathbf{A}^{(2)}$ in C^∞, which satisfies $\operatorname{div}\mathbf{B}^s = \operatorname{div}\mathbf{B}^r = 0$. We have $\mathbf{B}^r(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-3/2-2\delta})$ and $\partial_i B^r_k(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2-3\delta})$. Define $\mathbf{A}^r$ as the transversal gauge vector potential belonging to $\mathbf{B}^r$, then $\mathbf{A}^r \in C^\infty$ with $\mathbf{A}^r(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1/2-2\delta})$. By differentiating (16) under the integral and an analogous estimate, $\partial_i A^r_k(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1-3\delta})$ is obtained. Now $\mathbf{A} := \mathbf{A}^s + \mathbf{A}^r$ yields the desired gauge. Note that $\operatorname{div}\mathbf{A} = \operatorname{div}\mathbf{A}^r - \operatorname{div}\mathbf{A}^{(2)} \in C^\infty$ decays integrably. ∎

3 Inversion of X-Ray Transforms

From the asymptotics of the scattering operator S, we will know the line integral of $\mathbf{A}$ along all straight lines in $\mathbb{R}^\nu$, up to adding a function of the direction $\boldsymbol{\omega} \in S^{\nu-1}$:

$$a(\boldsymbol{\omega},\, \mathbf{x}) := \int_{-\infty}^{\infty} \boldsymbol{\omega} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\omega} t)\, dt \; . \tag{45}$$

Proposition 3.1 (Inversion of X-Ray Transforms)
Suppose $\mathbf{A}$ *is an unknown vector potential of medium range, and the line integral* (45) *is given for all* $\mathbf{x} \in \mathbb{R}^\nu$ *and* $\boldsymbol{\omega} \in S^{\nu-1}$*, up to adding a function* $f(\boldsymbol{\omega})$*. Then*

1. *The distribution* $\mathbf{B} := \mathbf{curl}\,\mathbf{A} \in \mathcal{S}'$ *is determined uniquely.*
2. *Assume in addition that* $\mathbf{B}$ *is a magnetic field of medium decay. On a.e. plane, the X-ray transform of the normal component of* $\mathbf{B}$ *is obtained from* (52) *below.*

Under stronger decay assumptions on $\mathbf{A}$, item 1 is due to [17, 18], and under stronger regularity assumptions, item 2 is found, e.g., in [16, 25].

Proof: 1. Given $\phi \in \mathcal{S}(\mathbb{R}^\nu,\, \mathbb{R}^{\nu'})$, we will need a vector field $\psi \in C^\infty(\mathbb{R}^\nu,\, \mathbb{R}^\nu)$ with

$$\mathbf{curl}\,\phi(z) = 2\int_{\mathbb{R}^\nu} \frac{\mathbf{y}\,\mathbf{y}\cdot\psi(\mathbf{z}-\mathbf{y})}{|\mathbf{y}|^{\nu+1}}\, dy \; . \tag{46}$$

By Fourier transformation, this equation is equivalent to

$$\mathrm{i}\mathbf{p} \times \widehat{\phi}(\mathbf{p}) = c_\nu\, \frac{1}{|\mathbf{p}|^3}\Big(|\mathbf{p}|^2\widehat{\psi}(\mathbf{p}) - \mathbf{p}\,\mathbf{p}\cdot\widehat{\psi}(\mathbf{p})\Big) \tag{47}$$

for some $c_\nu > 0$. Choose the following solution $\psi \in \mathbb{C}^\infty$:

$$\begin{aligned} \widehat{\psi}(\mathbf{p}) \;&:=\; \mathrm{i}c_\nu^{-1}|\mathbf{p}|\,\mathbf{p}\times\widehat{\phi}(\mathbf{p}) \;=\; c_\nu^{-1}|\mathbf{p}|^{-1}\Big(|\mathbf{p}|^2\,\mathrm{i}\mathbf{p}\times\widehat{\phi}(\mathbf{p})\Big)\;, \\ \psi(x) \;&=\; -c_\nu'\int_{\mathbb{R}^\nu} |\mathbf{x}-\mathbf{u}|^{-(\nu-1)}\,\Delta\,\mathbf{curl}\,\phi(\mathbf{u})\,du\;. \end{aligned} \tag{48}$$

To determine the decay properties of ψ, split $|\mathbf{x}|^{-(\nu-1)} = f_1(\mathbf{x}) + f_2(\mathbf{x})$ such that f_1 has compact support and $f_2 \in C^\infty$, and split $\psi = \psi_1 + \psi_2$ according to (48). Integrating by parts three times shows that ψ_2 is the convolution of ϕ with a C^∞-kernel, which is $\mathcal{O}(|\mathbf{x}|^{-(\nu+2)})$, thus in L^1. Therefore $\psi \in L^1$. Now suppose that $a(\boldsymbol{\omega},\, \mathbf{x}) = \int_{-\infty}^{\infty} \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega} t)\,dt$ is known up to a constant depending on $\boldsymbol{\omega}$, then

$$\int_{\mathbb{R}^\nu} a(\boldsymbol{\omega},\,\mathbf{x})\,\boldsymbol{\omega}\cdot\psi(\mathbf{x})\,dx = \int_{-\infty}^{\infty}\int_{\mathbb{R}^\nu} \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega} t)\,\boldsymbol{\omega}\cdot\psi(\mathbf{x})\,dx\,dt \tag{49}$$

is known: the unknown constant is canceled, since $\widehat{\psi}(0) = 0$ gives $\int_{\mathbb{R}^\nu}\psi\,dx = 0$. Consider polar coordinates $\mathbf{y} = \boldsymbol{\omega} t$, $dy = |\mathbf{y}|^{\nu-1}d\omega\,dt$ to obtain

$$\begin{aligned} \int_{S^{\nu-1}}\int_{\mathbb{R}^\nu} a(\boldsymbol{\omega},\,\mathbf{x})\,\boldsymbol{\omega}\cdot\psi(\mathbf{x})\,dx\,d\omega \;&=\; 2\int_{\mathbb{R}^\nu}\int_{\mathbb{R}^\nu}\frac{\mathbf{y}\cdot\mathbf{A}(\mathbf{x}+\mathbf{y})\,\mathbf{y}\cdot\psi(\mathbf{x})}{|\mathbf{y}|^{\nu+1}}\,dx\,dy \\ =\; 2\int_{\mathbb{R}^\nu}\int_{\mathbb{R}^\nu}\frac{\mathbf{y}\cdot\mathbf{A}(\mathbf{z})\,\mathbf{y}\cdot\psi(\mathbf{z}-\mathbf{y})}{|\mathbf{y}|^{\nu+1}}\,dy\,dz \;&=\; \int_{\mathbb{R}^\nu}\mathbf{A}(\mathbf{z})\cdot\mathbf{curl}\,\phi(\mathbf{z})\,dz \end{aligned}$$

by (46), thus the distribution $\mathbf{B} = \mathbf{curl}\,\mathbf{A} \in \mathcal{S}'$ has been computed.
2. Fix unit vectors $\boldsymbol{\omega}$, $\widetilde{\boldsymbol{\omega}} \in S^{\nu-1}$. If $\mathbf{A} \in C^1$, thus $\mathbf{B}$ is continuous, we have

$$\frac{\partial}{\partial u}\int_{-\infty}^{\infty} \boldsymbol{\omega} \cdot \mathbf{A}(\mathbf{x} + \widetilde{\boldsymbol{\omega}} u + \boldsymbol{\omega} t)\, dt = \int_{-\infty}^{\infty} (\widetilde{\boldsymbol{\omega}} \times \boldsymbol{\omega}) \cdot \mathbf{B}(\mathbf{x} + \widetilde{\boldsymbol{\omega}} u + \boldsymbol{\omega} t)\, dt\ . \tag{50}$$

(Proof by Stokes' Theorem, or with $(\widetilde{\boldsymbol{\omega}}\times\boldsymbol{\omega})\cdot(\nabla\times\mathbf{A}) = (\widetilde{\boldsymbol{\omega}}\cdot\nabla)(\boldsymbol{\omega}\cdot\mathbf{A}) - (\boldsymbol{\omega}\cdot\nabla)(\widetilde{\boldsymbol{\omega}}\cdot\mathbf{A})$.) Thus the X-ray transform of the component of $\mathbf{B}$ in the direction of $\widetilde{\boldsymbol{\omega}}\times\boldsymbol{\omega}$ is obtained on every plane normal to $\widetilde{\boldsymbol{\omega}}\times\boldsymbol{\omega}$. (It is natural, but not required, to assume $\widetilde{\boldsymbol{\omega}}\cdot\boldsymbol{\omega} = 0$.) The left hand side of (50) does not depend on the gauge of $\mathbf{A}$, since $\partial_u f(\boldsymbol{\omega}) = 0$. Now $\mathbf{B} \in L^p$, and $\mathbf{A}$ is of medium range. Then (50) remains true for a.e. $\mathbf{x} \in \mathbb{R}^\nu$ and a.e. $u \in \mathbb{R}$. The proof is given for $\nu = 3$, $\boldsymbol{\omega} = (0,\, 1,\, 0)^{\mathrm{tr}}$ and $\widetilde{\boldsymbol{\omega}} = (1,\, 0,\, 0)^{\mathrm{tr}}$: We have

$$\int_{-\infty}^{\infty} A_2(u,\, t,\, x_3)\, dt - \int_{-\infty}^{\infty} A_2(0,\, t,\, x_3)\, dt = \int_0^u \int_{-\infty}^{\infty} B_3(v,\, t,\, x_3)\, dt\, dv \tag{51}$$

for every u and almost every x_3, by integrating with respect to x_3, an approximation argument, and Fubini. For almost every x_3, both sides are well-defined, and the right hand side is weakly differentiable with respect to u. (50) may be rewritten as

$$(\widetilde{\boldsymbol{\omega}} \cdot \nabla)\, a(\boldsymbol{\omega},\, \mathbf{x}) = \int_{-\infty}^{\infty} (\widetilde{\boldsymbol{\omega}} \times \boldsymbol{\omega}) \cdot \mathbf{B}(\mathbf{x} + \boldsymbol{\omega} t)\, dt\ , \tag{52}$$

where $\widetilde{\boldsymbol{\omega}} \cdot \nabla$ denotes a weak directional derivative, which exists for a.e. $\mathbf{x}$. ■

$\mathbf{B}$ is reconstructed from the X-ray transform according to [14, 9, 18]. We have considered the normal component of $\mathbf{B}$ on almost every plane:

Remark 3.2 (Trace of B on a Plane)
Suppose $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ is a magnetic field of medium decay. If $\mathbf{B}$ is continuous except for a jump discontinuity transversal to a surface, the condition $\operatorname{div}\mathbf{B} = 0$ in $\mathcal{S}'$ implies that the normal component of $\mathbf{B}$ is continuous. If $\mathbf{B} \in L^p$ with $p > 3$, I do not know if there is a kind of trace operator, which defines the restriction of $\boldsymbol{\omega}\cdot\mathbf{B}$ to every plane normal to $\boldsymbol{\omega}$, in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ or in $L^2(\mathbb{R}^2)$. (This restriction is well-defined as a distribution in $\mathcal{S}'(\mathbb{R}^2,\, \mathbb{R})$, by employing a vector potential.)

4 The Direct Problem of Scattering Theory

Now vector potentials of medium range are applied to a nonrelativistic scattering problem: this section contains the proof of Thm. 1.4, item 1.

4.1 Definition of Hamiltonians

Our Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^\nu,\, \mathbb{C})$ for the Schrödinger equation, and $\mathcal{H} = L^2(\mathbb{R}^\nu,\, \mathbb{C}^2)$ for the Pauli equation. In the latter case, the Pauli matrices $\sigma_i \in \mathbb{C}^{2\times 2}$

are employed [36]. The free time evolution is generated by the free Hamiltonian $H_0 = -\frac{1}{2m}\Delta = \frac{1}{2m}\mathbf{p}^2$. It is self-adjoint with domain $D_{H_0} = W^2$, a Sobolev space. In an external electromagnetic field, the Pauli Hamiltonian is defined formally by the following expressions:

$$\begin{aligned} H &= \frac{1}{2m}\Big[(\mathbf{p}-\mathbf{A})^2 - \boldsymbol{\sigma}\cdot\mathbf{B}\Big] + A_0 &(53)\\ &= \frac{1}{2m}\Big[\mathbf{p}^2 - 2\mathbf{A}\cdot\mathbf{p} + \mathrm{i}\,\mathrm{div}\,\mathbf{A} + \mathbf{A}^2 - \boldsymbol{\sigma}\cdot\mathbf{B}\Big] + A_0\ . &(54)\end{aligned}$$

Domains will be specified below by employing perturbation theory of operators and quadratic forms, cf. [28, 27]. The following quadratic form is needed as well:

$$q_{\mathbf{A}}(\psi,\,\psi) := \frac{1}{2m}\Big\|(\mathbf{p}-\mathbf{A})\psi\Big\|^2 + \Big(\psi,\,(-\frac{1}{2m}\boldsymbol{\sigma}\cdot\mathbf{B} + A_0)\psi\Big)\ . \qquad (55)$$

Lemma 4.1 (Pauli- and Schrödinger Operators)
Suppose $\mathbf{B}$ *is a magnetic field of medium decay and* $\mathbf{A}$ *is a corresponding vector potential of medium range, and* A_0 *is a scalar potential of short range, cf.* (4). *The Pauli operator* H *is defined in item* 1*:*

1. *There is a unique self-adjoint operator* H*, such that its form domain is the Sobolev space* W^1*, and the quadratic form corresponding to* H *equals* $q_{\mathbf{A}}$ *according to* (55).
2. *Suppose* $\mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\lambda$ *is of medium range as well, and define* H' *in terms of* $q_{\mathbf{A}'}$ *analogously to item* 1. *Then* $D_{H'} = \mathrm{e}^{\mathrm{i}\lambda(\mathbf{x})}D_H$ *and* $H' = \mathrm{e}^{\mathrm{i}\lambda(\mathbf{x})}H\mathrm{e}^{-\mathrm{i}\lambda(\mathbf{x})}$.
3. *Suppose the distribution* $\mathrm{div}\,\mathbf{A} \in \mathcal{S}'$ *is a bounded function. Then* $D_H = W^2$*, and* H *satisfies* (54).

For the Schrödinger operator, the term $-\boldsymbol{\sigma}\cdot\mathbf{B}$ *is omitted, and* $\mathbf{curl}\,\mathbf{A} \in \mathcal{S}'$ *need not be a function.*

Proof: 1. The quadratic form (55) is well-defined on W^1, which is the form domain of H_0. It satisfies $|q_{\mathbf{A}}(\psi,\,\psi)| \le a(\psi,\,H_0\psi) + \mathcal{O}(\|\psi\|^2)$ for some $a < 1$: the bound for A_0 is < 1 since A_0 is short-range, and the bounds of the other terms are arbitrarily small. Now H is obtained from the KLMN Theorem.

2. $\mathbf{grad}\,\lambda = \mathbf{A}' - \mathbf{A}$ is bounded and continuous. The mapping $\psi \mapsto \mathrm{e}^{\mathrm{i}\lambda}\psi$ is unitary in $\mathcal{H}$ and sending W^1 to itself. On W^1 we have

$$\mathrm{e}^{\mathrm{i}\lambda(\mathbf{x})}\Big(\mathbf{p} - \mathbf{A}(\mathbf{x})\Big)\,\mathrm{e}^{-\mathrm{i}\lambda(\mathbf{x})} = \mathbf{p} - \mathbf{grad}\,\lambda(\mathbf{x}) - \mathbf{A}(\mathbf{x}) = \mathbf{p} - \mathbf{A}'(\mathbf{x})\ , \qquad (56)$$

thus $q_{\mathbf{A}'}(\psi,\,\psi) = q_{\mathbf{A}}(\mathrm{e}^{-\mathrm{i}\lambda}\psi,\,\mathrm{e}^{-\mathrm{i}\lambda}\psi)$ for $\psi \in W^1$. Since H and H' are determined uniquely by the quadratic forms, we have $H' = \mathrm{e}^{\mathrm{i}\lambda}H\mathrm{e}^{-\mathrm{i}\lambda}$.

3. Now a sum of distributions is a bounded function. Denoting it by $\mathrm{div}\,\mathbf{A}$, we have $\int_{\mathbb{R}^\nu} \mathbf{A}\cdot\mathbf{grad}\,\phi + (\mathrm{div}\,\mathbf{A})\phi\,dx = 0$ for $\phi \in \mathcal{S}$. (If the distributions $\mathbf{curl}\,\mathbf{A}$ and $\mathrm{div}\,\mathbf{A}$ are in L^2_{loc}, then all $\partial_i A_k \in L^2_{\mathrm{loc}}$ by elliptic regularity. But here we include the more general case of a Schrödinger operator as well, where $\mathrm{div}\,\mathbf{A}$ is a function but

$\mathbf{curl}\,\mathbf{A}$ is not.) The expression (54) is a symmetric operator on $\mathcal{S}$. By the Kato–Rellich Theorem, its closure is a self-adjoint operator $\widetilde{H}$ with $D_{\widetilde{H}} = D_{H_0} = W^2$. We compute $q_{\mathbf{A}}(\psi,\,\psi) = (\psi,\,\widetilde{H}\psi)$ for $\psi \in \mathcal{S}$. By the KLMN Theorem, $\mathcal{S}$ is a form core for H, thus $H = \widetilde{H}$. ∎

4.2 Existence of Wave Operators

Consider a Pauli operator H according to Lemma 4.1. We assume that $\mathbf{A}$ is given by item 4 of Thm. 1.2, i.e., $\mathbf{A} = \mathbf{A}^s + \mathbf{A}^r$ with $\mathbf{A}^s$ short-range, $\mathbf{A}^r$ transversal, and $\operatorname{div}\mathbf{A}$ short-range. Thus H is given by (54). Existence of the wave operators will be shown by estimating the Cook integral

$$\Omega_\pm\,\psi := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}Ht}\,\mathrm{e}^{-\mathrm{i}H_0t}\,\psi = \psi + \mathrm{i}\int_0^{\pm\infty} \mathrm{e}^{\mathrm{i}Ht}\,(H - H_0)\,\mathrm{e}^{-\mathrm{i}H_0t}\,\psi\,dt\ . \qquad (57)$$

The integral is well-defined on a finite t-interval if $\psi \in D_{H_0} = D_H = W^2$. We have

$$H - H_0 = -\frac{1}{m}\,\mathbf{A}^r\cdot\mathbf{p} + V^s\ , \quad V^s := \frac{1}{2m}\Big\{-2\mathbf{A}^s\cdot\mathbf{p} + \mathrm{i}\operatorname{div}\mathbf{A} + \mathbf{A}^2 - \boldsymbol{\sigma}\cdot\mathbf{B}\Big\} + A_0\ . \quad (58)$$

Fix $\psi \in \mathcal{S}$ with $\widehat{\psi} \in C_0^\infty(\mathbb{R}^\nu\setminus\{0\})$ and $\varepsilon > 0$ with $\widehat{\psi}(\mathbf{p}) \equiv 0$ for $|\mathbf{p}| < \varepsilon m$. Choose $g \in C_0^\infty(\mathbb{R}^\nu,\,\mathbb{R})$ with $g(\mathbf{p}) \equiv 1$ on $\operatorname{supp}(\widehat{\psi})$ and consider the decomposition

$$\begin{aligned}\Big\|A_0\,\mathrm{e}^{-\mathrm{i}H_0t}\,\psi\Big\| \;&\le\; \Big\|A_0\,g(\mathbf{p})\,F(|\mathbf{x}|\ge\varepsilon|t|)\Big\|\cdot\Big\|\mathrm{e}^{-\mathrm{i}H_0t}\,\psi\Big\| &(59)\\ &+\; \Big\|A_0\,g(\mathbf{p})\Big\|\cdot\Big\|F(|\mathbf{x}|\le\varepsilon|t|)\,\mathrm{e}^{-\mathrm{i}H_0t}\,\psi\Big\|\ . &(60)\end{aligned}$$

The first summand has an integrable bound since A_0 is a short-range potential (in (4), the resolvent may be replaced with $g(\mathbf{p})$). The second term can be estimated by any inverse power of $|t|$, by a standard nonstationary phase estimate for propagation into the classically forbidden region [36], since the speed is bounded below by ε. The remaining terms in V^s are treated analogously, observing the decay properties of $\mathbf{A}^s$, $\mathbf{A}^2$, $\operatorname{div}\mathbf{A}$, and $\mathbf{B}$. In the term $\mathbf{A}^s\cdot\mathbf{p}$ the space decomposition is introduced between $\mathbf{A}^s$ and $\mathbf{p}$. The term $\mathbf{A}^r\cdot\mathbf{p}$ in (58) is controlled with the technique of [21, 36]: write $\mathbf{A}^r(\mathbf{x}) = -\mathbf{x}\times\mathbf{G}(\mathbf{x})$ and note that $(\mathbf{G}\times\mathbf{x})\cdot\mathbf{p} = \mathbf{G}\cdot\mathbf{L}$ with the angular momentum $\mathbf{L} = \mathbf{x}\times\mathbf{p}$. Now $\mathbf{L}$ is commuting with H_0, and $\mathbf{G}$ is a short-range term, thus we have obtained an integrable bound $h(t)$ for the integrand in (57). The integral exists as a Bochner integral or as an improper Riemann integral. Thus the limit exists for a dense set of states ψ, and the wave operators exist as a strong limit on $\mathcal{H}$. (We have not used the fact that $\partial_i A_k$ decays integrably, but it will be needed in the relativistic case [19].) In an arbitrary gauge $\mathbf{A}'$ for the given $\mathbf{B}$, existence of the wave operators follows now from the transformation formula (65). The scattering operator is defined by $S := \Omega_+^*\,\Omega_-$. The Schrödinger equation is treated analogously, by omitting the term $-\boldsymbol{\sigma}\cdot\mathbf{B}$. (Existence of $\Omega_\pm$ can be shown for every medium-range $\mathbf{A}$, without any assumptions on $\operatorname{div}\mathbf{A}$ or $\mathbf{curl}\,\mathbf{A}$, by quadratic form techniques. This proof requires an additional regularization, and it is not suitable for obtaining a high-energy limit.)

4.3 Gauge Transformation

Suppose $\mathbf{A}$ and $\mathbf{A}'$ are vector potentials of medium range with $\mathbf{curl}\,\mathbf{A} = \mathbf{curl}\,\mathbf{A}'$ in $\mathcal{S}'$, thus $\mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\lambda$ and $\Lambda(\mathbf{x}) = \lim_{r\to\infty} \lambda(r\mathbf{x})$ is continuous on $\mathbb{R}^\nu \setminus \{0\}$ by Thm. 1.2. We claim

$$\operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}H_0 t}\lambda(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}H_0 t} = \Lambda(\pm\mathbf{p}) \ . \tag{61}$$

Note that $\Big(\lambda(\mathbf{x}) - \Lambda(\mathbf{x})\Big)\,(H_0 + \mathrm{i})^{-1}$ is compact, thus

$$\operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}H_0 t}\Big(\lambda(\mathbf{x}) - \Lambda(\mathbf{x})\Big)\,\mathrm{e}^{-\mathrm{i}H_0 t} = 0 \ . \tag{62}$$

Moreover, since Λ is 0-homogeneous, we may multiply the argument with $\pm m/t > 0$:

$$\begin{aligned} \mathrm{e}^{\mathrm{i}H_0 t}\Lambda(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}H_0 t} &= \Lambda(\mathbf{x} + t\mathbf{p}/m) \;=\; \Lambda(\pm m\mathbf{x}/t \pm \mathbf{p}) & (63)\\ &= \mathrm{e}^{-\mathrm{i}m\mathbf{x}^2/(2t)}\Lambda(\pm\mathbf{p})\,\mathrm{e}^{\mathrm{i}m\mathbf{x}^2/(2t)} \;\to\; \Lambda(\pm\mathbf{p}) & (64)\end{aligned}$$

strongly as $t \to \pm\infty$, since $\mathbf{x}^2/t \to 0$ pointwise for $\mathbf{x} \in \mathbb{R}^\nu$. This proves (61). Now consider the Hamiltonians H and $H' = \mathrm{e}^{\mathrm{i}\lambda(\mathbf{x})}H\mathrm{e}^{-\mathrm{i}\lambda(\mathbf{x})}$ according to Lemma 4.1, and suppose that $\Omega_\pm$ exist. Then the wave operators

$$\begin{aligned} \Omega'_\pm := \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}H't}\mathrm{e}^{-\mathrm{i}H_0 t} &= \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}\lambda(\mathbf{x})}\Big(\mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0 t}\Big)\Big(\mathrm{e}^{\mathrm{i}H_0 t}\mathrm{e}^{-\mathrm{i}\lambda(\mathbf{x})}\mathrm{e}^{-\mathrm{i}H_0 t}\Big)\\ &= \mathrm{e}^{\mathrm{i}\lambda(\mathbf{x})}\Omega_\pm\,\mathrm{e}^{-\mathrm{i}\Lambda(\pm\mathbf{p})} & (65)\end{aligned}$$

exist as well, and the scattering operators satisfy $S' = \mathrm{e}^{\mathrm{i}\Lambda(\mathbf{p})}S\,\mathrm{e}^{-\mathrm{i}\Lambda(-\mathbf{p})}$. The gauge transformation formula is employed, e.g., in [31, 35, 37, 40]. The proof (64) seems to be new. The analogous formula for the Dirac equation is found in [17, 18, 19]. If $\mathbf{A} - \mathbf{A}'$ is short-range, then Λ is constant, and $S' = S$.

4.4 Asymptotic Completeness of Wave Operators

The wave operators $\Omega_\pm$ are called asymptotically complete, if every "scattering state" in the continuous subspace of H is asymptotic to a free state, i.e., $\mathrm{Ran}(\Omega_-) = \mathrm{Ran}(\Omega_+) = \mathcal{H}^{cont}(H) = \mathcal{H}^{ac}(H)$, which implies that S is unitary. Consider again the special gauge $\mathbf{A} = \mathbf{A}^s + \mathbf{A}^r$: In the case of $\mathbf{A}^s = 0$ and $A_0 = 0$, completeness was shown in [21, 7] by the Enss geometric method. The proof shall extend to our case, since the additional short-range terms can be included with standard techniques, but I have not checked the details. In an arbitrary gauge $\mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\lambda$, completeness is carried over by the gauge transformation (65): we have $\mathrm{Ran}(\Omega'_\pm) = \mathrm{e}^{\mathrm{i}\lambda(\mathbf{x})}\,\mathrm{Ran}(\Omega_\pm) = \mathrm{e}^{\mathrm{i}\lambda(\mathbf{x})}\mathcal{H}^{ac}(H) = \mathcal{H}^{ac}(H')$. Under the stronger assumptions $\mathbf{B} \in L^4_{\mathrm{loc}}$ and $\mathbf{curl}\,\mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-(2+\delta)})$, completeness was shown by Arians [1] in the transversal gauge. Her proof employs a phase space cutoff of the form $f(\mathbf{p} - \mathbf{A}(\mathbf{x}))$, which can be defined by a Fourier transform.

4.5 Modified Wave Operators

For $\mathbf{A}$ of medium decay, the unmodified wave operators exist although $\mathbf{A}(\mathbf{x})$ need not decay integrably. We shall compare them to modified wave operators: These exist when $\mathbf{A}(\mathbf{x}) = \mathbf{A}^s(\mathbf{x}) + \mathbf{A}^l(\mathbf{x})$, where $\mathbf{A}^s$ is short-range, and $\mathbf{A}^l$ is C^∞, with $\mathbf{A}^l(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\delta})$, and with decay assumptions on its derivatives. The Dollard wave operators are obtained from a time-dependent modification. If $\mathbf{A}$ is of medium range, it could be done in the form

$$\Omega_\pm^D := \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}Ht}\, U^D(t) \ , \qquad U^D(t) := \exp\Big\{ -\mathrm{i}H_0 t + \mathrm{i}\int_0^t \mathbf{p}\cdot\mathbf{A}^l(s\mathbf{p})\, ds\Big\} \ , \quad (66)$$

thus $\Omega_\pm^D = \Omega_\pm \exp\Big\{\mathrm{i}\int_0^{\pm\infty} \mathbf{p}\cdot\mathbf{A}^l(s\mathbf{p})\, ds\Big\}$. By choosing $\mathbf{A}^l$ transversal, the modification is vanishing. The time-independent modification of Isozaki–Kitada is employed, e.g., in [35, 25, 30, 39]. With Fourier integral operators $J_\pm$ we have

$$\Omega_\pm^J := \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}Ht}\, J_\pm\, \mathrm{e}^{-\mathrm{i}H_0 t} \ , \qquad J_\pm \ : \ \mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{x}} \mapsto u_\pm^{\mathbf{q}}(\mathbf{x}) \quad (67)$$

for smooth $\mathbf{A}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\delta})$. Here $u_\pm^{\mathbf{q}}(\mathbf{x})$ is an approximate generalized eigenfunction of H with incoming or outgoing momentum $\mathbf{q}$, e.g., according to [39]:

$$u_\pm^{\mathbf{q}}(\mathbf{x}) := \exp\Big\{ \mathrm{i}\mathbf{q}\cdot\mathbf{x} + \mathrm{i}\int_0^{\pm\infty} \boldsymbol{\omega}\cdot\mathbf{A}(\boldsymbol{\omega}s) - \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}s)\, ds \Big\} \ , \quad \mathbf{q} = |\mathbf{q}|\boldsymbol{\omega} \ . \quad (68)$$

Under a change of gauge, $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\lambda$ with $\mathbf{grad}\,\lambda(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\delta})$, the FIOs and the modified wave operators are transformed according to

$$J_\pm' = \mathrm{e}^{\mathrm{i}\lambda(\mathbf{x}) - \mathrm{i}\lambda(0)}\, J_\pm \ , \qquad \Omega_\pm^{J\,\prime} = \mathrm{e}^{\mathrm{i}\lambda(\mathbf{x}) - \mathrm{i}\lambda(0)}\, \Omega_\pm^J \ , \quad (69)$$

and the modified scattering operator $S^J := \Omega_+^{J\,*}\, \Omega_-^J$ is gauge-invariant. (This is not the case when $\boldsymbol{\omega}\cdot\mathbf{A}(\boldsymbol{\omega}s)$ is omitted from the integrand in (68), or in the idealized Aharanov–Bohm experiment, where $\mathbf{A}(\mathbf{x})$ is unbounded at $\mathbf{x} = 0$ and $\lambda(\mathbf{x}) \equiv \Lambda(\mathbf{x})$ [25, 37, 30]. Then $\Omega_\pm^J = \Omega_\pm$ and $S^J = S$ transform according to (6).) When $\mathbf{A}$ is smooth and medium-range, [24, Lemma 2.2] implies

$$S^J = \mathrm{e}^{-\mathrm{i}a(\mathbf{p})}\, S\, \mathrm{e}^{\mathrm{i}a(-\mathbf{p})} \ , \qquad a(\mathbf{p}) := \int_0^\infty \mathbf{p}\cdot\mathbf{A}(s\mathbf{p})\, ds \ . \quad (70)$$

This may be taken as the definition of S^J when $\mathbf{A}$ is not smooth. Cf. Sec. 6.2.

5 High-Energy Limit and Inverse Scattering

Consider the scattering of a state $\mathrm{e}^{\mathrm{i}\mathbf{u}\mathbf{x}}\psi$, where $\mathbf{u} = u\boldsymbol{\omega}$. The position operator $\mathbf{x}$ is generating a translation by $\mathbf{u}$ in momentum space, and the limit of $\mathrm{e}^{-\mathrm{i}\mathbf{u}\mathbf{x}}\big(S\,\mathrm{e}^{\mathrm{i}\mathbf{u}\mathbf{x}}\psi\big)$ gives the high-energy asymptotics of the scattering process, as $u \to \infty$ for a fixed direction $\boldsymbol{\omega} \in S^{\nu-1}$.

5.1 High-Energy Limit of S

Applying the momentum-space translation by $\mathbf{u} = u\boldsymbol{\omega}$ to the free Hamiltonian gives

$$\mathrm{e}^{-\mathrm{i}\mathbf{ux}} H_0 \, \mathrm{e}^{\mathrm{i}\mathbf{ux}} = \frac{1}{2m}(\mathbf{p}+\mathbf{u})^2 = \frac{1}{2m}u^2 + \frac{u}{m}\Big(\boldsymbol{\omega}\cdot\mathbf{p} + \frac{m}{u}H_0\Big) \, . \tag{71}$$

In the free time evolution, the first term is a rapidly oscillating phase factor, which cancels with the corresponding term for H in the Cook integral (72). Rescaling the time $t' = ut/m$, the free time evolution is generated by $\boldsymbol{\omega}\cdot\mathbf{p} + \frac{m}{u}H_0 \to \boldsymbol{\omega}\cdot\mathbf{p}$ as $u \to \infty$. For the Hamiltonian H, note that $\mathbf{A}\cdot\mathbf{p}$ becomes $\mathbf{A}\cdot(\mathbf{p}+\mathbf{u})$ before rescaling, and H is replaced with $H_{\mathbf{u}} := \boldsymbol{\omega}\cdot(\mathbf{p}-\mathbf{A}) + \frac{m}{u}H$. Employing the special gauge $\mathbf{A} = \mathbf{A}^s + \mathbf{A}^r$ according to item 4 of Thm 1.2, consider

$$\begin{aligned} & \mathrm{e}^{-\mathrm{i}\mathbf{ux}} \, \Omega_\pm \, \mathrm{e}^{\mathrm{i}\mathbf{ux}} \, \psi \\ = \;& \psi + \mathrm{i}\int_0^{\pm\infty} \mathrm{e}^{\mathrm{i}H_{\mathbf{u}}t} \Big\{ -\boldsymbol{\omega}\cdot\mathbf{A} + \frac{m}{u}(H - H_0)\Big\} \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{\omega}\mathbf{p} + \frac{m}{u}H_0)t} \, \psi \, dt \end{aligned} \tag{72}$$

Assume $\psi \in \mathcal{S}$ with $\widehat{\psi} \in C_0^\infty$. The velocity operator corresponding to the translated and rescaled time evolution is $\boldsymbol{\omega} + \mathbf{p}/u$. Fix $0 < \varepsilon < 1$ and $u_0 > 0$, such that $\operatorname{supp}\widehat{\psi}$ is contained in the ball $|\mathbf{p}| \le u_0(1-\varepsilon)$, then the speed is bounded below by ε for $u \ge u_0$. By the standard techniques from Sec. 4.2, i.e., the decomposition (59)–(60), an integrable bound $h(t)$ is obtained for the integrand in (72), uniformly for $u \ge u_0$. The critical term is $-\mathbf{A}^r\cdot(\boldsymbol{\omega} + \mathbf{p}/u) = -\mathbf{G}\cdot[\mathbf{x}\times(\boldsymbol{\omega}+\mathbf{p}/u)]$ with $\mathbf{A}^r = \mathbf{G}\times\mathbf{x}$. Again, the translated angular momentum $\mathbf{x}\times(\boldsymbol{\omega}+\mathbf{p}/u)$ is commuting with the translated free time evolution, and $\mathbf{G}(\mathbf{x})$ is short-range. By the Dominated Convergence Theorem (for the $\mathcal{H}$-valued Bochner integral), the limit $u\to\infty$ is interchanged with the integration:

$$\lim_{u\to\infty} \mathrm{e}^{-\mathrm{i}\mathbf{ux}} \, \Omega_\pm \, \mathrm{e}^{\mathrm{i}\mathbf{ux}} \, \psi = \psi + \mathrm{i}\int_0^{\pm\infty} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}(\mathbf{p}-\mathbf{A})t} \, (-\boldsymbol{\omega}\cdot\mathbf{A}) \, \mathrm{e}^{-\mathrm{i}\boldsymbol{\omega}\mathbf{p}t} \, \psi \, dt \tag{73}$$

$$= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}(\mathbf{p}-\mathbf{A})t} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{\omega}\mathbf{p}t} \, \psi \tag{74}$$

$$= \exp\Big\{ -\mathrm{i}\int_0^{\pm\infty} \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t)\, dt\Big\} \, \psi \, . \tag{75}$$

We have employed the fact that $H_{\mathbf{u}} \to \boldsymbol{\omega}\cdot(\mathbf{p}-\mathbf{A})$ in the strong resolvent sense. The last step is verified from a differential equation [2, 18]. A density argument yields strong convergence of $\Omega_\pm$, and (7) follows from $S = \Omega_+^*\,\Omega_-$ and the strong convergence of Ω_+^*. (If Ω_u are isometric, Ω_∞ is unitary, and $\Omega_u \to \Omega_\infty$ strongly, then $\Omega_u^* \to \Omega_\infty^*$ strongly because $\Omega_u^* - \Omega_\infty^* = \Omega_u^*(\Omega_\infty - \Omega_u)\Omega_\infty^*$.) In an arbitrary medium-range gauge $\mathbf{A}'$ consider $\mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\lambda$, the limit Λ according to Thm. 1.2, and the transformation formula (6). We obtain

$$\begin{aligned} \operatorname*{s-lim}_{u\to\infty} \mathrm{e}^{-\mathrm{i}\mathbf{ux}} \, S' \, \mathrm{e}^{\mathrm{i}\mathbf{ux}} &= \mathrm{e}^{\mathrm{i}\Lambda(\boldsymbol{\omega})} \exp\Big\{\mathrm{i}\int_{-\infty}^{\infty} \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t)\, dt\Big\} \, \mathrm{e}^{-\mathrm{i}\Lambda(-\boldsymbol{\omega})} \\ &= \exp\Big\{\mathrm{i}\int_{-\infty}^{\infty} \boldsymbol{\omega}\cdot\mathbf{A}'(\mathbf{x}+\boldsymbol{\omega}t)\, dt\Big\} \, , \end{aligned}$$

since $\lambda(\mathbf{x}+\boldsymbol{\omega}t) \to \Lambda(\pm\boldsymbol{\omega})$ pointwise for $t \to \pm\infty$. Thus the high-energy limit (7) is established for an arbitrary gauge $\mathbf{A}$. Under stronger decay assumptions, error bounds are given in Cor. 5.3. The same proof applies to the Schrödinger equation by omitting the term $-\boldsymbol{\sigma}\cdot\mathbf{B}$.

The simplification of the scattering process to a mere phase change at high energies has a geometric interpretation, cf. [8, 10]: For a state ψ with momentum support in a ball of radius mR around $\mathbf{u} = u\boldsymbol{\omega}$ and large $u = |\mathbf{u}|$, translation dominates over spreading of the wave packet in the free time evolution. On the physical time scale t, the region of strong interaction is traveled in a time $t \simeq m/u$, and the effective diameter of the wave packet is increased by $\approx Rt \simeq 1/u$.

5.2 Reconstruction of B

In the inverse scattering problem, A_0 , $\mathbf{B}$ and $\mathbf{A}$ are unknown, and the high-energy limit of S is known up to a gauge transformation. The absolute phase of (7) is not gauge-invariant, but we assume that the relative phase is observable. The exponential function is 2πi-periodic, but by the continuity of $\mathbf{A}$, the integral transform of $\mathbf{A}$ is obtained up to a direction-dependent constant, and the magnetic field $\mathbf{B}$ is reconstructed according to Prop 3.1. This concludes the proof of Thm. 1.4 for the Pauli operator and the Schrödinger operator.

5.3 Reconstruction of A_0

For the Schrödinger equation with short-range electromagnetic fields, Arians [2] first reconstructs $\mathbf{B}$ from (7), and then she reconstructs A_0 from the $1/u$-term of the high-energy asymptotics. The following theorem is quite similar, but the proof will be modified. Similar results for $\mathbf{B} \in C^\infty$ are obtained by Nicoleau in [24] using Fourier integral operators.

Theorem 5.1 (Reconstruction of A_0 (Arians))
Consider a short-range electrostatic potential A_0 according to (4)*, and a magnetic field $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ or $\mathbf{B} : \mathbb{R}^2 \to \mathbb{R}$. Suppose $\mathbf{B}$ and $\mathbf{curl}\,\mathbf{B}$ are continuous and both decay as $\mathcal{O}(|\mathbf{x}|^{-\mu})$ with $\mu > 2$, and that the flux of $\mathbf{B}$ is vanishing if $\nu = 2$. $\mathbf{A}$ is a corresponding vector potential of short range. Set $a(\boldsymbol{\omega},\,\mathbf{x}) := \int_{-\infty}^{\infty} \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t)\,dt$. The scattering operator S for the Schrödinger- or Pauli equations has the following high-energy asymptotics:*

$$\begin{aligned}\lim_{u\to\infty} \frac{u}{m}\,\mathrm{e}^{-\mathrm{i}\mathbf{u}\mathbf{x}}\left(S-\mathrm{e}^{\mathrm{i}a}\right)\mathrm{e}^{\mathrm{i}\mathbf{u}\mathbf{x}}\,\psi \;&=\; -\mathrm{i}\,\mathrm{e}^{\mathrm{i}a}\int_{-\infty}^{\infty} A_0(\mathbf{x}+\boldsymbol{\omega}t)\,\psi\,dt \\ &\quad -\mathrm{i}\,\mathrm{e}^{\mathrm{i}a}\int_{-\infty}^{0} K_-^{\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}t,\,\mathbf{p})\,\psi\,dt \qquad (76)\\ &\quad -\mathrm{i}\int_{0}^{\infty} K_+^{\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}t,\,\mathbf{p})\,\mathrm{e}^{\mathrm{i}a}\,\psi\,dt\end{aligned}$$

for $\widehat{\psi} \in C_0^\infty$. The operators $K_\pm^{\boldsymbol{\omega}}(\mathbf{x},\, \mathbf{p})$ are defined in (81)*, they depend on* $\mathbf{B}$ *but not on* A_0*. Thus* $\mathbf{B}$ *and* A_0 *are reconstructed uniquely from* S*.*

Proof: For every $\boldsymbol{\omega} \in S^{\nu-1}$ and the signs $\pm$, consider the vector potentials

$$\mathbf{A}_\pm^{\boldsymbol{\omega}}(\mathbf{x}) := \int_0^{\pm\infty} \boldsymbol{\omega} \times \mathbf{B}(\mathbf{x} + \boldsymbol{\omega} s)\, ds = \mathbf{A}(\mathbf{x}) + \mathbf{grad}\, \lambda_\pm^{\boldsymbol{\omega}}(\mathbf{x}) \tag{77}$$

$$\text{with} \qquad \lambda_\pm^{\boldsymbol{\omega}}(\mathbf{x}) := \int_0^{\pm\infty} \boldsymbol{\omega} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\omega} s)\, ds\ . \tag{78}$$

This definition is motivated by [35, 2, 24], and the transformation (77) is verified with $\boldsymbol{\omega} \times (\nabla \times \mathbf{A}) = \nabla(\boldsymbol{\omega} \cdot \mathbf{A}) - (\boldsymbol{\omega} \cdot \nabla)\mathbf{A}$. $\mathbf{A}_\pm^{\boldsymbol{\omega}}$ and $\mathbf{grad}\, \lambda_\pm^{\boldsymbol{\omega}}$ are bounded and continuous. $\mathbf{A}_\pm^{\boldsymbol{\omega}}(\mathbf{x})$ decays as $|\mathbf{x}|^{-(\mu-1)}$ in the half-space $\boldsymbol{\omega} \cdot \mathbf{x} \to \pm\infty$, but in general it does not decay for $\boldsymbol{\omega} \cdot \mathbf{x} \to \mp\infty$, thus $\mathbf{A}_\pm^{\boldsymbol{\omega}}$ is not of medium range. We have $\mathbf{curl}\, \mathbf{A}_\pm^{\boldsymbol{\omega}} = \mathbf{B}$ and $\operatorname{div} \mathbf{A}_\pm^{\boldsymbol{\omega}}(\mathbf{x}) = -\int_0^{\pm\infty} \boldsymbol{\omega} \cdot \mathbf{curl}\, \mathbf{B}(\mathbf{x} + \boldsymbol{\omega} s)\, ds$ in $\mathcal{S}'$. The latter function is bounded, continuous, and decays as $|\mathbf{x}|^{-(\mu-1)}$ for $\boldsymbol{\omega} \cdot \mathbf{x} \to \pm\infty$. The Hamiltonian $H_\pm^{\boldsymbol{\omega}}$ is defined by (54) with $\mathbf{A}_\pm^{\boldsymbol{\omega}}$ instead of $\mathbf{A}$, it satisfies $H_\pm^{\boldsymbol{\omega}} = \mathrm{e}^{\mathrm{i}\lambda_\pm^{\boldsymbol{\omega}}(\mathbf{x})}\, H\, \mathrm{e}^{-\mathrm{i}\lambda_\pm^{\boldsymbol{\omega}}(\mathbf{x})}$. Analogously to the gauge transformation formula (65) we have

$$\Omega_\pm\, \psi = \mathrm{e}^{-\mathrm{i}\lambda_\pm^{\boldsymbol{\omega}}(\mathbf{x})} \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}H_\pm^{\boldsymbol{\omega}} t}\, \mathrm{e}^{-\mathrm{i}H_0 t}\, \psi\ , \tag{79}$$

when the support of $\widehat{\psi}$ is contained in the half-space $\pm\boldsymbol{\omega} \cdot \mathbf{p} > 0$. To verify that the analog of $\Lambda(\pm\mathbf{p})$ is vanishing, a nonstationary phase estimate is employed for a dense set of states. For $\widehat{\psi} \in C_0^\infty$, consider the translated and rescaled Cook integral

$$\begin{aligned} &\frac{u}{m}\, \mathrm{e}^{-\mathrm{i}u\mathbf{x}} \Big(\mathrm{e}^{\mathrm{i}\lambda_\pm^{\boldsymbol{\omega}}(\mathbf{x})}\, \Omega_\pm - 1\Big)\, \mathrm{e}^{\mathrm{i}u\mathbf{x}}\, \psi \\ &= \mathrm{i} \int_0^{\pm\infty} \mathrm{e}^{\mathrm{i}(\boldsymbol{\omega}\mathbf{p} + \frac{m}{u} H_\pm^{\boldsymbol{\omega}})t} \Big\{A_0(\mathbf{x}) + K_\pm^{\boldsymbol{\omega}}(\mathbf{x},\, \mathbf{p})\Big\}\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{\omega}\mathbf{p} + \frac{m}{u} H_0)t}\, \psi\, dt \end{aligned} \tag{80}$$

with the symmetric operators (omit $-\boldsymbol{\sigma} \cdot \mathbf{B}$ in the Schrödinger case)

$$K_\pm^{\boldsymbol{\omega}}(\mathbf{x},\, \mathbf{p}) := \frac{1}{2m} \Big(-2\mathbf{A}_\pm^{\boldsymbol{\omega}}(\mathbf{x}) \cdot \mathbf{p} + \mathrm{i}\, \operatorname{div} \mathbf{A}_\pm^{\boldsymbol{\omega}}(\mathbf{x}) + (\mathbf{A}_\pm^{\boldsymbol{\omega}}(\mathbf{x}))^2 - \boldsymbol{\sigma} \cdot \mathbf{B}(\mathbf{x}) \Big)\ . \tag{81}$$

Note that $\boldsymbol{\omega} \cdot \mathbf{A}_\pm^{\boldsymbol{\omega}} = 0$, thus the main contribution from (72) is missing, and a common factor $1/u$ was extracted from the remaining terms. By the same uniform estimates as in Sec. 5.1, the limit $u \to \infty$ can be performed under the integral. Since the translated and rescaled generators of the time evolutions are converging to $\boldsymbol{\omega} \cdot \mathbf{p}$ in the strong resolvent sense, (80) is converging to

$$\mathrm{i} \int_0^{\pm\infty} \Big\{A_0(\mathbf{x} + \boldsymbol{\omega} t) + K_\pm^{\boldsymbol{\omega}}(\mathbf{x} + \boldsymbol{\omega} t,\, \mathbf{p})\Big\}\, \psi\, dt\ . \tag{82}$$

The relation $a(\boldsymbol{\omega},\, \mathbf{x}) = \lambda_+^{\boldsymbol{\omega}}(\mathbf{x}) - \lambda_-^{\boldsymbol{\omega}}(\mathbf{x})$ gives (76) as a weak limit. Moreover, we have $\mathrm{e}^{-\mathrm{i}u\mathbf{x}}\, \Omega_\pm^*\, \mathrm{e}^{\mathrm{i}u\mathbf{x}} \to \mathrm{e}^{\mathrm{i}\lambda_\pm^{\boldsymbol{\omega}}(\mathbf{x})}$ strongly, and the strong limit in (76) is obtained from the decomposition

$$S - \mathrm{e}^{\mathrm{i}a} = \Omega_+^* \Big(\Omega_- - \mathrm{e}^{-\mathrm{i}\lambda_-^{\boldsymbol{\omega}}}\Big) + \Omega_+^* \Big(\mathrm{e}^{-\mathrm{i}\lambda_+^{\boldsymbol{\omega}}} - \Omega_+\Big)\, \mathrm{e}^{\mathrm{i}a}\ . \tag{83}$$

For the limit of Ω_+, we are applying (80) to $\mathrm{e}^{\mathrm{i}a}\,\psi \in W^2$ instead of ψ, which requires two additional arguments: First, the regularization of A_0 is not done with $\psi = g(\mathbf{p})\psi$, but with $\mathrm{e}^{\mathrm{i}a}\,\psi = (H_0 + \mathrm{i})^{-1}\Big((H_0 + \mathrm{i})\,\mathrm{e}^{\mathrm{i}a}\,\psi\Big)$. Second, in the proof of the nonpropagation property, $(\mathrm{e}^{\mathrm{i}a}\,\psi)^\wedge$ does not have the desired compact support. But $a(\boldsymbol{\omega},\, \mathbf{x})$ is constant in the direction $\boldsymbol{\omega}$, thus the support of $\widehat{\psi}$ is enlarged only orthogonal to $\boldsymbol{\omega}$, and remains bounded in the direction of $\boldsymbol{\omega}$. (Use only the directional derivative $\boldsymbol{\omega}\cdot\nabla_{\mathbf{p}}$ in the proof of the nonstationary phase estimate [36, Thm. 1.8].)

Now suppose that S is known (it is invariant under short-range gauge transformations), thus the absolute phase of its high-energy limit is known. Then $\mathbf{B}$ is reconstructed by Thm. 1.4, and any corresponding short-range $\mathbf{A}$ gives $a(\boldsymbol{\omega},\, \mathbf{x})$. Since $K_\pm^{\boldsymbol{\omega}}(\mathbf{x},\, \mathbf{p})$ can be computed from $\mathbf{B}$, the X-ray transform of A_0 is obtained from (76). A_0 is reconstructed in the second step according to [14], at least under stronger regularity assumptions. In the general case, the potential is regularized by translated test functions [9], or it is considered in $\mathcal{S}'$ [18]. ■

The main difference to Arians' original proof [2] is the adaptive gauge transformation $\mathbf{A}_\pm^{\boldsymbol{\omega}} = \mathbf{A} + \mathbf{grad}\,\lambda_\pm^{\boldsymbol{\omega}}$: since $\boldsymbol{\omega}\cdot\mathbf{A}_\pm^{\boldsymbol{\omega}}(\mathbf{x}) \equiv 0$, the limit (82) is read off easily from (80) after showing the uniform bound, and the expression (81) for $K_\pm^{\boldsymbol{\omega}}(\mathbf{x},\, \mathbf{p})$ is obtained from $H_\pm^{\boldsymbol{\omega}} - H_0$ without calculation. In [3], magnetic fields with compact support and nonvanishing flux are considered as well, by employing a family of transversal gauges with adapted reference point. Cf. Cor. 2.8. Analogously we have

Remark 5.2 (Generalization (Arians))
Suppose $\nu = 2$ and A_0, $\mathbf{B}$ satisfy the assumptions of Thm. 5.1, except the flux is not vanishing. Then (76) and the proof remain valid, if $\mathbf{A}(\mathbf{x})$ decays integrably in the half-plane $\boldsymbol{\omega}\cdot\mathbf{x} > 0$ and in a sector around $-\boldsymbol{\omega}$. When $\mathbf{A}$ is fixed, we may consider a family of gauge transformations according to Cor. 2.8 to satisfy the decay requirements, and the right hand side of (76) is modified.

As noted in [2, 3], the uniform estimate of the Cook integral (80) and Remark 5.2 give error bounds for the high-energy limit of S according to Thm. 1.4:

Corollary 5.3 (Error Bounds (Arians))
1. *Under the short-range assumptions of Thm.* 5.1*, the limit* (7) *has an explicit error bound for* $\widehat{\psi} \in C_0^\infty$ *, which is of the form*

$$\mathrm{e}^{-\mathrm{i}u\mathbf{x}}\, S\, \mathrm{e}^{\mathrm{i}u\mathbf{x}}\, \psi = \mathrm{e}^{\mathrm{i}a(\boldsymbol{\omega},\, \mathbf{x})}\, \psi + \mathcal{O}(1/u)\ . \tag{84}$$

2. *If* $\nu = 2$ *and the flux of* $\mathbf{B}$ *is not vanishing, then an analogous weak estimate remains valid for* $\widehat{\phi},\, \widehat{\psi} \in C_0^\infty$ *and medium-range* $\mathbf{A}$ *with* $\mathbf{A}(\mathbf{x}) = \mathcal{O}(1/|\mathbf{x}|)$*:*

$$\Big(\phi,\, \mathrm{e}^{-\mathrm{i}u\mathbf{x}}\, S\, \mathrm{e}^{\mathrm{i}u\mathbf{x}}\, \psi\Big) = \Big(\phi,\, \mathrm{e}^{\mathrm{i}a(\boldsymbol{\omega},\, \mathbf{x})}\, \psi\Big) + \mathcal{O}(1/u)\ . \tag{85}$$

Error bounds for (76) would require stronger decay assumptions on A_0 [9], or stronger regularity assumptions [24]. The right hand side of (76) contains a multiplication operator times $\mathbf{p}$. In the short-range case, it can be rewritten according to

$$\int_{-\infty}^{0} \mathbf{A}_{-}^{\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}t)\,dt+\int_{0}^{\infty} \mathbf{A}_{+}^{\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}t)\,dt = \int_{-\infty}^{\infty} t\boldsymbol{\omega}\times\mathbf{B}(\mathbf{x}+\boldsymbol{\omega}t)\,dt = \nabla_{\boldsymbol{\omega}}\,a(\boldsymbol{\omega},\,\mathbf{x})\,. \quad (86)$$

Only the last expression remains meaningful, if $\mathbf{B}$ is smooth but does not not decay faster than $|\mathbf{x}|^{-2}$: an asymptotic expansion of $\mathrm{e}^{-\mathrm{i}\mathbf{u}\mathbf{x}}\,S\,\mathrm{e}^{\mathrm{i}\mathbf{u}\mathbf{x}}$ in powers of $1/u$ has been obtained in [24] for $\mathbf{B} \in C^\infty$, $\partial^\alpha \mathbf{B}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-\mu-|\alpha|})$, $\mu > 3/2$.

In the special case of $\mathbf{B} = 0$ and $\mathbf{A} = 0$, thus $H = H_0 + A_0$, the uniform estimate of (80) together with $S - 1 = \Omega_+^*(\Omega_- - \Omega_+)$ and the strong limit of Ω_+^* yield

$$\lim_{u\to\infty} \frac{u}{m}\left(\mathrm{e}^{-\mathrm{i}\mathbf{u}\mathbf{x}}\,S\,\mathrm{e}^{\mathrm{i}\mathbf{u}\mathbf{x}} - 1\right)\psi \;=\; -\mathrm{i}\int_{-\infty}^{\infty} A_0(\mathbf{x}+\boldsymbol{\omega}t)\,\psi\,dt \qquad (87)$$

for $\widehat{\psi} \in C_0^\infty$. This limit was obtained by Enss and Weder in a series of papers, cf. [8, 9], introducing the time-dependent geometric method, and including also the cases of N-particle scattering, inverse two cluster scattering, and long-range electrostatic potentials. Error bounds for the weak formulation of (87) were established in [9] under stronger decay assumptions. The strong limit is due to [17].

6 Concluding Remarks

A geometric interpretation of the scattering process at high energies is given in [8, 10], cf. Sec. 5.1. Here we shall discuss the implications of gauge invariance and the question of measurable quantities in a scattering process, as well as the possible application of the inverse scattering problem.

6.1 Gauge Invariance

The vector potential $\mathbf{A}$ is determined only up to a gradient by the magnetic field $\mathbf{B}$ (and by additional discrete values of fluxes, when the domain is multiply connected). It is not eliminated easily from the Schrödinger equation $\mathrm{i}\dot{\psi} = H\psi$. (For the nonlinear hydrodynamic formalism, cf. [33, 31, 26].) The Schrödinger equation or Pauli equation is invariant under the simultaneous gauge transformation of $\mathbf{A}$ and ψ,

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\lambda \qquad \psi \to \psi' = \mathrm{e}^{\mathrm{i}\lambda}\,\psi\;. \qquad (88)$$

(If the electromagnetic field was time-dependent, we would have $A_0' = A_0 - \dot{\lambda}$ in addition.) One interpretation is, that the transformation of $\mathbf{A}$ comes from $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$, and it needs to be compensated for by the transformation of ψ. Following Weyl, this argument can be reversed: if we assume that the local phase of the wave function ψ is not observable, the theory should be invariant under a local $U(1)$ transformation, and ψ must be coupled to a vector potential. It is assumed in general that $\mathbf{A}$ and $\mathbf{A}'$ describe the same magnetic field, and that all observable physical effects are independent of the chosen gauge. Thus an electron in an electromagnetic field is described by an equivalence class of pairs $(\mathbf{A},\, \psi)$, where $(\mathbf{A},\, \psi) \sim (\mathbf{A}',\, \psi')$ iff there is a λ with (88).

Gauge invariance implies that not every self-adjoint operator corresponds to a physical observable: Suppose that the self-adjoint operator $F = F(\mathbf{p},\, \mathbf{A}(\mathbf{x}),\, \ldots)$ is constructed from a function $f(\mathbf{p},\, \mathbf{A}(\mathbf{x}),\, \ldots)$ by some quantization procedure (since the ordinary functional calculus does not apply due to $[\mathbf{x},\, \mathbf{p}] \neq 0$). Then the expectation value $(\psi,\, F\psi)$ is gauge-invariant, i.e., it depends only on the equivalence class of $(\mathbf{A},\, \psi)$, iff

$$\mathrm{e}^{\mathrm{i}\lambda(\mathbf{x})}\, F(\mathbf{p},\, \mathbf{A}(\mathbf{x}),\, \ldots)\, \mathrm{e}^{-\mathrm{i}\lambda(\mathbf{x})} = F(\mathbf{p},\, \mathbf{A}(\mathbf{x}) + \mathbf{grad}\, \lambda(\mathbf{x}),\, \ldots)\ . \qquad (89)$$

This restriction is a "superselection rule" in the general sense of [33]. At least if f is polynomial in $\mathbf{p}$, this means that it depends on $\mathbf{p}$ and $\mathbf{A}$ only in the combination $\mathbf{p} - \mathbf{A}$. Examples of nonobservable operators are $\mathbf{A}(\mathbf{x})$, the canonical momentum operator $\mathbf{p}$, the free Hamiltonian $H_0 = \frac{1}{2m}\mathbf{p}^2$, and the canonical angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. The following operators are among the observables: $\mathbf{x}$, $A_0(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, the kinetic momentum $m\dot{\mathbf{x}} = \mathbf{p} - \mathbf{A}$, the kinetic energy $\frac{1}{2m}(\mathbf{p} - \mathbf{A})^2$, the Hamiltonian H, and the kinetic angular momentum $\mathbf{x} \times m\dot{\mathbf{x}} = \mathbf{x} \times (\mathbf{p} - \mathbf{A}) = \mathbf{L} - \mathbf{x} \times \mathbf{A}$. See [33, 19] for a discussion of vector potentials and gauge invariance in the context of the Aharanov–Bohm effect [26].

6.2 The Scattering Cross Section

When $\mathbf{B} = 0$, or for the time evolution of asymptotic configurations, it is natural to set $\mathbf{A} = 0$ by convention. In the scattering theory with medium-range vector potentials, we must assume that $\mathbf{A}$ and $\mathbf{A}'$ describe the same physical system, as soon as $\mathbf{curl}\, \mathbf{A} = \mathbf{curl}\, \mathbf{A}'$. Then $\mathbf{A}' - \mathbf{A}$ need not be short-range, and the scattering operator S is not gauge-invariant, but it transforms according to (6). Given a scattering state $\psi \in \mathrm{Ran}(\Omega_+)$, the particle is found in a cone $\mathcal{C}$ for $t \to +\infty$ with probability

$$\lim_{t\to+\infty} \Big\|\, F(\mathbf{x} \in \mathcal{C})\, \mathrm{e}^{-\mathrm{i}Ht}\, \psi \,\Big\|^2 \;=\; \Big\|\, F(\mathbf{p} \in \mathcal{C})\, \Omega_+^*\, \psi \,\Big\|^2 \qquad (90)$$

according to Dollard [28, Thm. IX.31]. Now $\Omega_+'^*\psi' = \mathrm{e}^{\mathrm{i}\Lambda(\mathbf{p})}\, \Omega_+^*\psi$ by (6), thus this number is gauge-invariant. By the correspondence between subsets of $S^{\nu-1}$ and cones in $\mathbb{R}^\nu$ (with apex 0), (90) defines a measure on $S^{\nu-1}$. The differential cross section $d\sigma/d\omega$ for incident momentum $\mathbf{q}$ is obtained when $\phi = \Omega_-^*\psi$ is approaching a plane wave, rescaled such that its Fourier transform $\widehat{\phi}(\mathbf{p})$ is approaching "$\sqrt{\delta(p_1 - q)}\, \delta(p_2)\, \delta(p_3)$" when $\mathbf{q} = (q,\, 0,\, 0)^{\mathrm{tr}}$. If the momentum support of an incoming asymptotic configuration ϕ is concentrated at $\mathbf{p} \approx \mathbf{q}$, or at $\mathbf{p} \in \mathbf{q}[0,\, \infty)$, we have

$$S'\phi = \mathrm{e}^{\mathrm{i}\Lambda(\mathbf{p})}\, S\, \mathrm{e}^{-\mathrm{i}\Lambda(-\mathbf{p})}\, \phi \approx \mathrm{e}^{\mathrm{i}\Lambda(\mathbf{p})}\, S\, \mathrm{e}^{-\mathrm{i}\Lambda(-\mathbf{q})}\, \phi = \mathrm{e}^{\mathrm{i}\Lambda(\mathbf{p})\, -\, \mathrm{i}\Lambda(-\mathbf{q})}\, S\, \phi\ . \quad (91)$$

The phase factor in momentum space does not influence the probability of finding the outgoing state in a cone, thus $d\sigma/d\omega$ is gauge-invariant. By the same argument, i.e., replacing Λ with a, we may compute it from the gauge-invariant modified scattering operator S^J, which was defined in (70). See also [35, 30].

6.3 The Phase of the Scattering Amplitude

In short-range scattering theory, the differential cross section is $\dfrac{d\sigma}{d\omega} = \Big|f_{\mathbf{q}}(\boldsymbol{\omega})\Big|^2$, where $f_{\mathbf{q}}(\boldsymbol{\omega})$ is the scattering amplitude for incident momentum $\mathbf{q}$ and outgoing direction $\boldsymbol{\omega}$. It is obtained from the T-matrix, i.e., the integral kernel of $S-1$ on the energy shell. Probably this relation remains valid in the medium-range situation, although the latter kernel will be more singular on the diagonal [39]. In scattering experiments, usually the differential cross section is observed for an incident beam of particles, which is modeled as a plane wave. Information on the phase of the scattering amplitude is not available directly, but it is required for solving the inverse scattering problem, e.g., with (87).

For a small, central, scalar potential in $\mathbb{R}^3$, $f_{\mathbf{q}}$ can be reconstructed from $\Big|f_{\mathbf{q}}(\boldsymbol{\omega})\Big|^2$ by employing the unitarity of S, see [27, Sec. V.6.D] and the references in [20]. If this approach is extended to $\mathbb{R}^2$, it will work for rotationally symmetric $\mathbf{B}$ as well. Phase information would be available experimentally, if it was possible to localize the incoming particles more precisely. It could be reconstructed as well, if the location of the unknown scatterer is kept fixed, and the location of a known additional potential is varied while measuring the cross sections [20]. In some cases, phase information is obtained from interference between the scattered beam and a coherent reference beam [26].

6.4 Two-Particle Scattering

If two nonrelativistic particles are interacting via a pair potential $A_0(\mathbf{x}_2 - \mathbf{x}_1)$, their relative motion is equivalent to an external field problem for one particle with the reduced mass $m = \frac{m_1 m_2}{m_1 + m_2}$. The pair potential has a physical justification only if it is a central potential, or if, say, $m_2 \ll m_1$: Suppose particle 1 is a molecule with a dipole field given by $A_0(\mathbf{x}_2 - \mathbf{x}_1)$, and particle 2 is an electron. The molecule will be rotated by interacting with the electron, but the corresponding rotation of A_0 is neglected in the model. This simplification is justified if $m_1 \gg m_2$, and in this case we might assume as well that the molecule is generating a magnetic field. If the orientation of particle 1 is unknown, the scattering cross section may be defined by averaging over the orientations. But the phase information required for solving the inverse problem is unlikely to be recovered.

6.5 Inverse Scattering and Error Bounds

The high-energy asymptotics (7), (76), (87) might be applied independently from inverse scattering. But consider the inverse scattering problem for a particle in an external electromagnetic field, or for two-particle scattering under the restrictions from Sec. 6.4. The uniqueness of the solution is interesting from a theoretical point of view, because in the analogous situation of particle physics, the models are based mainly on scattering data. In our situation, two additional questions must be addressed, before a field can be reconstructed from a scattering experiment: the problem of obtaining phase information, cf. Sec. 6.3, and the effects of a high but finite energy. When a-priori bounds on A_0, $\mathbf{B}$, and $\mathbf{A}$ are given, it is possible to estimate the approximation error in (7) according to Cor. 5.3. Thus we can check in principle, if the required energy is available in the experimental setup, and if the nonrelativistic model makes sense for this high energy (note also that the scattering operators for the Pauli- and Dirac equations (positive energy) coincide if $A_0 = 0$, cf. [36, 19]).

If the high-energy limit was observed for a suitable family of states ψ, we could obtain the required X-ray transforms as multiplication operators. At a high but finite energy, we do not get the operator of multiplication with an approximate X-ray transform, and it is not clear how to obtain the latter. (If the X-ray transform was obtained approximately, we could apply regularization techniques to provide an approximate inversion of the X-ray transform, whose exact inversion is ill-posed.) For A_0 of compact support, it is suggested in [34] to consider (87) for a single chosen ψ. Or we might specify an inversion procedure for the X-ray transform and apply it to the high-energy asymptotics, to check if the resulting composition of operators is converging. It may be possible as well, to obtain A_0 at a lower energy by a recursive approach, or by considering more terms of the asymptotic expansion.

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