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UNIQUENESS AND SYMMETRY IN PROBLEMS OF OPTIMALLY DENSE PACKINGS

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Abstract. Part of Hilbert's eighteenth problem is to classify the symmetries of the densest packings of bodies in Euclidean and hyperbolic spaces, for instance the densest packings of balls or simplices. We prove that when such a packing problem has a unique solution up to congruence then the solution must have cocompact symmetry group, and we prove that the densest packing of unit disks in the Euclidean plane is unique up to congruence. We also analyze some densest packings of polygons in the hyperbolic plane.

I. Introduction

The objects of our study are the densest packings, particularly of balls and polyhedra, in a space of infinite volume; for a survey see the classic texts [Feje] and [Roge], and the review [FeKu]. Most interest has centered on densest packings in the Euclidean spaces $\mathbb{E}^n$, notably when the dimension n is 2 or 3, but we will see that packing problems in hyperbolic spaces $\mathbb{H}^n$ can clarify some issues for problems set in Euclidean spaces so we consider the more general problem in the n dimensional spaces $\mathbb{X}^n$, where $\mathbb{X}^n$ will stand for either $\mathbb{E}^n$ or $\mathbb{H}^n$. (It would be reasonable to generalize our considerations further, to symmetric spaces, and even to include infinite graphs, but as we have no

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noteworthy results in that generality we felt it would be misleading to couch our considerations in that setting.) We will give results of two types. We prove (Theorem 2) that when a packing problem in $\mathbb{X}^n$ has a solution which is unique up to congruence then that solution must have symmetry group cocompact in the isometry group of $\mathbb{X}^n$; and we prove (Theorem 1) that the densest packing of unit disks in $\mathbb{E}^2$ is unique up to congruence. In Section IV we analyze the symmetries of some densest packings of polygons in the hyperbolic plane. This will suggest a modified form of uniqueness for the solution of a densest packing problem.

We now introduce some notation and basic features of density. We will be concerned with "packings" of "bodies" in $\mathbb{X}^n$. By a body we mean a connected compact set in $\mathbb{X}^n$ with dense interior and boundary of volume 0. Assume given some finite collection $\mathcal{B}$ of bodies in $\mathbb{X}^n$, for instance a single ball. By a packing of bodies we then mean a collection P of bodies, each congruent under the isometry group of $\mathbb{X}^n$ to some body in $\mathcal{B}$, such that the interiors of bodies in P do not intersect. Denoting by $B_r(p)$ the closed ball in $\mathbb{X}^n$ of radius r and center p, we define the "density relative to $B_r(p)$" of a packing P as:

$$D_{B_r(p)}(P) \equiv \frac{\sum_{\beta \in P} m_{\mathbb{X}^n}[\beta \cap B_r(p)]}{m_{\mathbb{X}^n}[B_r(p)]}, \tag{1}$$

where $m_{\mathbb{X}^n}$ is the usual measure on $\mathbb{X}^n$. Then, assuming the limit exists, we define the "density" of P as:

$$D(P) \equiv \lim_{r \to \infty} \frac{\sum_{\beta \in P} m_{\mathbb{X}^n}[\beta \cap B_r(p)]}{m_{\mathbb{X}^n}[B_r(p)]}. \tag{2}$$

It is not hard to construct packings P for which the limiting density $D(P)$ does not exist, for instance by the adroit choice of arbitrarily large empty regions so that the relative density oscillates with r instead of having a limit. (In hyperbolic space the limit could exist but depend on p, which we also consider unacceptable.) The possible nonexistence of the limit of (2) is an essential feature of analyzing density in spaces of infinite volume; density is inherently a *global* quantity, and fundamentally requires a formula somewhat like (2) for its definition [Feje], [FeKu]. We discuss this further below.

The most important examples for which we know the densest packings are those for balls of fixed radius in $\mathbb{E}^n$ for $n = 2$ and 3. (For a recent survey of this problem in higher dimensions see [CoGS]). It will be useful in discussing these problems to make use of the notion of "Voronoi cell", defined for each body β in a packing P as the closure of the set of those points in $\mathbb{X}^n$ closer to β than to any other body

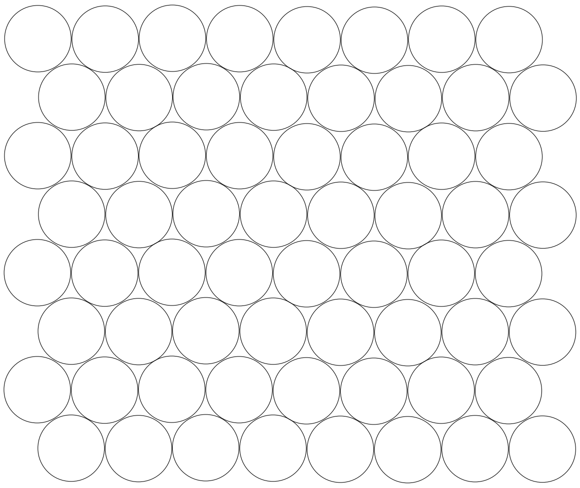

FIGURE 1. The densest packing of equal disks in the Euclidean plane

in P. A noteworthy feature of the $n = 2$ example is, then, that in the optimal packing (see Figure 1) the Voronoi cell of every disk (the smallest regular hexagon that could contain the disk) has the property that the fraction of the area of this cell taken up by the disk is strictly larger than for any other Voronoi cell in any packing by such disks. (Intuitively, the optimal configuration is simultaneously optimal in all local regions.) As for $n = 3$, it is generally felt that the densest lattice packing (i.e., the face centered cubic) achieves the optimum density among all possible packings, along with all the other packings made by layering hexagonally packed planar configurations, such as the hexagonal close packed structure; see [Roge]. There are claims in the literature by Hsiang [Hsia] and by Hales [Hale] for proofs of this, and there is hope that the problem will soon be generally accepted as solved.

Less well known but perhaps next in significance as examples of optimal density (see [Miln]) are the various "aperiodic tilings", especially the "Penrose kite & dart tilings", the tilings of $\mathbb{E}^2$ by congruent copies of the two polygons of Figure 2. (A portion of a kite & dart tiling is shown in Figure 3.) A key feature of these bodies is that the *only* way to tile the plane with them is with a tiling whose symmetry group does not have a fundamental domain of finite volume; this situation is the defining characteristic of "aperiodicity", and has led to renewed study of the symmetry of tilings (and thus packings); see [Radi]. There are other significant symmetry features of this example which will be

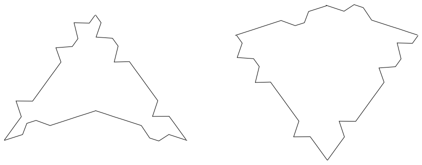

FIGURE 2. The Penrose kite & dart tiles of the Euclidean plane

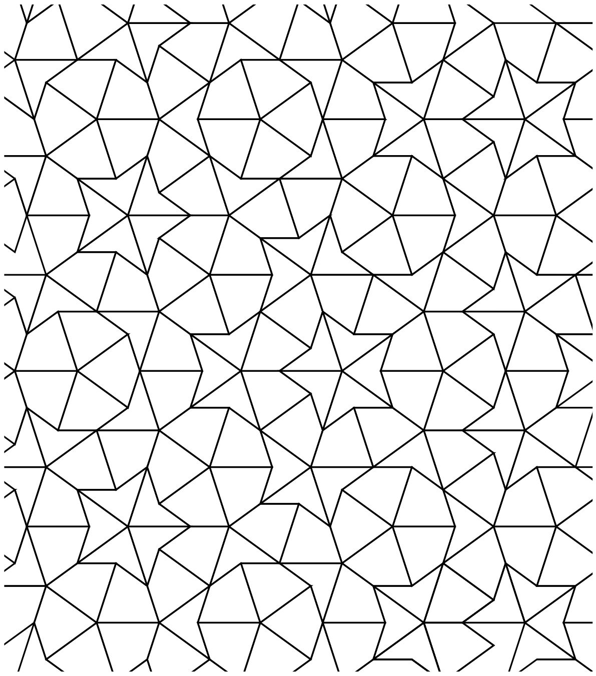

FIGURE 3. A Penrose kite & dart tiling of the Euclidean plane

discussed in Section V, where they will help to develop an appropriate notion of equivalence among optimally dense packings.

In practice it is almost impossible to actually determine optimally dense packings – for instance, there is nothing yet proven qualitatively of the densest packings in $\mathbb{E}^2$ by regular pentagons of fixed size (see however [KuKu]) – and we follow the lead of Hilbert [Hilb] and others in concentrating on general features of optima, such as their geometric symmetries. (It is because of their qualitative symmetry features that we attributed high significance to the aperiodic tilings.) We will discuss in Sections III and IV some optimization results for packings in hyperbolic spaces, by balls in $\mathbb{H}^n$, and by a certain polygon in $\mathbb{H}^2$.

We note that, for optimization in a Euclidean space, there is no difficulty proving *existence* of an optimum *density*, even though, since the limit of (2) does not exist for some packings, we are not able to make a comparison among *all* packings. One way to understand this is to use the fact that the relative densities $D_{B_r(p)}(P)$ of *any* packing P can be well approximated by packings P' that have compact fundamental domains; intuitively, the supremum of the densities of such symmetric packings P' is the desired optimum density, and can be shown to be achieved, *in the sense of* (2), by some packing which is a limit of such symmetric packings. (See [BoR2] for a complete argument.)

The situation for hyperbolic space packings is much more complicated, due to the fact that the volume of a ball of radius R grows exponentially in R ([Bear], [Kato]). This has the consequence that a significant fraction of the volume is near the surface of the ball, so it is by no means clear that one could make a useful approximation using a packing with cofinite symmetry group. Once one is prevented from reducing the problem to such symmetric packings, one is confronted with the difficulty of showing the *existence* of a limit such as (2) for a purported optimal packing; see [BoR2] for a history of this difficulty. In summary, lack of proof of the existence of appropriate limiting densities was an impediment to progress in the study of optimal density (and, *a fortiori*, optimally dense packings) in hyperbolic spaces for many years [BoR2]. This led to a search for alternatives to the notion of optimal density. The best known of these are that of "solid" packings, and "completely saturated" packings [FeKu]. Both notions are defined through local properties of packings. The quest for a local approach/alternative to optimal density is perhaps reasonable given that the only practical way yet devised to prove that a packing is optimal is to prove that it is locally optimal in all local regions (as noted above for disk packing in $\mathbb{E}^2$, and as the basis for the methods for ball packings in $\mathbb{E}^3$ [Laga]). However, no local alternative has proven satisfactory [FeKu], [Bowe].

So far we have concentrated on the question of *existence* of solutions (i.e., optimally dense packings) for our general optimization problem, especially the difficulties for packings in a hyperbolic space. This existence problem was solved recently ([BoR1], [BoR2]), the existence of limiting densities proven in an ergodic theory formalism outlined in the next section. One goal of this paper is analysis of the *uniqueness* of solutions for our optimization problems. Currently, there are difficulties even for Euclidean problems. For instance, even though it would be intuitively satisfying to declare that the problem of optimally dense packings of $\mathbb{E}^2$ by disks of fixed radius has the "unique" solution discussed above (Figure 1), there has been no satisfactory way to exclude some other packings of the same density, for instance those obtained by deleting a finite number of disks from this packing [CoGS]. This has been a serious obstacle to treatment of optimal density as one treats other optimization problems [Kupe], and will be a useful guide for our approach to a general understanding of the qualitative features of optimally dense packings.

The situation is significantly more complicated, and interesting, for the optimal packings (tilings) by kites & darts in $\mathbb{E}^2$ (Figure 2) than it is for equal disks. It can be proven that there are uncountably many pairwise noncongruent such tilings, but that every finite region in any one tiling appears in every other such tiling, so they are, in some sense, "locally indistinguishable" [Gard]. It is natural to want to declare that this optimization problem also has a "unique" solution, and this has, in effect, been the practice of those studying aperiodicity, a practice we will follow.

The notion of uniqueness must differentiate between the situations for densest packings of equal balls in $\mathbb{E}^3$ and that of $\mathbb{E}^2$; for the former the expected solution is intuitively far from unique, containing for instance the face centered cubic and also the hexagonal close packed structures, which must be considered different if the notion is used at all. However at a deeper level it must also give a useful criterion for when two such optimal structures are "the same", that is, it must give some useful notion of the geometric symmetry of optimally dense packings, for instance of the kite & dart tilings. In Section V we will discuss the connection between our approach and the use, by Connes and others, of noncommutative topology to understand the symmetry of structures such as the kite & dart tilings; see the expository works [Conn], [KePu] and references they contain.

As we hope to demonstrate, study of the uniqueness problem for packings of hyperbolic space will be useful even for understanding packings in Euclidean space. Much of this paper consists of the analysis of a specific family of examples of optimal density in a hyperbolic space, the first aperiodic examples in hyperbolic space for which explicit optimally dense packings have been determined. (See [BoR2] for an analysis of other examples of tilings in $\mathbb{H}^2$, such as those in [MaMo], [Moze] and [Good]). These examples exhibit features not seen in Euclidean examples, and will help us draw some conclusions about the general features of optimum density problems.

Optimal density problems have a long tradition but have never been treated as have other classes of optimization problems – for instance by analysis of conditions for existence and uniqueness of solutions. In this paper we consider the question of uniqueness, guided by symmetry properties of solutions. We analyze such problems as dynamical systems, with the group of isometries of the ambient space ($\mathbb{E}^n$ or $\mathbb{H}^n$) acting on the (compact, metrizable) space of all possible packings. We are led to classify such systems up to topological conjugacy, and use geometric features as invariants.

More specifically, Theorems 1 and 2 will lead us to partition the class of optimal density problems with unique solution into two classes: the periodic and the aperiodic. The study of *periodic* optimization problems in $\mathbb{E}^n$, for which the solutions have cofinite symmetry group, led, many years ago, to classification of the discrete subgroups of the isometry group of Euclidean space in low dimensions. Aperiodic tilings have led to related work; among problems set in $\mathbb{E}^3$, the study of the quaquaversal aperiodic tilings [CoRa] led to new results on classification of certain (dense) subgroups of $SO(3)$ [RaS1], [RaS2], [CoRS]. Similarly, the classification results in Section IV of our optimization problems in $\mathbb{H}^2$ naturally lead to *noncofinite* subgroups of PSL($2\mathbb{R}$), such as the symmetry groups of fixed-sum tilings (Theorem 6), as well as questions about Hecke groups.

In summary, classifying aperiodic optimization problems amounts to studying the "symmetries" of the packing solutions in senses related to, but different from, the manner appropriate for the well studied periodic structures. The mathematics that is generated by such analysis, a mixture of dynamics, operator algebras and Lie groups, is perhaps the main significance of the study of optimal density problems.

II. The Ergodic Theory Formalism for Optimal Packing

One concern of this paper is with the uniqueness of solutions to optimal density packing problems. While proving existence of solutions for problems set in Euclidean spaces did not require giving the problems a formal structure, the question of existence of solutions for problems set in hyperbolic space definitely did require introducing a formal structure, and we will see that this same structure is useful for handling questions of uniqueness, even in Euclidean space. We follow [BoR1], [BoR2] in introducing an ergodic theory structure into our optimization problems, in order to control the existence of limits such as (2).

Using the notation of section I, consider the space $\mathcal{P}_\mathcal{B}$ of all possible packings of $\mathbb{X}^n$ by bodies from $\mathcal{B}$, and put a metric on $\mathcal{P}_\mathcal{B}$ such that convergence of a sequence of packings corresponds to uniform convergence on compact subsets of $\mathbb{X}^n$. Such a metric makes $\mathcal{P}_\mathcal{B}$ compact, and makes continuous the natural action on $\mathcal{P}_{\mathcal{B}}$ of the (connected) group $\mathcal{G}^n$ of rigid motions of $\mathbb{X}^n$ [RaWo].

We next consider Borel probability measures on $\mathcal{P}_{\mathcal{B}}$ which are invariant under $\mathcal{G}^n$. (To see that such measures exist, consider any packing P for which the symmetry group has fundamental domain of finite volume, and identify the orbit $O(P)$ of P under $\mathcal{G}^n$ as the quotient of $\mathcal{G}^n$ by that symmetry group. One can then project Haar measure from $\mathcal{G}^n$ to an invariant probability measure on $O(P)$ and then extend it to all of $P_{\mathcal{B}}$ so that the complement of $O(P)$ has measure zero.)

We define the "density of the (invariant) measure μ" on $\mathcal{P}_{\mathcal{B}}$, $D(\mu)$, by $D(\mu) \equiv \mu(A)$, where A is the following set of packings:

$$A \equiv \{P \in \mathcal{P}_{\mathcal{B}} \,|\, \text{the origin } \mathcal{O} \text{ of } \mathbb{X}^n \text{ is in a body in } P\}. \tag{3}$$

(It is easy to see from the invariance of μ that $\mu(A)$ is independent of the choice of origin.) We may now introduce the notion of optimal density.

Definition 1. A probability measure $\bar{\mu}$ on the space $\mathcal{P}_{\mathcal{B}}$ of packings, ergodic under rigid motions, is "optimally dense" if $D(\bar{\mu}) = \sup_\mu D(\mu) = \sup_\mu \mu(A)$; the number $\sup_\mu \mu(A)$ is the "optimal density" for packing bodies from $\mathcal{B}$.

In [BoR1] the existence of such optimal measures is proven for any given $\mathcal{B}$.

Finally, the use of optimally dense measures in our optimum density problem is justified as follows. First we rewrite the right hand side of (1) as:

$$\lim_{r\to\infty} \frac{1}{\nu[\mathcal{G}^n(r,p)]} \int_{\mathcal{G}^n(r,p)} \chi_A[g(P)]\, d\nu(g), \tag{4}$$

where χ_A is the indicator function for A, ν is Haar measure on $\mathcal{G}^n$ and

$$\mathcal{G}^n(r,p) = \{g \in \mathcal{G}^n \,|\, d_{\mathbb{X}^n}[g(p),p] < r\}, \tag{5}$$

where $d_{\mathbb{X}^n}$ is the distance function on $\mathbb{X}^n$. It follows from G.D. Birkhoff's pointwise ergodic theorem [Walt] that for $\mathbb{X}^n = \mathbb{E}^n$ and any ergodic μ there is a set of P's, of full μ-measure and invariant under $\mathcal{G}^n$, for which the limit in (4) exists. This has been extended to $\mathbb{X}^n = \mathbb{H}^n$ by Nevo et al.: [Nevo], [NeSt] (with the invariance of the set of P's proven in [BoR2]). We may conclude then that "most" of the packings in the support of a fixed ergodic measure have the same well defined density in the sense of (2); so as one varies the measure one sees $\mathcal{P}_{\mathcal{B}}$ decomposed into packings of various densities, with those of optimal density being the ones in which we are interested. Formally we define optimally dense packings (slightly more stringently than in [BoR2]) as follows.

Definition 2. A packing P is "optimally dense" if it is "generic" for some optimally dense μ. That is, it is in the support of μ and:

$$\int_{\mathcal{P}_{\mathcal{B}}} f(Q)\, d\mu(Q) = \lim_{r\to\infty} \frac{1}{\nu[\mathcal{G}^n(r,p)]} \int_{\mathcal{G}^n(r,p)} f[g(P)]\, d\nu(g), \tag{6}$$

for every $p \in \mathbb{X}^n$ and every continuous f on $\mathcal{P}_{\mathcal{B}}$. The set of all optimally dense packings for bodies in $\mathcal{B}$ will be denoted $\mathcal{P}^o_{\mathcal{B}}$.

We note that the set of packings generic for the invariant measure μ is of full measure with respect to μ; this follows from the ergodic theorem and the fact that the space of continuous functions on $\mathcal{P}_{\mathcal{B}}$ is separable in the uniform norm.

Lemma 1. *If P is generic for the ergodic measure μ (which is not necessarily optimally dense) then:*

$$\begin{aligned} \lim_{r\to\infty} \frac{1}{\nu[\mathcal{G}^n(r,p)]} \int_{\mathcal{G}^n(r,p)} \chi_A[g(P)]\, d\nu(g) &= \int_{\mathcal{P}_{\mathcal{B}}} \chi_A(Q)\, d\mu(Q) \\ &= D(\mu) \end{aligned} \tag{7}$$

for every $p \in \mathbb{X}^n$.

Proof. Let

$$A' \equiv \{P \in \mathcal{P}_{\mathcal{B}} \,|\, \mathcal{O} \text{ is in the interior of a body in } P\}. \tag{8}$$

Define continuous f_k on $\mathcal{P}_{\mathcal{B}}$ by:

$$f_k(P) = \begin{cases} 1, & \text{on } A \\ 0, & \text{on } \{P \in A^c \,|\, d_{\mathbb{X}^n}(\mathcal{O}, \partial P) \ge \frac{1}{k}\} \\ 1 - kc, & \text{on } \{P \in A^c \,|\, d_{\mathbb{X}^n}(\mathcal{O}, \partial P) = c < \frac{1}{k}\}, \end{cases} \tag{9}$$

where ∂P denotes the union of the boundaries of the bodies in P. Note that the f_k decrease pointwise to χ_A. Similarly define continuous g_k on $\mathcal{P}_{\mathcal{B}}$ by:

$$g_k(P) = \begin{cases} 0, & \text{on } A'^c \\ 1, & \text{on } \{P \in A' \,|\, d_{\mathbb{X}^n}(\mathcal{O}, \partial P) \ge \frac{1}{k}\} \\ kc, & \text{on } \{P \in A' \,|\, d_{\mathbb{X}^n}(\mathcal{O}, \partial P) = c < \frac{1}{k}\}. \end{cases} \tag{10}$$

Note that the g_k increase pointwise to $\chi_{A'}$. Now, given $\epsilon > 0$, choose $K > 0$ such that

$$\begin{aligned} &0 < \int f_K \, d\mu - \int \chi_A \, d\mu < \epsilon/2 \quad \text{and} \\ &0 < \int \chi_{A'} \, d\mu - \int g_K \, d\mu < \epsilon/2. \end{aligned} \tag{11}$$

Define, for $R > 0$, the measure $\nu(R, P)$ on $\mathcal{P}_{\mathcal{B}}$ by

$$\int f \, d\nu(R, P) \equiv \frac{1}{\nu[\mathcal{G}^n(R, \mathcal{O})]} \int_{\mathcal{G}^n(R,\mathcal{O})} f[g(P)] \, d\nu(g), \tag{12}$$

for continuous f, where as before ν is Haar measure on $\mathcal{G}^n$. Then choose $\tilde{R} > 0$ such that

$$\begin{aligned} &\left| \int f_K \, d\nu(R, P) - \int f_K \, d\mu \right| < \epsilon/2 \quad \text{and} \\ &\left| \int g_K \, d\nu(R, P) - \int g_K \, d\mu \right| < \epsilon/2 \end{aligned} \tag{13}$$

for all $R > \tilde{R}$. We then have:

$$\begin{aligned} \int \chi_{A'} \, d\mu - \epsilon < \int g_K \, d\nu(R, P) < \int \chi_{A'} \, d\nu(R, P) \\ \leq \int \chi_A \, d\nu(R, P) < \int f_K \, d\nu(R, P) < \int \chi_A \, d\mu + \epsilon. \end{aligned} \tag{14}$$

However, from the ergodic theorem $\int \chi_A - \chi_{A'} \, d\mu = \int \chi_{A/A'} \, d\mu = 0$, so $\int \chi_{A'} \, d\nu(R, P)$ and $\int \chi_A \, d\nu(R, P)$ both converge to $\int \chi_A \, d\mu = D(\mu)$ as $R \to \infty$. □

Next we note a useful tool for computing optimal densities. For those $P \in \mathcal{P}_{\mathcal{B}}$ such that the point $p \in S^n$ is contained in the interior of a Voronoi cell (which cell we denote by $V_p(P)$), we define $F_p(P)$ to be the relative volume of $V_p(P)$ occupied by the bodies of P. (We note that F_p is defined μ-almost everywhere for any invariant μ.)

Definition 3. For invariant measures μ we define the "average Voronoi density for μ", $D_V(\mu)$, as $\int_{\mathcal{P}_{\mathcal{B}}} F_p(P) \, d\mu(P)$. (Note that $D_V(\mu)$ does not depend on p because of the invariance of μ.)

The notion of average Voronoi density is useful, as it has been shown [BoR1, BoR2] that, for any invariant measure μ, the average Voronoi density $D_V(\mu)$ equals the average density $D(\mu)$.

Summarizing the above, we have sketched a formalism through which one proves *existence* of solutions to the general problem of densest packings of $\mathbb{X}^n$ by congruent copies of bodies from $\mathcal{B}$. Our next goal is to consider uniqueness, but we will first need to discuss symmetry further (section III), and a new family of examples (section IV).

III. Symmetry in the Problem of Optimally Dense Packings

We will use the following common terms: a packing is called "periodic" (resp. "nonperiodic") if it has (resp. does not have) a symmetry group with fundamental domain of finite volume, and we say an optimal packing problem is "aperiodic" if *all* its optimally dense packings are nonperiodic.

We now use the formalism of section II to solve the old problem of making sense of uniqueness for the problem for disks of fixed radius in $\mathbb{E}^2$.

Theorem 1. *There is only one optimally dense packing in* $\mathbb{E}^2$ *for disks of fixed radius, up to rigid motion.*

Proof. Assume μ is any optimally dense measure for this problem. Using the fact that the hexagonal Voronoi cells of P^o (the hexagonal packing of Figure 1) are, up to rigid motion, the unique cells of optimal density (see [Feje]), and using the basic result on average Voronoi cells [BoR1], we see that for μ-almost every packing P, the cell containing a particular point p must be this regular hexagon. Repeating this argument for a countable dense set of p's, we see that for μ-almost every packing P *every* Voronoi cell is, up to rigid motion, this regular hexagon, i.e., μ-almost every packing P is P^o, up to rigid motion. It is shown in [BoR1] that there is a unique invariant measure with support in the (closed) orbit of any periodic packing, and therefore the orbit of P^o consists of all the optimal packings. □

The ergodic theory formalism automatically makes sense of the uniqueness of this optimization problem, by its subjugation of packings to invariant measures on packings. Aperiodic problems, such as the kites & darts, are more subtle. Our next result is a connection between aperiodicity and the uniqueness of packing problems.

Theorem 2. *If there is only one optimally dense packing of* $\mathbb{X}^n$*, up to congruence, by copies of bodies from some fixed, finite collection* $\mathcal{B}$*, then that packing must have a symmetry group with compact fundamental domain.*

Proof. By assumption there exists a probability measure μ, invariant under $\mathcal{G}^n$, for which the orbit $O(P)$ of some packing P has measure one. Since $O(P)$ can be identified with the quotient of $\mathcal{G}^n$ by the symmetry group Γ_P of P, it follows from the uniqueness of Haar measure on $\mathcal{G}^n$ that Γ_P is cofinite.

Since the volume of $\mathbb{X}^n/\Gamma_P$ is finite, only a finite number of bodies can appear in any fundamental domain, and in particular the bodies lie in a compact region of $\mathbb{X}^n/\Gamma_P$. If Γ_P were cofinite but not cocompact (something possible only if $\mathbb{X}^n$ is hyperbolic), then the preimage in $\mathbb{X}^n$ of the ends of $\mathbb{X}^n/\Gamma_P$ would not contain any bodies. However, the preimage of a hyperbolic end contains arbitrarily large balls, so there is room in our packing P to add additional bodies.

This contradicts the fact that optimally dense packings must be "saturated", meaning that one cannot add another body to the uncovered regions. In fact it was proven in [Bowe] that the set of *completely* saturated packings have full measure with respect to any optimally dense measure. (A "completely saturated packing" is one in which it is impossible to remove any finite number of bodies and replace them with bodies of larger total volume.) $\square$

One thing we can conclude from these two theorems is that the problems of optimal density decompose naturally into two classes: those allowing periodic optima (such as with balls of fixed size in $\mathbb{E}^2$ or $\mathbb{E}^3$), and the class of aperiodic problems. The former no longer pose any difficulty as to classifying their uniqueness, leaving us now to understand the more interesting class of aperiodic problems.

Aperiodicity is not an unnatural circumstance – it may even be generic in some sense; see [MiRa] for a related problem. In Euclidean space aperiodicity has only been discovered so far in packings of complicated polyhedra, whereas in hyperbolic space it already appears in packings of balls.

Theorem 3. [BoR1]. *For all but countably many fixed radii R the ball packing problem in $\mathbb{H}^n$ has only aperiodic solutions.*

For packings/tilings in Euclidean space there has not yet been a significant attempt to understand the uniqueness problem. We will postpone our attempt at a formal definition until after considering the following new examples, the first aperiodic problem in hyperbolic space for which we can determine explicit solutions.

IV. A New Example: the Modified Binary Tile in $\mathbb{H}^2$

Using the upper half plane model of $\mathbb{H}^2$, consider the "binary tile" τ of Figure 4 (introduced by Roger Penrose in 1978 [Penr]). We define as the "core" γ of τ that shape with four edges – two segments of geodesics, and two segments of horocycles (one twice the length of

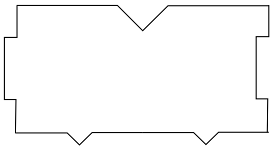

FIGURE 4. The simple binary tile in the upper half plane model of the hyperbolic plane

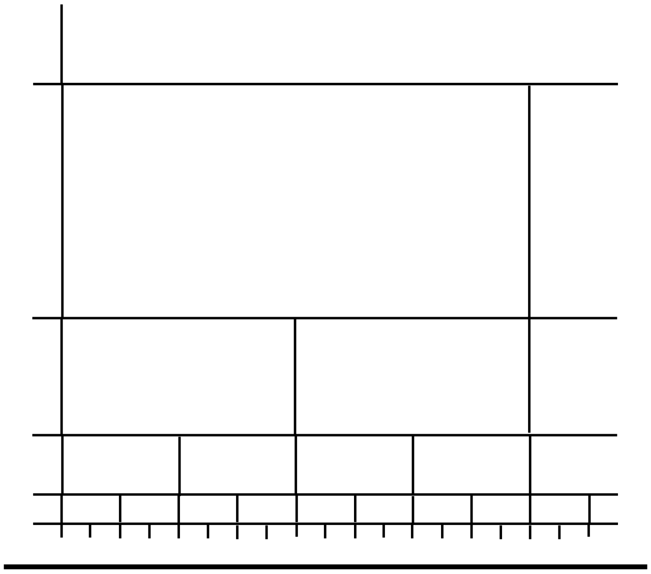

FIGURE 5. The binary tiling of the upper half plane

the other) – obtained by omitting the bumps and dents of τ. More specifically we could take the coordinates of the vertices of γ to be $i,\ i+2, 2i$, and $2i+2$. Congruent copies of τ can only tile $\mathbb{H}^2$ as in Figure 5. In fact it is useful to classify these tilings as follows. Once the location of one tile is known, the bumps on the geodesic edges of (the core of) τ force the positions of tiles filling out the region between two concentric horocycles (the ones containing the horocyclic edges of the (core of the) tile). Consider now the possible tiles abutting the ones in this "horocyclic strip". There is only one way to fill an abutting strip which is "further" from the common point at infinity of the horocycles, and two ways to fill the strip which is "closer". This fully classifies the possible tilings of $\mathbb{H}^2$ by τ.

We now construct a new tile $\bar{\tau}$ based on the same core γ, which will permit some new tilings. On the geodesic edges of γ we add the same bumps and dents as before, but we enlarge each of the other original

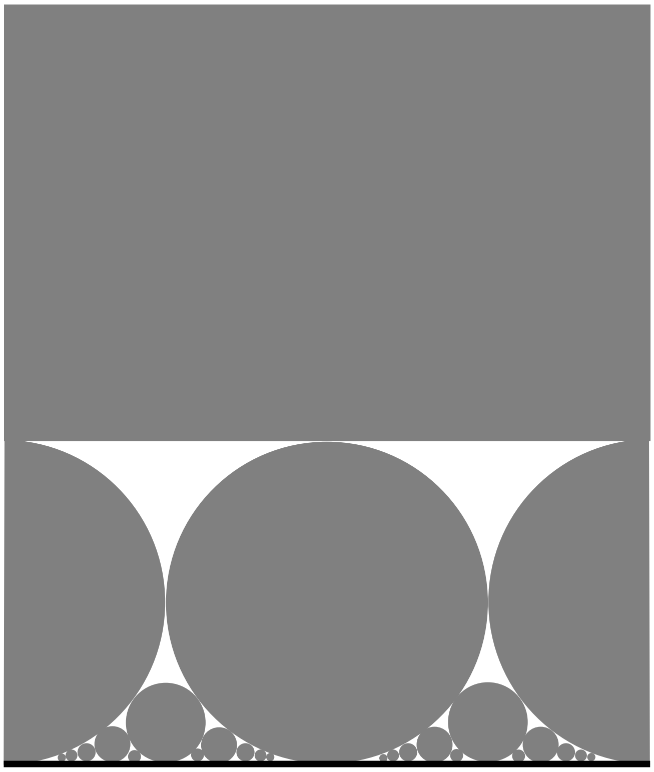

FIGURE 6. A packing of the upper half plane by horoballs

bumps and dents as follows. Consider a (densest [Boro]) packing of the hyperbolic plane by horoballs, as illustrated in Figure 6, and consider three abutting horoballs. Divide the (white) region between the horoballs into 3 congruent regions by means of three geodesics, each drawn from the center of three-fold symmetry of the region to the point where a pair of horoballs touch. The new tile $\bar{\tau}$ is illustrated in Figure 7. From the construction we see that, as with τ, in any tiling of the plane by $\bar{\tau}$, once we know the location of a specific tile we can uniquely fill in a horocyclic strip, and then have two choices for filling, consecutively, each of those strips which are closer to their point at infinity, thus filling in a horoball. However, there is now a second way to fill the abutting strip which is further from the point at infinity, in which the bumps from three tiles abut to fill in a white region in Figure 6. If this latter method is used, the only way to complete a tiling of the plane is to fill in each of the horoballs defined by the two new tiles, then use the same method to extend beyond these horoballs to more horoballs, etc. Intuitively, these new tilings are obtained from a densest horoball packing by tiling each horoball with copies of the tile $\bar{\tau}$.

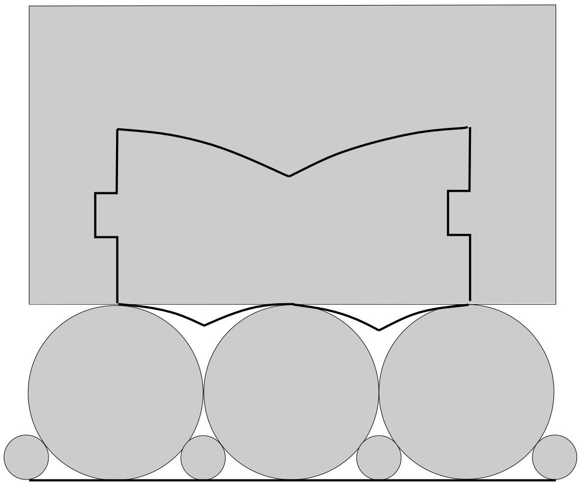

FIGURE 7. The modified binary tile in the upper half plane

Let $\mathcal{T}(\bar{\tau})$ be the set of all tilings of the plane by $\bar{\tau}$. We call such a tiling "degenerate" if the cores of the tiles themselves tile the hyperbolic plane, and "non-degenerate" if the union of the cores corresponds to a (densest) packing $\bar{P}$ of $\mathbb{H}^2$ by horoballs. (Such a horoball packing has symmetry group conjugate to $\mathrm{PSL}(2, \mathbb{Z})$, which is cofinite but not cocompact.) We know from [BoR1] that the set $\mathcal{T}_{deg}$ of degenerate tilings has measure 0 with respect to any invariant probability measure on $\mathcal{T}(\bar{\tau})$. As a result, degenerate tilings do not qualify as optimally dense packings of $\bar{\tau}$ in the sense of Definition 2. However, invariant measures on $\mathcal{T}(\bar{\tau})$ do exist.

To construct such a measure we consider the internal structures of the different horoballs. This structure is related to a choice of the two lower bumps on the tile $\bar{\tau}$, henceforth called "prongs". For each horoball H and each triangle that touches the horoball, the internal structure of H is associated to a dyadic integer, that is a formal sum $\sum_{i=0}^{\infty} a_i 2^i$, with $a_i \in \{0, 1\}$, where two dyadic integers are considered close if their first N terms agree, with N large. The first digit tells whether the left or right prong of a tile from H sticks into the triangle, the next digit tells whether that tile emerges from the left or right prong of its "parent", and so on. (We will eventually use the algebraic structure of these quantities.) We let each digit be an independent random variable, with equal probability of being 0 or 1. Let the internal structures of distinct horoballs be independent, and be independent of the location of the horoballs, which is given by Lebesgue measure on $\mathbb{H}^2/\mathrm{PSL}(2, \mathbb{Z})$.

From the existence of an invariant measure, it follows [BoR1] that there are tilings in $\mathcal{T}(\bar{\tau})$ which are optimally dense packings of $\bar{\tau}$ in the sense of Definition 2. It follows that *all* optimally dense packings of $\bar{\tau}$ are tilings [BoR1]. We now see that the optimal density problem for the modified binary $\bar{\tau}$ is aperiodic: the presence of the horoballs immediately implies that the symmetry group of an optimal packing/tiling is at most cofinite, not cocompact, and, as argued in the proof of Theorem 2, a tiling by compact bodies cannot have a symmetry group which is cofinite but not cocompact.

The presence of the closed invariant set $\mathcal{T}_{deg}$ is a new feature in optimal density problems. Since it is a subset of the orbit closure of every tiling, it is in the support of every invariant measure on $\mathcal{T}(\bar{\tau})$, even the ergodic ones. However, there is no invariant measure on $\mathcal{T}_{deg}$. This is not possible for problems set in a Euclidean space, since if the Euclidean group acts on a compact metric space a standard fixed point argument ([Radi]) guarantees the existence of an invariant probability measure on that set. We will discuss this feature of $\mathcal{T}_{deg}$ further below, when we consider various types of conjugacy for the dynamical systems in which we are couching our optimization problems.

We now turn to the construction of uniquely ergodic invariant subsets of $\mathcal{T}(\bar{\tau})$, and the measures they support. Let w be a dyadic integer. For each w, let $\mathcal{T}_w$ be the closure of the class of tilings for which the sum of the three dyadics at each triangle is w.

Theorem 4. *$\mathcal{T}_w$ is uniquely ergodic under the action of $\mathcal{G}^2 = PSL(2,\mathbb{R})$.*

Proof. For $T \in \mathcal{T}_w - \mathcal{T}_{deg}$, there is naturally associated to T a horoball packing $h(T)$. For $N > 0$ and $T \in \mathcal{T}_w - \mathcal{T}_{deg}$, we let $\Theta_N(T)$ be the packing obtained from T by removing all but the N horocyclic rows of tiles closest to the boundary of any horoball in $h(T)$. If $T \in \mathcal{T}_{deg}$, we let $\Theta_N(T)$ be the empty packing, $\emptyset$. Note that Θ_N defines a continuous map from $\mathcal{T}_w$ onto a compact space $\mathcal{P}_N$ of packings and that Θ_N commutes with the action of $\mathcal{G}^2$.

From Lemma 2 below it follows that $\mathcal{P}_N$ admits only two ergodic measures; one concentrated on $\emptyset$ and the other being derived from Haar measure on the space $\mathcal{G}^n/H$ where H is the symmetry group of $\Theta_N(T)$ for any $T \in \mathcal{T}_w - \mathcal{T}_{deg}$ ([BoR2]). Since $\mathcal{T}_{deg}$ has μ-measure zero with respect to any invariant measure μ on $\mathcal{T}_w$ ([BoR2]), the empty packing is measure zero with respect to the pushforward of μ. Therefore there is only one possibility for the pushforward of μ.

These pushforwards uniquely determine the measure μ. Let A_N be the set of tilings T for which one of the tiles of $\Theta_N(T)$ contains the origin, and let $S \subset \mathcal{T}_w$ be any cylinder set, consisting of tilings in which the tiles in some finite neighborhood of the origin appear in a given pattern. Since $\mu(\mathcal{T}_{deg}) = 0$, and since $\mathcal{T}_w - \mathcal{T}_{deg} = \cup_{N=1}^{\infty} A_N$, we have $\mu(S) = \mu(S \cap (\mathcal{T}_w - \mathcal{T}_{deg})) = \lim_{N\to\infty} \mu(S \cap A_N)$. However, $\mu(S \cap A_N)$ is determined by the pushforward of μ by some $\Theta_{N'}$, where N' depends on N and S. Since the cylinder sets generate the Borel σ-algebra of $\mathcal{T}_w$, it follows that $\mathcal{T}_w$ is uniquely ergodic. It remains only to prove the following lemma.

Lemma 2. *For any $T, T' \in \mathcal{T}_w - \mathcal{T}_{deg}$, $\Theta_N(T)$ has cofinite symmetry group, $\Theta_N(T)$ is in the orbit of $\Theta_N(T')$ and $\mathcal{P}_N$ is equal to this orbit union the empty packing.*

Proof. The symmetry group of the horoball packing $h(T)$ for $T \in \mathcal{T}_w - \mathcal{T}_{deg}$ is conjugate to $\mathrm{PSL}(2,\mathbb{Z})$. The packing $h(T)$ naturally corresponds to an infinite trivalent tree with the triangles of the packing corresponding to the vertices of the tree. We can navigate around the tree with two fundamental operations.

If a "state" is a vertex together with a choice of one of the three edges leading out from that vertex, then the two operations on states are

$$C = \text{Rotate counterclockwise by 120 degrees} \tag{15}$$

$$L = \text{Go forwards to the next vertex and bear left.} \tag{16}$$

C and L obviously generate the entire symmetry group of the tree. In terms of $\mathrm{PSL}(2,\mathbb{Z})$, C is the elliptic element

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \tag{17}$$

or $z \mapsto 1/(1-z)$, and L is the parabolic element

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{18}$$

or $z \mapsto z+1$, and together they generate all of $\mathrm{PSL}(2,\mathbb{Z})$.

Now we consider filling the horoballs with tiles $\bar{\tau}$, so that to each triangle we can associate three dyadic integers, one for each of the horoballs that meet at the triangle. Different triangles that touch the same horoball will not have the same dyadic integer; rather, moving along the edge of the horoball counterclockwise (as seen from inside the horoball) will increase the dyadic number by one each step.

If we apply the condition that the three numbers at each triangle must add up to w (a fixed dyadic integer), then two dyadic numbers at a triangle determine all the rest. If you know that a given vertex has numbers a and b then the third number must be $c = w - a - b$.

One neighboring vertex has numbers $b-1$ and $a+1$, so its third vertex must be c. Each vertex determines its neighbors, and so determines the entire tree. Thus a tiling in standard position (that is, with a choice of preferred vertex and preferred edge directed out from the preferred vertex) can be associated to a pair (a, b) of dyadic numbers, where a is the index of the horoball to the left of the preferred outgoing edge, and b is the index of the horoball to the right.

The action of L and C is easy to compute, namely:

$$L : (a, b) \mapsto (a + 1, c) \tag{19}$$

$$C : (a, b) \mapsto (c, a), \tag{20}$$

where $c = w - a - b$. It is not hard to check the following elements:

$$L^2 : (a, b) \mapsto (a + 2, b - 1) \tag{21}$$

$$CL^2C^2 : (a, b) \mapsto (a - 1, b + 2) \tag{22}$$

$$L^4CL^2C^2 : (a, b) \mapsto (a + 3, b) \tag{23}$$

$$L^2CL^4C^2 : (a, b) \mapsto (a, b + 3). \tag{24}$$

Now consider the effect of $\mathrm{PSL}(2, \mathbb{Z})$ on pairs (a, b) as above but taken modulo 2^N. Since 3 and 2^N are relatively prime, some power of $L^4CL^2C^2$ sends (a, b) to $(a + 1, b) \pmod{2^N}$, and some power of $L^2CL^4C^2$ sends (a, b) to $(a, b + 1) \pmod{2^N}$. Thus $\mathrm{PSL}(2, \mathbb{Z})$ acts transitively on the space of pairs $(a, b) \pmod{2^N}$.

Therefore, for all $T, T' \in \mathcal{T}_w - \mathcal{T}_{deg}$, $\Theta_N(T)$ is congruent to $\Theta_N(T')$. Also, the subgroup that preserves the pair $(a, b) \pmod{2^N}$ (and hence the first N rows of each horoball in the tiling) is an index 2^{2N} subgroup of $\mathrm{PSL}(2, \mathbb{Z})$, and hence is a cofinite subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Finally, P_N is the union of this orbit and the image of $\mathcal{T}_{deg}$, which is the empty packing. □

We next give another property of these "fixed-sum" classes of tilings, in terms of dynamical conjugacy. For convenience we recall some common terms. A topological group G acts continuously on a compact metric space X if there is a map $\phi : (g, x) \in G \times X \mapsto g(x) \in X$ which is continuous and satisfies $h[g(x)] = [hg](x)$ for all $g, h \in G$ and $x \in X$. Assuming G acts continuously on X and Y, the actions are called "topologically conjugate" if there is a homeomorphism $\alpha : x \in X \mapsto \alpha[x] \in Y$ such that $\alpha[g(x)] = g(\alpha[x])$. Assume further the existence on X and Y of Borel probability measures μ_X and μ_Y which are invariant under the corresponding actions of G. These two actions are said "measurably (or metrically) conjugate" if there are invariant subsets $X_0 \subset X$ and $Y_0 \subset Y$, each of measure zero, and an invertible map $\alpha' : x \in X/X_0 \mapsto \alpha'[x] \in Y/Y_0$ such that $\alpha'[g(x)] = g(\alpha'[x])$ which, together with its inverse, is measure preserving. Finally we introduce an intermediate form of conjugacy (related to "almost topological conjugacy" [AdMa]) as follows. The actions of G on X and Y will be called "almost conjugate" if there are invariant subsets $X_0 \subset X$ and $Y_0 \subset Y$, each of measure zero with respect to all invariant measures, and a homeomorphism $\alpha' : x \in X/X_0 \mapsto \alpha'[x] \in Y/Y_0$ such that $\alpha'[g(x)] = g(\alpha'[x])$.

Theorem 5. *$\mathcal{T}_w$ and $\mathcal{T}_{w'}$ are topologically conjugate if and only if $w - w' \in 3\mathbb{Z}$. $\mathcal{T}_w$ and $\mathcal{T}_{w'}$ are almost conjugate for any w, w'.*

Proof. If $w' - w = 3e$, where $e \in \mathbb{Z}$, then we construct a conjugacy by leaving the location of all the horoballs fixed, and simply adding e to the dyadic index of each horoball. In the N-th layer of a horoball the conjugacy essentially acts by translation by $e/2^N$, so points deep within a horoball are moved only slightly. In the case of degenerate tilings, the conjugacy leaves the entire tiling fixed.

If $w - w'$ is not 3 times a rational integer, then $(w - w')/3$ is still a dyadic integer, since 3 is a unit in the ring of dyadic integers. Adding $(w - w')/3$ to the index of each horoball is a continuous map on the complement of $\mathcal{T}_{deg}$, but is not uniformly continuous and does not extend to all of $\mathcal{T}_w$. This shows that $\mathcal{T}_w$ and $\mathcal{T}_{w'}$ are almost conjugate for any w, w'.

The proof that $w - w' \in 3\mathbb{Z}$ is necessary for topological conjugacy is harder, and will consist of four lemmas.

Lemma 3. *Any topological conjugacy ϕ between $\mathcal{T}_w$ and $\mathcal{T}_{w'}$ must preserve the points on the sphere at infinity that are tangent to horoballs. Furthermore, the "radii" of the horoballs can only change by a finite amount. That is, there exists a constant R (depending only on ϕ) such that, if $T \in \mathcal{T}_w$ is a tiling and H is a horoball in $h(T)$, and H' is the corresponding horoball in $h[\phi(T)]$ (that is, with the same tangent point), then H is contained in an R-neighborhood of H' and vice-versa.*

Lemma 4. *The R of the previous lemma is actually zero; topological conjugacies preserve the locations of horoballs exactly.*

Lemma 5. *Let H be any horoball in $h(T)$ for any $T \in \mathcal{T}_w$, and let a be its dyadic index (measured from a particular triangle). Let a' be the dyadic index of $H' \subset h[\phi(T)]$ measured from the same triangle. Then the set of differences $a' - a$ (for all such horoballs H in all such tilings T) is a bounded subset of $\mathbb{Z}$.*

Lemma 6. *There is a triangle in T, with indices a, b and c, such that $a' - a = b' - b = c' - c = (w' - w)/3$.*

Proof of Lemma 3. A topological conjugacy is uniformly continuous, so if $\phi : \mathcal{T}_w \mapsto \mathcal{T}_{w'}$ is a topological conjugacy, then for every $r' > 0$ and $\epsilon > 0$ there is a radius r such that, if the neighborhoods of two points agree to radius r, then from uniform continuity and conjugacy their images agree out to radius r', up to an "ϵ wiggle". More precisely, for any two tilings $T_1, T_2 \in \mathcal{T}_w$ and points p_1, $p_2 \in \mathbb{H}^2$, if there is an isometry of $\mathbb{H}^2$ that sends a ball of radius r of p_1 in T_1 exactly onto a ball of radius r of p_2 in T_2, then the same isometry sends a ball of radius r' of p_1 in $\phi(T_1)$ to an ϵ-small distortion of a ball of radius r of p_2 in $\phi(T_2)$, where an "ϵ-small distortion" means an isometry that moves each point in the neighborhood a distance ϵ or less.

Now take r' to be greater than the diameter of a triangle and ϵ to be much less than the diameter of a triangle. Take any tiling T for which $\phi(T)$ has a triangle centered at p_2. We claim that T contains a triangle centered at a point p_1 at distance at most $r+1$ from p_2. For if not, then the r-neighborhood of p_2 (call it U) lies completely within a horoball of $h(T)$. But then there is a constant, $\tilde{r}$, say such that every ball of size $\tilde{r}$ contains an r-ball such that T restricted to that r-ball is isometrically conjugate to T restricted to U. (Here T is thought of as the function from the plane to the tile $\bar{\tau}$ that is induced by the tiling T). This implies that every ball of size $\tilde{r} + \epsilon$ contains a triangle of $\phi(T)$. But this contradicts the fact that there are points, deep within a horoball of $h(\phi(T))$ that are at least a distance $\tilde{r} + \epsilon$ away from any triangle in the complement of $h(\phi(T))$. But this contradicts the fact that $\phi(T)$ is made up of horoballs, some points of which are arbitrarily far from triangles.

Now let H be any horoball in $h(T)$. Since all points in H that are farther than $r + 1$ from the boundary of H are mapped into a horoball in $h[\phi(T)]$ (i.e., not into a triangle), and since this set of points is connected, H lies within an r-neighborhood of a specific horoball $H' \subset h[\phi(T)]$. This implies that H and H' have the same tangent point on the sphere at infinity.

To obtain the fixed bound R, just repeat the argument for ϕ^{-1} and take R to be the larger of the two constants $r + 1$. $\square$

Proof of Lemma 4. Let R be as before. We know that ϕ preserves the location of each horoball, and changes its radius by at most R. The question is which horoballs grow and which shrink, and by how much. Consider a triangle in T, with center point p, where horoballs H_1, H_2 and H_3 meet. It is impossible for two of these horoballs to grow, or one to grow while a second does not change, lest they overlap. If two stay fixed, then the entire pattern is fixed. Thus, if there are *any* changes *anywhere*, then at each triangle either one horoball grows (or stays fixed) and the other two shrink, or all three shrink and one or more other horoballs $H_4, H_5, \ldots$ grow to fill up the space. There are only a finite number of horoballs within distance R of the triangle, so only a finite number of possible directions where the tile containing p in $\phi(T)$ can be pointing. By continuity there is a number N such that knowing all tiles within the Nth collar of the triangle determines which of H_1, H_2 and H_3 grow and which shrink. In particular, knowing the first N digits of the dyadic labels for $H_{1,2,3}$ determines which grow and shrink. In essence, all our labels should be counted mod 2^N.

There are 2^{2N} possible triples (a, b, c) of numbers $\pmod{2^N}$ that add up to $w \pmod{2^N}$. These correspond to labels for horoballs that meet at a triangle, counting clockwise. For each one, either all triangles with this label have the "a" tile shrink, or none do. Let S_a be the set of labels for which the "a" tile shrinks, let S_b be the set for which the "b" tile shrinks, and let S_c be the set for which the "c" tile shrinks. We will show all three sets are the whole set of triples, so all horoballs shrink, which is a contradiction.

Each horoball actually meets an infinite number of triangles. By comparing adjacent triangles with the same horoball we see that $(a, b, c) \in S_a$ if and only if $(a + 1, c - 1, b) \in S_a$. Continuing this process, we get that either the entire orbit $\{(a + 2n, b - n, c - n)\} \cup \{(a + 2m + 1, c - m - 1, b - m)\}$, is in S_a or the entire orbit is out. Note that 3 is a unit in $\mathbb{Z}_{2^N}$, so we can take $n = (b + 1 - a)/3$ and $m = (b - a - 2)/3$. This means that both

$$\left(\frac{a+2b+2}{3}, \frac{a+2b-1}{3}, \frac{c+a-b-1}{3}\right) \tag{25}$$

and

$$\left(\frac{a+2b-1}{3}, \frac{c+a-b-1}{3}, \frac{a+2b+2}{3}\right) \tag{26}$$

are in the same orbit as (a, b, c). If $(a, b, c) \notin S_a$, then in any triangle with indices $(a + 2b + 2)/3$, $(a + 2b - 1)/3$, and $(c + a - b - 1)/3$, in clockwise cyclic order, the horoballs with indices $(a + 2b + 2)/3$ and $(a + 2b - 1)/3$ must both grow (or stay the same size), which is impossible taking into account the first paragraph. Thus $(a, b, c) \in S_a$. But the triple (a, b, c) was arbitrary, so every triple is in S_a, and likewise in S_b and S_c. □

Proof of Lemma 5. First we show that for each horoball $a' - a$ must be an integer. For each dyadic integer x let $\pi_m(x)$ be the fractional part of $2^{-m}x$, and let $\sigma_m(x)$ be the integer part. The effect of adding $a' - a$ on the mth layer of the horoball is to translate the locations of the tiles by $\pi_m(a' - a)$ and to change the pattern of "ancestor" tiles by $\sigma_m(a' - a)$. If $a' - a$ is not an integer, then $\pi_m(a' - a)$ does not converge, so different layers deep in the horoball get shifted by different amounts, which contradicts uniform continuity, insofar as each piece of each layer looks like a piece of every other layer.

Thus for each horoball, the difference $a' - a$ is an integer. If these differences are not bounded, we can pick a sequence of horoballs H_m (with indices a_m, and possibly in different tilings) such that $\pi_m(a'_m - a_m)$ does not converge. Since with radius $m\log(2)$ every patch centered on the mth layer of H_m is replicated in the Mth layer of H_M, for every $M > m$, this lack of convergence of $\pi_m(a'_m - a_m)$ contradicts uniform continuity. $\square$

Proof of Lemma 6. Since the differences $(a' - a)$ take values in a finite set, the values of $(a' - a)$ can be determined by knowing the first N digits of (a, b, c). But at some triangles, it happens that $a = b = c \pmod{2^N}$ (since all allowable triples mod 2^N do occur, and since 3 is a unit when working mod 2^N). At such triangles, we must have $a' - a = b' - b = c' - c = (w' - w)/3$ by symmetry. By Lemma 5 this common difference must be a (rational) integer, so $w' - w \in 3\mathbb{Z}$, which completes the proof of this lemma, and the theorem. $\square$

A consequence of Lemma 6, together with the fact that two adjacent horoballs determine the entire tiling, is:

Proposition 1. *There is a unique topological conjugacy from $\mathcal{T}_w$ to $\mathcal{T}_{w'}$ when $w - w' \in 3\mathbb{Z}$. Equivalently, there are no nontrivial automorphisms of $\mathcal{T}_w$.*

Although the tilings in $\mathcal{T}(\bar{\tau})$ – in particular those in any $\mathcal{T}_w$ – cannot have a cofinite symmetry group, those in any $\mathcal{T}_w$ in fact do have a nontrivial symmetry group, as we see next.

Let $T_0 \in \mathcal{T}_w - \mathcal{T}_{deg}$ and consider the set $h^{-1}[h(T_0)]$ of nondegenerate tilings having the same associated horoball packing as T_0. We shall describe the symmetry group of $h^{-1}[h(T_0)]$, i.e., the subgroup of $\mathrm{PSL}(2, \mathbb{R})$ consisting of those elements which fix every tiling in $h^{-1}[h(T_0)]$. Up to conjugacy, this group is independent of w and T_0.

The symmetry group of the horoball packing $h(T_0)$ is conjugate in $\mathrm{PSL}(2, \mathbb{R})$ to $\mathrm{PSL}(2, \mathbb{Z})$; choosing a particular conjugacy is the same as choosing a triangle and a distinguished vertex in $h(T_0)$, i.e., a state in the trivalent tree. Fix a conjugacy and identify the symmetries of $h^{-1}[h(T_0)]$ with the elements of $\mathrm{PSL}(2,\mathbb{Z})$ which fix every pair of dyadic numbers. We recall some terminology from Lemma 2 concerning particular elements of $\mathrm{PSL}(2,\mathbb{Z})$.

Theorem 6. *The elements L^2 and $R^2 \equiv C^2LC^2L$ freely generate a subgroup $\mathcal{E}$ of $PSL(2,\mathbb{Z})$ of index 6. The symmetry group of $h^{-1}[h(T_0)]$ is the kernel of the abelianization $\langle L^2, R^2\rangle \mapsto \mathbb{Z}\oplus\mathbb{Z}$.*

Proof. The actions of L^2 and R^2 on pairs of dyadics are given by

$$R^2 : (a,b) \mapsto (a+1, b-2) \quad \text{and} \quad L^2 : (a,b) \mapsto (a+2, b-1). \tag{27}$$

One readily checks that the operations $L^2, R^2, L^{-2}, R^{-2}, R^2L^{-2}$ and L^2R^{-2} are precisely the ones which move from a vertex in the trivalent tree to a vertex two edges away and induce maps of the form $(a,b) \mapsto (a+k, b+\ell)$, $k,\ell \in \mathbb{Z}$ on pairs of dyadic numbers. It follows that the index of $\mathcal{E}$ in $\mathrm{PSL}(2,\mathbb{Z})$ is 6.

Freeness follows from the fact that distance from the starting point does not decrease as we follow some sequence of the basic operations $L^{\pm 2}, R^{\pm 2}$ unless one of the operations is followed immediately by its inverse.

Since the vectors $(1,-2)$ and $(2,-1)$ are linearly independent, the symmetry group of $h^{-1}[h(T_0)]$ consists of those words in $L^{\pm 2}, R^{\pm 2}$ for which the sums of the powers of R and L are both zero, i.e., the kernel of the abelianization of $\mathcal{E}$. □

We now note that our use of the densest packing by horoballs, Figure 6 (or, using the Poincaré disk, Figure 8), was not critical in the above method. An infinite family of generalizations can be made from other such horoball packings, as we now argue.

To generalize our "triangular" tilings, based on Figure 8, we consider tilings of $\mathbb{H}^2$ constructed as follows. First pack $\mathbb{H}^2$ by horoballs such that five horoballs meet along regular "pentagons" (rather than triangles), as in Figure 9. The symmetry group of such a packing, the Hecke group G_5, is a cofinite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by $z \mapsto -1/z$ and $z \mapsto z + \lambda$, where $\lambda = (1+\sqrt{5})/2$ is the golden mean.

We tile each horoball with (differently) modified binary tiles, where now we need an appropriate width so we can arrange that the prongs sticking out of such a tile each fill up a fifth of a pentagon; see Figure 10. Relative to such a pentagon, the tiling of a horoball with modified binary tiles is associated to a dyadic integer. The first digit tells whether we are on the left or right prong of the tile, the next digit tells whether that tile emerges from the left or right prong of its parent, and so on.

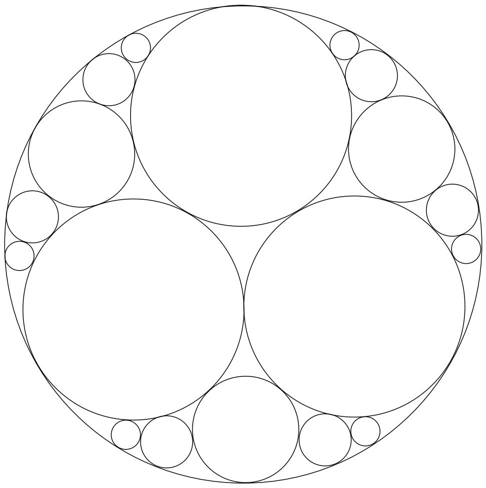

FIGURE 8. A packing of the Poincaré disk by horoballs

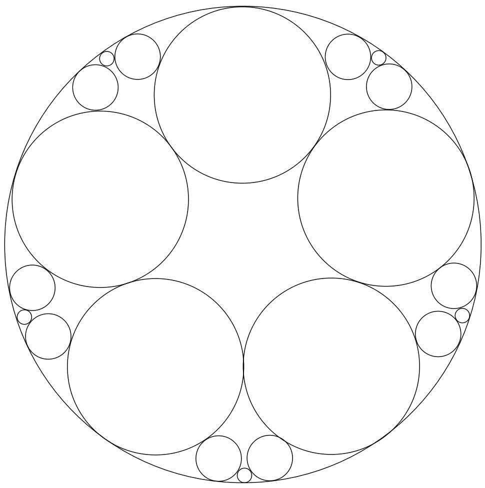

FIGURE 9. Another packing of the Poincaré disk by horoballs

Pick a rational integer k, once and for all. Let the dyadic integers around a pentagon, counting clockwise, be a, b, c, d and e. The "a" and "b" horoballs also meet at another pentagon, and we assume that the five dyadic integers representing these horoballs, counting *counter-clockwise*, are $a+1$, $b-1$, $c-k$, d and $e+k$. This rule for relating patterns around adjacent pentagons is a generalization of the "fixed sum" rule for triangular tilings. Notice also that the sum around the pentagons *is* fixed, and we let $\mathcal{T}_{k,w}$ be the closure of the class of tilings that follow the "k-rule" and for which the sum of the five dyadics at

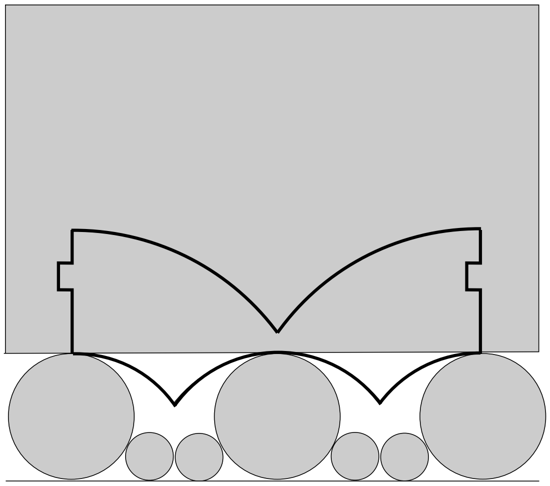

FIGURE 10. A "pentagonal" binary tile

each pentagon add up to w. Again we denote by $\mathcal{T}_{deg}$ the degenerate tilings.

Theorem 7. *$\mathcal{T}_{k,w}$ is uniquely ergodic under the action of $\mathcal{G}^2 = PSL(2,\mathbb{R})$.*

Proof. This proof is nearly identical to the proof of Theorem 4. As before, we approximate tilings in $\mathcal{T}_{k,w}$ by packings in which each horoball has only N layers, and show that such packings have cofinite symmetry groups.

As before let $\Theta_N(T)$ be the packing obtained from T by removing all but the N horocyclic rows of tiles closest to the boundary of any horoball in $h(T)$, and let P_N be the range of $\Theta_N(T)$.

In place of Lemma 2, we need to prove:

Lemma 7. *For any $T\,T' \in \mathcal{T}_{k,w} - \mathcal{T}_{deg}$, $\Theta_N(T)$ has cofinite symmetry group, $\Theta_N(T)$ is in the orbit of $\Theta_N(T')$ and $\mathcal{P}_N$ is equal to this orbit union the empty packing.*

Proof of Lemma 7. A "pentagonal" horoball packing of $\mathbb{H}^2$ corresponds to an infinite 5-valent tree, with the pentagons of the packing corresponding to the vertices of the tree. We can navigate around the tree with two fundamental operations. If a "state" is a vertex together with a choice of one of the five edges leading out from that vertex, then the two operations are

$$P = \text{Rotate counterclockwise by 72 degrees} \tag{28}$$

$$L = \text{Go forwards to the next vertex and bear hard left.} \tag{29}$$

Together these generate G_5. In terms of $\mathrm{PSL}(2,\mathbb{R})$, P is the elliptic element $z \mapsto 1/(\lambda - z)$ while L is the parabolic element $z \mapsto z + \lambda$.

We list the horoballs around a vertex in counterclockwise order, starting with the one to the right of the chosen edge. We need only list the first four of the five horoballs, since if their dyadic integers are a, b, c, and d, then the last one must be $e = w - a - b - c - d$.

The actions of L and P are easy to compute, namely:

$$L : (a,b,c,d) \mapsto (a+1, e+k, d, c-k) \tag{30}$$

$$P : (a,b,c,d) \mapsto (e,a,b,c). \tag{31}$$

It is not hard to check the following elements of G_5:

$$L^2 : (a,b,c,d) \mapsto (a+2, b-1+k, c-k, d-k) \tag{32}$$

$$PL^2P^4 : (a,b,c,d) \mapsto (a+k-1, b+2, c-1+k, d-k) \tag{33}$$

$$P^2L^2P^3 : (a,b,c,d) \mapsto (a-k, b+k-1, c+2, d+k-1) \tag{34}$$

$$P^3L^2P^2 : (a,b,c,d) \mapsto (a-k, b-k, c+k-1, d+2). \tag{35}$$

Thus the possible values of (a,b,c,d) differ by (among others) the elements of the sub-lattice of $\mathbb{Z}^4$ generated by $(2, k-1, -k, -k)$, $(k-1, 2, k-1, -k)$, $(-k, k-1, 2, k-1)$ and $(-k, -k, k-1, 2)$.

Since

$$\det \begin{pmatrix} 2 & k-1 & -k & -k \\ k-1 & 2 & k-1 & -k \\ -k & k-1 & 2 & k-1 \\ -k & -k & k-1 & 2 \end{pmatrix} = 5[(k-2)(k-1)k(k+1)+1] \tag{36}$$

is odd, G_5 acts transitively on the space of quadruples (a,b,c,d) (mod 2^N), and the subgroup that preserves a given quadruple (and hence the first N rows of each horoball in the tiling), is an index 2^{4N} subgroup of G_5, and hence is a cofinite subgroup of $\mathrm{PSL}(2,R)$. The remainder of the lemma, and the theorem, follow as in triangular case: Lemma 2 and Theorem 4. $\square$

Next we consider the question of conjugacy for these systems. Again, we just modify the argument that worked for triangle tilings.

Theorem 8. *$\mathcal{T}_{k,w}$ and $\mathcal{T}_{k,w'}$ are topologically conjugate if and only if $w - w' \in 5\mathbb{Z}$. $\mathcal{T}_w$ and $\mathcal{T}_{w'}$ are almost conjugate for any w, w'.*

Proof. The proof is essentially the same as the proof of Theorem 5, in particular that of almost conjugacy, which we do not discuss further. If $w - w' \in 5\mathbb{Z}$, the conjugacy is simply adding $(w-w')/5$ to each dyadic index. The converse follows from the analogues of Lemmas 33–6. The proofs of Lemmas 3, 5, and 6 carry over almost word-for-word.

Lemma 4 was algebraic, and used specific properties of the fixed-sum rule for triangles. In its place we have the following two lemmas that are specific to pentagonal horoball packings and the Hecke group G_5.

Lemma 8. *The only elements of* $\mathbb{Q}[\sqrt{2}]$ *that appear as elements of matrices in* G_5 *are* -1, 0 *and* 1.

Proof. Viewed as matrices, the pentagonal Hecke group is generated by $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The matrix elements are manifestly elements of $\mathbb{Z}[\lambda]$, and $\mathbb{Z}[\lambda] \cap \mathbb{Q}[\sqrt{2}] = \mathbb{Z}$. We will show that the only integers that actually appear as matrix elements are 0 and ± 1.

Rosen [Rose] showed that an element of $\mathbb{Z}[\lambda]$ is an element of a matrix in the group if and only if it is (up to sign) the denominator of a finite approximant of the continued fraction

$$r_0\lambda + \cfrac{\epsilon_1}{r_1\lambda + \frac{\epsilon_2}{r_2\lambda + \cdots}}, \tag{37}$$

where $\epsilon_i = \pm 1$ and each r_i is a positive integer (except possibly r_0, which may be zero). The continued fraction expansion of a real number is not unique, but can always be expressed in a unique "reduced form", one of whose requirements is that if $r_n = 1$, then $\epsilon_{n+1} r_{n+1} \neq -1$. The denominators Q_n of the successive approximants to the (possibly infinite) continued fraction satisfy the recursion:

$$Q_n = r_n \lambda Q_{n-1} + \epsilon_n Q_{n-2}, \tag{38}$$

and we may take $Q_{-1} = 0$ and $Q_0 = 1$. Writing $Q_n = a_n\lambda + b_n$, the recursion becomes:

$$a_n = r_n(a_{n-1} + b_{n-1}) + \epsilon_n a_{n-2} \tag{39}$$

$$b_n = r_n a_{n-1} + \epsilon_n b_{n-2} \tag{40}$$

We claim that the coefficients satisfy three properties: α) $a_n \geq a_{n-1}$, β) $b_n \geq 0$ and γ) $a_n \geq b_{n-1}$. These are easily checked for $n = 1, 2, 3$. We prove these hold for all n by induction. Suppose they hold for n up to k. We have $a_{k+1} = r_{k+1}(a_k + b_k) + \epsilon_{k+1} a_{k-1}$. If $\epsilon_{k+1} = 1$ this is manifestly at least a_k. If $r_{k+1} > 1$ and $\epsilon_{k+1} = -1$ then $a_{k+1} - a_k = r_{k+1} b_k + (r_{k+1} - 1) a_k - a_{k-1}$, which is non-negative since $a_k - a_{k-1} \geq 0$. Finally, if $r_{k+1} = 1$ and $\epsilon_{k+1} = -1$, then $r_k \geq 2$, and by property γ we have

$$\begin{aligned} a_{k+1} = a_k + b_k - a_{k-1} &= a_k + (r_k - 1) a_{k-1} + \epsilon_k b_{k-2} \\ &\geq a_k + a_{k-1} - b_{k-2} \geq a_k, \end{aligned} \tag{41}$$

which is the needed induction for α. Next, $b_{k+1} = r_{k+1} a_k + \epsilon_{k+1} b_{k-1} \geq a_k - b_{k-1} \geq 0$, which is the needed induction for β. For γ we note $a_{k+1} \geq a_k + b_k - a_{k-1} \geq b_k$, which completes the induction. Finally, since the sequence a_k is nondecreasing and since a_1 is positive a_k is never zero and Q_k is never rational for $k \geq 1$. □

Lemma 9. *A conjugacy between $\mathcal{T}_{k,w}$ and $\mathcal{T}_{k,w'}$ must preserve the locations of horoballs exactly.*

Proof. We have already shown that the points on the sphere at infinity where the horoballs touch are not changed, that their radii change by a bounded amount, and that there are only a finite number of possible values for that change in radius. This implies that knowing the first N digits of all five indices at a pentagon will determine which horoballs grow and shrink, and by how much. Note also that the deep interiors of all horoballs are identical, so the change in radius is the same for all horoballs, modulo $\log(2)$.

Now consider a tiling whose associated horoball packing is as follows: One horoball is the set $\{x + iy \,|\, y \geq 1\}$, and the others are its images under the Hecke group G_5. For each matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ in the group there is a horoball tangent to the x-axis at α/γ with Euclidean diameter $1/\gamma^2$.

Note that the dyadic indices of the tiling modulo 2^N are unchanged by the transformation $z \mapsto z + 2^{N+1}\lambda$, so that ϕ of this tiling corresponds to a packing that is invariant under $z \mapsto z + 2^{N+1}\lambda$. However, if the packing contains the horoball $\{x+iy \,|\, y \geq \nu\}$, then it is invariant only under addition of multiples of $\nu\lambda$. Thus ν must divide 2^{N+1}, and in particular must be rational.

Now consider a horoball that meets the horoball at infinity in the new packing (but did not meet the horoball at infinity in the original packing). Since its (hyperbolic) radius has changed by $-\log(\nu)$ (mod $\log(2)$), and since ν is rational, its Euclidean radius must have been a power of 2 times the square of a rational to begin with. However, this implies that there is an element of the Hecke group with γ of the form of a product of a rational and a power of $\sqrt{2}$, and not equal to 0 or ± 1. By Lemma 8 no such element exists. □

Next we consider some differences between optimization problems with other variations on our basic tile. So far we have considered packings of $\mathbb{H}^2$ by horoballs which meet either in "triangles" (the densest packing of horoballs) or in "pentagons", and modified our basic tile to have 2 prongs, each of which is either a third of a triangle (Figure 7) or a fifth of a pentagon (Figure 10). One can easily allow horoball packings defined by other regular n-gons, and also consider tiles to have more prongs, one for each neighboring n-gon; for $n = 3$ and $m = 3$ see Figure 11.

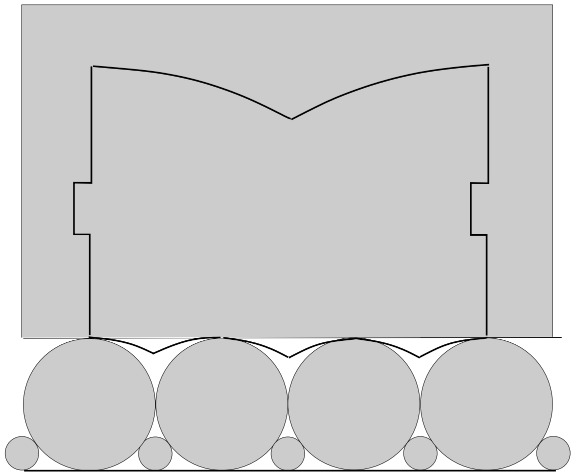

FIGURE 11. A triangular tile with 3 prongs

Let $\mathcal{T}(n,m)$ denote the space of all tilings by m-pronged tiles, each prong congruent to one of the n isosceles triangles dividing an n-gon as defined above. Let X be a closed, $\mathrm{PSL}(2,\mathbb{R})$-invariant subset of $\mathcal{T}(n,m)$ and let Y be a closed, $\mathrm{PSL}(2,\mathbb{R})$-invariant subset of $\mathcal{T}(n',m')$. We want to show that, under various assumptions, X and Y cannot be topologically conjugate.

Lemma 10. *If $\phi : X \mapsto Y$ is a topological conjugacy, ϕ maps degenerate tilings to degenerate tilings and nondegenerate tilings to nondegenerate tilings. Moreover, for nondegenerate tilings the points of tangency of the horoballs at the sphere at infinity are not changed by ϕ.*

Proof. The proof is essentially that of Lemma 3. The fact that we were dealing with fixed-sum tilings (with $n = n' = 3$ and $m = m' = 2$) was never used.

Theorem 9. *Let $n = 3$ and $n' = 5$. Suppose $X \subseteq \mathcal{T}(n,m)$ contains nondegenerate tilings, is invariant under $\mathcal{G}^n$ and closed, and suppose $Y \subseteq \mathcal{T}(n',m')$ is invariant under $\mathcal{G}^n$ and closed. Then the actions of $\mathcal{G}^n$ on X and Y are not topologically conjugate.*

Proof. For a nondegenerate tiling in $\mathcal{T}(3,m)$, the "cusp point set", the set of points at infinity that meet horoballs, is conjugate, by some fixed element of $\mathrm{PSL}(2,\mathbb{R})$ to $\mathbb{Q} \cup \{\infty\}$. The cusp point set for the corresponding (nondegenerate) tiling in $\mathcal{T}(5,m')$ is conjugate to $\mathbb{Q}[\lambda] \cup \infty$, where λ is the golden mean. This contradicts Lemma 10. $\square$

Theorem 10. *Suppose $X \subseteq \mathcal{T}(n,m)$ is invariant under $\mathcal{G}^n$ and closed, and $Y \subseteq \mathcal{T}(n',m')$ is invariant under $\mathcal{G}^n$ and closed. If $m \neq m'$ then the actions of $\mathcal{G}^n$ on X and Y are not topologically conjugate.*

Proof. Assume without loss of generality that $m < m'$. X necessarily contains all the degenerate tilings, and in particular contains a tiling invariant under the map $z \mapsto mz$. By Lemma 10, any conjugacy would have to take this to a degenerate tiling in $\mathcal{T}(n',m')$ that is also invariant under $z \mapsto mz$. However, under this symmetry, points along the y-axis are only moved a distance $\ln(m)$, while any symmetry of a degenerate tiling in $\mathcal{T}(n',m')$ must move points at least a distance $\ln(m')$. □

Note that Theorem 9 requires that X contain nondegenerate tilings; if X consists only of degenerate tilings, then n is irrelevant. Theorem 10, however, only depends on the existence of degenerate tilings. These arise automatically since they are in the orbit closure of every nondegenerate tiling. Generalizing Theorem 9 would require knowing more than we do about cusp point sets for Hecke groups.

V. Isomorphism and Uniqueness in Problems of Optimally Dense Packings

Optimally dense packings, especially tilings of $\mathbb{E}^2$ and $\mathbb{E}^3$, have been important for many years in classifying certain geometric properties of patterns; we refer here to the classification through symmetry groups, the so-called crystallographic symmetries.

At heart the formalism consists of treating the structures of interest as subsets of $\mathbb{X}^n$, with the action on them of $\mathcal{G}^n$ – that is, one uses the structure of dynamical systems. Consider for instance two ball packings in $\mathbb{E}^3$, P^{fcc}, the face centered cubic, and P^{hcp}, the hexagonal close packed. When we choose to distinguish P^{fcc} from P^{hcp} on symmetry grounds what we are saying is that the subgroup of $\mathcal{G}^3$ (the connected Euclidean group) which acts trivially on every element of one orbit is different from the symmetry group of the elements of the other orbit. This implies the systems are not conjugate: there is no bijection, between the two orbits under $\mathcal{G}^3$, which intertwines the action of $\mathcal{G}^3$. So in this simple situation we see that conjugacy can detect differences in symmetry.

The case of the kite & dart tilings (Figure 3) is instructive. As we noted in section III, it is natural to want to think of all these tilings as equivalent. This is true even though the tilings can actually have different symmetry groups; for instance there are two noncongruent kite & dart tilings with a point of 5-fold rotational symmetry, which the other kite & dart tilings do not have [Gard]. Furthermore, the symmetry of the two special tilings actually do not play an essential role, for two reasons. First, the 5-fold rotational symmetry appears in regions of arbitrarily large size in every tiling, and this could replace the exact symmetry of the special tilings. Furthermore the Penrose tilings have a statistical form of 10-fold rotational symmetry, which is expressed by the 10-fold rotational symmetry of all the translation invariant measures on the space of Penrose tilings [Radi]. We also note an analogy between the different "symmetry" of elements of tilings with different fixed-sums, as evidenced by nonconjugacy, and the different (5-fold rotational) symmetry that appears among kite & dart tilings.

So we are led to relax the strict form of equivalence whereby two packings are equivalent if they are in the same orbit under $\mathcal{G}^n$. From the example of the kite & dart tilings one might be led to replace this by having optimally dense packings equivalent if they are generic for the same measure. But from the various examples of section IV we will go one step further.

Definition 4. We say that two optimally dense packings p, p' are "weakly equivalent" if the optimal measures μ_p, $\mu_{p'}$ for which they are generic have the following property: the set $M(p)$ of all optimal measures the support of which intersects the support of μ_p coincides with the set $M(p')$. An optimal density problem will be said to have a "unique solution" if there is only one weak equivalence class of optimally dense packings.

As we saw in section III, there are simple examples of optimally dense packing problems, in particular that for disks of fixed radius in $\mathbb{E}^2$, for which the solution is unique in the sense of consisting of a single orbit of $\mathcal{G}^n$, or, put another way, in the sense that the quotient $P^o_{\mathcal{B}}/\mathcal{G}^n$ consists of a single point. Because of the aperiodicity of the kite & dart tilings, and the modified binary tilings, we have been led to divide $P^o_{\mathcal{B}}$ by a cruder equivalence relation.

This paper is a continuation of a long tradition of classifying a pattern through the dynamical system associated with it by the action of the isometry group of the ambient space of the pattern. A common step taken when following the dynamics approach is to settle on a form of conjugacy, typically either measurable conjugacy or topological conjugacy – and in effect declare two optimization (or tiling) problems equivalent if their dynamical systems are conjugate in the chosen sense. Prominent in this vein is the analysis by Connes, Putnam, Kellendonk et al. noted above, in which invariants of aperiodic tilings are sought through operator algebras associated with their dynamical systems.

Definition 5. We declare two optimal density problems, associated with finite sets $\mathcal{B}$ and $\mathcal{B}'$ of bodies in some fixed $\mathbb{X}^n$, to be "equivalent" if there is a topological conjugacy between their dynamical systems, $(\bar{P}^o_{\mathcal{B}}, \mathcal{G}^n)$ and $(\bar{P}^o_{\mathcal{B}}, \mathcal{G}^n)$, where $\bar{P}^o_{\mathcal{B}}$ denotes the closure in $P_{\mathcal{B}}$ of $P^o_{\mathcal{B}}$.

In terms of this notion of equivalence the proofs of Theorems 9 and 10 show how the optimization problems, for different variations of our modified binary tile, can be distinguished by geometric features.

The operator algebra approach noted above is a powerful way to obtain the desired invariants for topological conjugacy. Associated to a dynamical system (X, G) is the crossed-product C^*-algebra $C(X) \times_\alpha G$, where $C(X)$ is the C^*-algebra of continuous complex-valued functions on X and α is the action of G on $C(X)$. (The crossed-product is the completion of the algebraic tensor product in a certain norm; for this and other terms in operator algebras we refer to [Blac].) The K-theoretic invariants of the crossed-product algebra are topological conjugacy invariants for the dynamical system.

A common way to compute invariants is to associate an AF algebra with the dynamical system, "large" in an algebra Morita equivalent to the crossed-product of interest. (Morita equivalence preserves the K-theoretic invariants.) We do not see how to do that here. Alternatively one could try to compute K_0 using tools such as the Pimsner-Voiculescu 6-term exact sequence; this has been a practical route at least when the dynamical group is $\mathbb{R}$ or $\mathbb{R}^2$, but this seems to be harder for groups such as $\mathrm{PSL}(2, \mathbb{R})$.

In short, the operator algebra methods used to produce dynamical invariants for aperiodic systems in Euclidean space seem to need extension for this more general aperiodic setting.

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