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SPACE AVERAGES AND HOMOGENEOUS FLUID FLOWS

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Abstract. The relation between space averages of vector fields in $L^1_{\text{loc}}(\mathbb{R}^3)$ and averages with respect to homogeneous measures on such vector fields is examined. The space average, obtained by integration over balls in space, is shown to exist almost always and, whenever the measure is ergodic or the correlation decays, to equal the ensemble average.

1. Introduction

In statistical theories of turbulence the velocity vector field $u(t, x)$ of a fluid is taken for each $t > 0$, $x \in \mathbb{R}^3$ to be a random variable on some (usually unspecified) probability space, see [McC] for example. $u(t, x)$ is also required to solve the Navier-Stokes equations in t and x. In more mathematical formulations, the flow is described by a measure on a function space, the support of the measure consisting of solutions of the Navier-Stokes equations, see [VF] or [FT].

Of particular interest are homogeneous flows: Flows with statistical properties independent of shifts in the x argument, alternatively flows described by measures that are invariant under shifts of the argument.

In his "Theory of Homogeneous Turbulence," G.K. Batchelor considers for fixed time the space average of a quantity F depending on the velocity field u of a homogeneous flow

$$\lim_{|A|\to\infty} \frac{1}{|A|} \int_A F(u(t, x))\, dx, \tag{1.1}$$

claims that this average is the same for almost all realizations of the field, and equals the probability average at (any) x:

$$\int F(u(t, x))\, du, \tag{1.2}$$

see [B], page 16. This note is motivated by Batchelor's claim and provides rigorous proofs of his claim within the setting of homogeneous measures on function spaces. It adopts the point of view of [VF], [FT], [FMRT], in particular their definition of averages at a point in space over all flow realizations. The conditions on F will always be compatible with this definition.

It is first shown that the space average exists for almost all realizations of the flow u (Theorem 3.1) with respect to a homogeneous measure. This follows from the standard ergodic theorem after (3.4) has expressed space averages as averages of a 3-dimensional family of measure-preserving transformations. Then yet another application of the ergodic theorem and a factorization lemma (Lemma 4.3) show that the limit of the space averages equals the probability average at a point in space, as defined in [VF], [FT], [FMRT], provided that the homogeneous measure is also ergodic (Theorem 4.4).

Since it is not clear that Batchelor was assuming ergodicity of measure, but he certainly argued later that correlations decay, it is also shown here that the space average equals the probability average at a point in space if the correlation function with respect to a homogeneous measure decays as the distance of separation increases (Theorem 4.6).

The main results, Theorems 3.1, 4.4 and 4.6, follow from classical ergodic theorems as in [Kh], [Ko], [P], and [W]. They are discussed in the setting of homogeneous flows in section 5.

2. Background

Throughout, B_R denotes the ball in $\mathbb{R}^3$ of center 0 and radius R, $|B_R|$ its volume, and dx or $d\lambda$ the Lebesgue measure on $\mathbb{R}^3$. (Everything that follows applies equally well on any $\mathbb{R}^n$.)

The Ergodic Theorem for continuous, 3-dimensional families of measure preserving transformations, cf. [P], [W], will be used repeatedly as follows:

Theorem 2.1. *For $(\mathcal{X},\mu)$ a probability space, F a μ-integrable function on $\mathcal{X}$, and $\{T_\lambda, \lambda \in \mathbb{R}^3\}$ a 3-dimensional family of measure-preserving transformations on $\mathcal{X}$ such that $T_\lambda T_h = T_{\lambda+h}$ for all $\lambda, h \in \mathbb{R}^3$, the limit*

$$F_1(u) := \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} F(T_\lambda u)\, d\lambda \tag{2.1}$$

exists for almost all u in $\mathcal{X}$, and $F_1(u)$ is invariant under T_λ: If $F_1(u_0)$ exists, then $F_1(T_\lambda u_0)$ exists for any λ and equals $F_1(u_0)$. F_1 is integrable with respect to μ, and

$$\int_{\mathcal{X}} F_1(u)\, d\mu(u) = \int_{\mathcal{X}} F(u)\, d\mu(u). \tag{2.2}$$

Moreover, if $F \in L_p(\mu)$ for some $1 \leq p < \infty$ then $F_1 \in L_p(\mu)$ and the convergence (2.1) holds in $L_p(\mu)$ as well.

It is also standard that if μ is ergodic with respect to the T_λ's (i.e. the only invariant subsets are either of full or zero measure) then (2.1) is constant, therefore (2.2) becomes

$$\lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} F(T_\lambda u)\, d\lambda = \int_{\mathcal{X}} F(u)\, d\mu(u). \tag{2.3}$$

Now for a vector field $u = (u_1, u_2, u_3)$ on $\mathbb{R}^3$ whose components u_i are distributions in $L^1_{\text{loc}}(\mathbb{R}^3)$, T_λ will be the translation operation defined by

$$\int_{\mathbb{R}^3} (T_\lambda u)(x)\phi(x)\, dx = \int_{\mathbb{R}^3} u(x)\phi(x-\lambda)\, dx \tag{2.4}$$

for all $\phi : \mathbb{R}^3 \to \mathbb{R}$ smooth and with compact support.

Definition 2.2. *Let $\mathcal{X}$ be a space of vector fields with $T_\lambda : \mathcal{X} \to \mathcal{X}$ for each λ. A measure μ on $\mathcal{X}$ is homogeneous if it is translation invariant: For any μ-integrable F on $\mathcal{X}$ and λ in $\mathbb{R}^3$,*

$$\int_{\mathcal{X}} F(u)\ d\mu(u) = \int_{\mathcal{X}} F(T_\lambda u)\ d\mu(u), \tag{2.5}$$

for all λ.

Assumptions 2.3. *For the rest of this note the following will hold:*

(1) *$\mathcal{X}$ will be a subspace of the Fréchet space of vector fields u on $\mathbb{R}^3$ such that u and ∇u belong to $L^1_{loc}(\mathbb{R}^3)$.*
(2) *The measure μ will be a homogeneous Borel measure on $\mathcal{X}$.*
(3) *For any λ in $\mathbb{R}^3$ and u in $\mathcal{X}$, $T_\lambda(u)$ will also be in $\mathcal{X}$.*
(4) *$\hat{f}(v_0, v_1, v_2, v_3), v_i \in \mathbb{R}^3$, will satisfy the following:*

$$\begin{gathered} f(u(x)) = \hat{f}(u(x), \nabla u(x)) \in L^p_{loc}(\mathbb{R}^3), \\ \int_{\mathcal{X}} \int_{B_R} |f(u(x))|^p\ dx\ d\mu(u) < \infty \end{gathered} \tag{2.6}$$

for some p, $1 \leq p < \infty$, all $u \in \mathcal{X}$, and all $R > 0$.

Example: Of interest are probability spaces $\mathcal{X}$ with a Banach space structure and μ a Borel measure on $\mathcal{X}$. L^p spaces do not support homogeneous measures other than the Dirac measure at 0, see [VF]. However, $\mathcal{X} = \mathcal{H}^0(r)$, the space of vector fields on $\mathbb{R}^3$ with finite $(0, r)$-norm

$$\|u\|^2_{0,r} = \int_{\mathbb{R}^3} \left(1 + |x|^2\right)^r |u(x)|^2\ dx, \quad r < -\frac{3}{2}, \tag{2.7}$$

supports homogeneous measures. See [VF] or section 5 below for more.

Example: Functions usually studied in stochastic turbulence are products of components of u and its derivatives: $(f \circ u)(x) = \partial^{k_1}_{j_1} u_{i_1}(x) \cdots \partial^{k_n}_{j_n} u_{i_n}(x)$.

3. An Ergodic Theorem

Theorem 3.1. *Let $(\mathcal{X}, \mu)$ and f satisfy the assumptions 2.3. Then for μ-almost all u in $\mathcal{X}$*

$$S(u) = \lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R} f(u(\lambda))\ d\lambda \tag{3.1}$$

exists, defines a function in $L^p(\mu)$, and is translation invariant:

$$S(u) = S(T_h(u)) \text{ for all } h \in \mathbb{R}^3. \tag{3.2}$$

Moreover, (3.1) holds in $L^p(\mu)$ as well.

Similar results appear in [T] (Theorems 6.1, 6.9 and Corollary 8.1)and [NZ] (Corollary 4.9, Proposition 4.23). The (short, direct) proof here appeals only on the classical Ergodic Theorem as in [P] and [W].

Proof of Theorem 3.1. Let $(\mathcal{X}, \mu)$ and $f : \mathbb{R}^3 \to \mathbb{R}$ as in the statement of the theorem. Writing f as $f = f^+ - f^-$, it suffices to prove the result for $f : \mathbb{R}^3 \to [0, \infty)$ satisfying (2.6).

Notice that in Theorem 2.1 the function F is defined on the probability space $\mathcal{X}$, but in Theorem 3.1 the function f is defined on the values of u for any $u \in \mathcal{X}$. Define therefore $F : \mathcal{X} \to \mathbb{R}$ by

$$F(u) := \liminf_{n \to \infty} n^3 \int_{[0, \frac{1}{n}]^3} f(u(x)) dx. \tag{3.3}$$

Since for every $u \in \mathcal{X}$, $f \circ u \in L^1_{\text{loc}}(\mathbb{R}^3)$, the Lebesgue differentiation theorem implies that for almost all $\lambda \in \mathbb{R}^3$,

$$F(T_\lambda(u)) = \liminf_{n\to\infty} n^3 \int_{[0,\frac{1}{n}]^3} f(u(x+\lambda))dx = f(u(\lambda))$$

where T_λ is defined in (2.4). Thus for $R > 0$,

$$\frac{1}{|B_R|}\int_{B_R} f(u(\lambda))d\lambda = \frac{1}{|B_R|}\int_{B_R} F(T_\lambda u)d\lambda. \tag{3.4}$$

Since the transformations T_λ ($\lambda \in \mathbb{R}^3$) are measure preserving, Theorem 2.1 finishes the proof once it is shown that $F \in L_p(\mu)$ for the same p that (2.6) holds. In order to show that $F \in L_p(\mu)$,

$$\begin{aligned}
\|F\|_p^p &= \int_{\mathcal{X}} \left(\lim_{n\to\infty} \inf_{m\geq n} m^3 \int_{[0,\frac{1}{m}]^3} f(u(x))dx \right)^p d\mu(u) \\
&= \int_{\mathcal{X}} \lim_{n\to\infty} \inf_{m\geq n} \left(m^3 \int_{[0,\frac{1}{m}]^3} f(u(x))dx \right)^p d\mu(u) \text{ (since } f \geq 0) \\
&\leq \liminf_{n\to\infty} \int_{\mathcal{X}} \inf_{m\geq n} \left(m^3 \int_{[0,\frac{1}{m}]^3} f(u(x))dx \right)^p d\mu(u) \text{ (by Fatou's lemma)} \\
&\leq \liminf_{n\to\infty} \int_{\mathcal{X}} \left(n^3 \int_{[0,\frac{1}{n}]^3} f(u(x))dx \right)^p d\mu(u) \\
&\leq \liminf_{n\to\infty} \int_{\mathcal{X}} \int_{[0,\frac{1}{n}]^3} f(u(x))^p \frac{dx}{n^{-3}} d\mu(u) \qquad (3.5)
\end{aligned}$$

where the last inequality holds by Jensen's inequality since the function $[0,\infty) \ni x \mapsto x^p$ is convex and $\frac{dx}{n^{-3}}$ is a probability measure on $[0, \frac{1}{n}]^3$. Fix $n \in \mathbb{N}$ and let $I = \{(i/n, j/n, k/n) : i, j, k \in \{0, 1, \ldots, n-1\}\}$. Notice that for every $(a, b, c), (a', b', c') \in I$,

$$\int_{\mathcal{X}} \int_{(a,b,c)+[0,\frac{1}{n}]^3} f(u(x))^p dx d\mu(u) = \int_{\mathcal{X}} \int_{(a',b',c')+[0,\frac{1}{n}]^3} f(u(x))^p dx d\mu(u)$$

by the homogeneity of μ. Hence, since $\#I = n^3$,

$$\begin{aligned}
\int_{\mathcal{X}} \int_{[0,\frac{1}{n}]^3} f(u(x))^p dx d\mu(u) &= \frac{1}{n^3} \sum_{(a,b,c)\in I} \int_{\mathcal{X}} \int_{(a,b,c)+[0,\frac{1}{n}]^3} f(u(x))^p dx d\mu(u) \\
&= \frac{1}{n^3} \int_{\mathcal{X}} \int_{[0,1]^3} f(u(x))^p dx d\mu(u). \qquad (3.6)
\end{aligned}$$

Equation (3.6) gives that the right hand side of (3.5) is bounded by a constant independent of n, which implies that $F \in L_p(\mu)$. □

4. Establishing Batchelor's Claim

Batchelor claims that (under appropriate assumptions on the measure μ or the function f) for μ-almost all realizations u in the probability space, the ensemble average of f equals the limit of the space averages of $f \circ u$ (see equation (4.11)). In the present section Batchelor's claim is established. The main results are Theorems 4.4 and 4.6.

First recall from [VF] or [FMRT] the definition of the ensemble average for homogeneous measures at any point in space. Let (X, μ) and f satisfy the assumptions 2.3 for some

$1 \le p < \infty$. Note that after applying Hölder and Jensen's inequalities f satisfies (2.6) for $p = 1$, too. Then the linear functional I defined by

$$I(\phi) := \int_{\mathcal{X}} \int_{\mathbb{R}^3} f(u(\lambda))\phi(\lambda)\ d\lambda\ d\mu(u) \text{ for any } \phi \in C_0^\infty(\mathbb{R}^3, \mathbb{R}), \tag{4.1}$$

is a continuous distribution on $C_0^\infty(\mathbb{R}^3, \mathbb{R})$ and invariant under translations of the ϕ's:

$$\begin{aligned} I(T_x\phi) &= \int_{\mathcal{X}} \int_{\mathbb{R}^3} f(u(\lambda))\phi(\lambda + x) d\lambda d\mu(u) = \int_{\mathcal{X}} \int_{\mathbb{R}^3} f((T_{-x}u)(\lambda))\phi(\lambda) d\lambda \\ &= \int_{\mathcal{X}} \int_{\mathbb{R}^3} f(u(\lambda))\phi(\lambda) d\lambda \text{ (since } \mu \text{ is homogeneous)} \\ &= I(\phi). \end{aligned}$$

Since there is only one translation invariant measure on $\mathbb{R}^3$ up to a multiplicative constant, there is a constant $E(f)$ such that for all $\phi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ and for all $x \in \mathbb{R}^3$,

$$\int_{\mathcal{X}} \int_{\mathbb{R}^3} f(u(\lambda))\phi(\lambda + x)\ d\lambda\ d\mu(u) = E(f) \int_{\mathbb{R}^3} \phi(\lambda) d\lambda. \tag{4.2}$$

By homogeneity, the constant $E(f)$ does not depend on $x \in \mathbb{R}^3$. Then use $\int_{\mathcal{X}} f(u(x)) d\mu(u)$ as an alternative notation for $E(f)$.

Definition 4.1. *The ensemble average of f at (any) $x \in \mathbb{R}^3$ is*

$$\int_{\mathcal{X}} f(u(x)) d\mu(u). \tag{4.3}$$

When μ is ergodic, the following proposition gives that the ensemble average of f can be computed as the limit of the spatial averages of $f \circ u$ for almost all u's:

Proposition 4.2. *Let $(\mathcal{X}, \mu)$ and f satisfy the assumptions 2.3. Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a fixed smooth function with compact support. Then*

$$S_\phi(u) = \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} \int_{\mathbb{R}^3} f(u(x + \lambda))\phi(x)\ dx\ d\lambda \tag{4.4}$$

exists for almost all u. S_ϕ is translation invariant and in $L^1(\mathcal{X})$. In addition, if μ is ergodic then for almost all u

$$S_\phi(u) = \int_{\mathcal{X}} \int_{\mathbb{R}^3} f(u(x))\phi(x)\ dx\ d\mu. \tag{4.5}$$

Proof. After noticing again that if (2.6) is valid for some $1 \le p < \infty$ then it is valid for $p = 1$, this is an immediate application of the standard multidimensional ergodic Theorem 2.1 for the integrable function

$$u \mapsto \int_{\mathbb{R}^3} f(u(x))\phi(x)\ dx \tag{4.6}$$

and the one 3-dimensional family of measure preserving transformations T_λ defined in (2.4). $\square$

Lemma 4.3. *Let $(\mathcal{X}, \mu)$ and f satisfy the assumptions 2.3. Let ϕ be in $C_0^\infty(\mathbb{R}^3)$ with $\phi \ge 0$. Then for μ almost every $u \in \mathcal{X}$,*

$$\begin{aligned} &\lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} \int_{\mathbb{R}^3} f(u(x + \lambda))\phi(x) dx\ d\lambda \\ &= \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f(u(\lambda)) d\lambda \left(\int_{\mathbb{R}^3} \phi(x)\ dx \right). \end{aligned} \tag{4.7}$$

Proof. Fix $\phi \in C_0^\infty$ with $\phi \geq 0$. Writing again $f = f^+ - f^-$, assume that $f : \mathbb{R}^3 \to [0, \infty)$ and satisfies (2.6). Notice that the limits in the left and right hand side of (4.7) exist and are finite for μ almost all $u \in \mathcal{X}$ by Proposition 4.2 and Theorem 3.1 respectively.

Since $f \geq 0$ and $\phi \geq 0$ Fubini's theorem applies:

$$\lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} \int_{\mathbb{R}^3} f(u(x+\lambda))\phi(x)\; dx d\lambda = \lim_{R\to\infty} \int_{\mathbb{R}^3} \frac{1}{|B_R|} \int_{B_R} f(u(x+\lambda))\phi(x) d\lambda\; dx \tag{4.8}$$

and Fatou's lemma on any sequence of R's going to infinity estimates the right hand side of (4.8) as:

$$\begin{aligned}
\lim_{R\to\infty} \int_{\mathbb{R}^3} \frac{1}{|B_R|} \int_{B_R} f(u(x+\lambda))\phi(x) d\lambda\; dx &\geq \int_{\mathbb{R}^3} \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f(u(x+\lambda))\phi(x) d\lambda\; dx \\
&= \int_{\mathbb{R}^3} \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f(u(x+\lambda)) d\lambda \phi(x)\; dx \\
&= \int_{\mathbb{R}^3} \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f(u(\lambda)) d\lambda \phi(x)\; dx \qquad (4.9)\\
&\quad \text{(by Theorem 3.1)} \\
&= \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f(u(\lambda)) d\lambda \int_{\mathbb{R}^3} \phi(x)\; dx.
\end{aligned}$$

On the other hand, since ϕ has compact support, assume that $x \in B_{R_0}$ for some $R_0 > 0$ and change variable to $x + \lambda = y$. Then estimate the right hand side of (4.8) as:

$$\begin{aligned}
\lim_{R\to\infty} \int_{B_{R_0}} \frac{1}{|B_R|} \int_{x+B_R} f(u(y))\phi(x) dy\; dx &\leq \lim_{R\to\infty} \int_{B_{R_0}} \frac{1}{|B_R|} \int_{B_{R+R_0}} f(u(y))\phi(x) dy\; dx \\
&\quad \text{(since } f \geq 0 \text{ and } \phi \geq 0) \\
&= \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_{R+R_0}} f(u(y)) dy \int_{B_{R_0}} \phi(x)\; dx \\
&= \lim_{R\to\infty} \frac{|B_{R+R_0}|}{|B_R|} \frac{1}{|B_{R+R_0}|} \int_{B_{R+R_0}} f(u(y)) dy \int_{B_{R_0}} \phi(x)\; dx \qquad (4.10)\\
&= \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f(u(y)) dy \int_{\mathbb{R}^3} \phi(x)\; dx \\
&\quad \left(\text{since } \lim_{R\to\infty} \frac{|B_{R+R_0}|}{|B_R|} = 1.\right)
\end{aligned}$$

□

The following is precisely Batchelor's claim for an ergodic measure μ.

Theorem 4.4. *Let $(\mathcal{X}, \mu)$ and f satisfy the assumptions 2.3. If μ is an ergodic probability measure then for almost all u in $\mathcal{X}$ the following holds:*

$$\lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f(u(x+\lambda))\; d\lambda = \int_{\mathcal{X}} f(u(x))\; d\mu(u) \tag{4.11}$$

for all x in $\mathbb{R}^3$.

Proof. Fix $\phi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ with $\phi \geq 0$ and $\int_{\mathbb{R}^3} \phi(x) dx = 1$. Then by Proposition 4.2, since μ is ergodic, the ensemble average of f (at any $x \in \mathbb{R}^3$) is given by

$$E(f) = \int_{\mathcal{X}} f(u(x)) dx = \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} \int_{R^3} f(u(x+\lambda))\phi(x) dx d\lambda. \tag{4.12}$$

By Lemma 4.3 since $\phi \geq 0$ and $\int_{\mathbb{R}^3} \phi(x)dx = 1$, one obtains that for μ almost every $u \in \mathcal{X}$,

$$\lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} \int_{R^3} f(u(x+\lambda))\phi(x)dxd\lambda = \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f(u(\lambda))d\lambda. \tag{4.13}$$

Theorem 3.1 implies that for every $x \in \mathbb{R}^3$ and μ almost all $u \in \mathcal{X}$,

$$\lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f(u(\lambda))d\lambda = \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f(u(x+\lambda))d\lambda. \tag{4.14}$$

Thus (4.11) follows from (4.12), (4.13) and (4.14). □

Batchelor has not explicitly indicated the existence of an ergodic measure in [B]. However, it is likely that he was assuming decay of correlations. In fact, [BP] argues that correlations should decay with rate r^{-5}, where r is the distance of separation. The following shows that the original claim in [B] is correct when correlations decay.

Define correlations following [VF] as follows: Let $(\mathcal{X}, \mu)$ and f satisfy the assumptions 2.3 for $p = 2$. Then use Definitions (2.4) and (4.1) to define the correlation $\mathcal{R}_f : \mathbb{R}^3 \to \mathbb{R}$ of the function f by

$$\mathcal{R}_f(h) = E((f - E(f))((f \circ T_h) - E(f \circ T_h))). \tag{4.15}$$

Since $E(f \circ T_h) = E(f)$ for all $h \in \mathbb{R}^3$,

$$\mathcal{R}_f(h) = E(f(f \circ T_h)) - (E(f))^2.$$

Notice that if f satisfies (2.6) for $p = 2$, then for every $h \in \mathbb{R}^3$ and for every $\phi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$, Hölder's inequality gives that

$$\int_{\mathcal{X}} \int_{\mathbb{R}^3} |f(u(x))f(u(x+h))\phi(x)|dxd\mu(u) \leq$$
$$\left[\int_{\mathcal{X}} \int_{\mathbb{R}^3} |f(u(x))|^2 |\phi(x)|dxd\mu(u)\right]^{1/2} \left[\int_{\mathcal{X}} \int_{\mathbb{R}^3} |f(u(x+h))|^2 |\phi(x)|dxd\mu(u)\right]^{1/2} < \infty.$$

The following is needed in the proof of the main result of this section, Theorem (4.6):

Lemma 4.5. *There exists a constant $C > 0$ such that if g on $(\mathcal{X}, \mu)$ satisfies the assumptions 2.3 for $p = 2$, then for every $R \geq 1$,*

$$\int_{\mathcal{X}} \frac{1}{|B_R|} \int_{B_R} g(u(x))^2 dxd\mu(u) \leq C \int_{\mathcal{X}} \int_{[0,1]^3} g(u(x))^2 dxd\mu(u). \tag{4.16}$$

Proof. Fix $R > 0$ and let C_R to denote the smallest set $C = \cup\{(a,b,c)+[0,1)^3 : (a,b,c) \in F\}$ such that F is a finite subset of $\mathbb{Z}^3$ and $B_R \subseteq C$. Then

$$\begin{aligned}
\int_{\mathcal{X}} \frac{1}{|B_R|} \int_{B_R} g(u(x))^2 dxd\mu(u) &\leq \int_{\mathcal{X}} \frac{1}{|B_R|} \int_{C_R} g(u(x))^2 dxd\mu(u) \\
&= \frac{1}{|B_R|} \int_{\mathcal{X}} \sum_{(a,b,c)\in F} \int_{(a,b,c)+[0,1]^3} g(u(x))^2 dxd\mu(u) \\
&\leq \frac{1}{|B_R|} \sum_{(a,b,c)\in F} \int_{\mathcal{X}} \int_{(a,b,c)+[0,1]^3} g(u(x))^2 dxd\mu(u).
\end{aligned}$$

Notice that for every $(a,b,c), (a',b',c') \in F$,

$$\int_{\mathcal{X}} \int_{(a,b,c)+[0,1]^3} g(u(x))^2 dxd\mu(u) = \int_{\mathcal{X}} \int_{(a',b',c')+[0,1]^3} g(u(x))^2 dxd\mu(u)$$

by the homogeneity of μ. Hence

$$\begin{aligned}\frac{1}{|B_R|}\sum_{(a,b,c)\in F}\int_{\mathcal{X}}\int_{(a,b,c)+[0,1]^3} g(u(x))^2 dx d\mu(u) &= \frac{\#F}{|B_R|}\int_{\mathcal{X}}\int_{[0,1]^3} g(u(x))^2 dx d\mu(u)\\ &= \frac{|C_R|}{|B_R|}\int_{\mathcal{X}}\int_{[0,1]^3} g(u(x))^2 dx d\mu(u).\end{aligned}$$

Now set $C := \sup\{\frac{|C_R|}{|B_R|} : R \geq 1\}$ to obtain (4.16). This finishes the proof of Lemma 4.5. $\square$

The next result establishes Batchelor's claim when the correlation of f tends to zero at infinity, cf. [Kh], p. 68.

Theorem 4.6. *Let $(\mathcal{X}, \mu)$ and f satisfy the assumptions 2.3 for $p = 2$. Assume that*

$$\mathcal{R}_f(h) \to 0 \ \text{as}\ |h| \to \infty. \tag{4.17}$$

Then for almost all u in $\mathcal{X}$ the following holds:

$$\lim_{R\to\infty}\frac{1}{|B_R|}\int_{B_R} f(u(x+\lambda))\ d\lambda = \int_{\mathcal{X}} f(u(x))\ d\mu(u) \tag{4.18}$$

for all x in $\mathbb{R}^3$.

Proof. Recall that

$$S(u) := \lim_{R\to\infty}\frac{1}{|B_R|}\int_{B_R} f(u(\lambda))d\lambda$$

exists for μ almost all $u \in \mathcal{X}$ by Theorem 3.1. To show that $S(u) = E(f)$ for μ almost all $u \in \mathcal{X}$, it will be shown that

$$\|S - E(f)\|_2 = 0. \tag{4.19}$$

(Notice that $S \in L^2(\mu)$ by Theorem 3.1, since f satisfies (2.6) for $p = 2$). For $R > 0$ let

$$S_R(u) := \frac{1}{|B_R|}\int_{B_R} f(u(\lambda))d\lambda.$$

Then

$$\|S - E(f)\|_2 \leq \|S - S_R\|_2 + \|S_R - E(f)\|_2. \tag{4.20}$$

By Theorem 3.1,

$$\|S - S_R\|_2 \to 0 \text{ as } R \to \infty. \tag{4.21}$$

To estimate $\|S_R - E(f)\|_2$, let $\varepsilon > 0$ and choose $\Lambda > 0$ such that $|\mathcal{R}_f(\lambda)| < \varepsilon$ for all $|\lambda| > \Lambda$. For $R > \max(\Lambda/2, 1)$ use Fubini's Theorem and change the variable λ to $x + \lambda$ to obtain

$$\|S_R - E(f)\|_2^2$$

$$= \int_{\mathcal{X}} \left(\frac{1}{|B_R|} \int_{B_R} f(u(x)) - E(f) dx \right)^2 d\mu(u)$$

$$= \frac{1}{|B_R|^2} \int_{\mathcal{X}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (f(u(x)) - E(f))\chi_{B_R}(x)(f(u(\lambda)) - E(f))\chi_{B_R}(\lambda) dx d\lambda d\mu(u)$$

$$= \frac{1}{|B_R|^2} \int_{\mathcal{X}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (f(u(x)) - E(f))\chi_{B_R}(x)(f(u(x+\lambda)) - E(f))\chi_{B_R}(x+\lambda) dx d\lambda d\mu(u)$$

(4.22)

$$= \frac{1}{|B_R|^2} \int_{\mathcal{X}} \int_{B_\Lambda} \int_{\mathbb{R}^3} (f(u(x)) - E(f))\chi_{B_R}(x)(f(u(x+\lambda)) - E(f))\chi_{B_R}(x+\lambda) dx d\lambda d\mu(u)$$

$$+ \frac{1}{|B_R|^2} \int_{\mathcal{X}} \int_{B_\Lambda^c} \int_{\mathbb{R}^3} (f(u(x)) - E(f))\chi_{B_R}(x)(f(u(x+\lambda)) - E(f))\chi_{B_R}(x+\lambda) dx d\lambda d\mu(u).$$

Notice that $\chi_{B_R}(x)\chi_{B_R}(x+\lambda) = 0$ if $|\lambda| > 2R$, thus the last term of (4.22) is equal to

(4.23)

$$\frac{1}{|B_R|^2} \int_{\mathcal{X}} \int_{B_{2R}\backslash B_\Lambda} \int_{\mathbb{R}^3} (f(u(x)) - E(f))\chi_{B_R}(x)(f(u(x+\lambda)) - E(f))\chi_{B_R}(x+\lambda) dx d\lambda d\mu(u).$$

Notice that for every fixed $\lambda \in B_{2R}\backslash B_\Lambda$ the function

$$\phi_\lambda(x) = \frac{\chi_{B_R}(x)\chi_{B_R}(x+\lambda)}{|B_R \cap (B_R - \lambda)|}$$

is integrable and $\int_{\mathbb{R}^3} \phi_\lambda(x) dx = 1$. Since the function ϕ_λ can be perturbed to be equal to a smooth function except on a set of arbitrarily small measure, the correlation of f can be expressed as

$$\mathcal{R}_f(\lambda) = \int_{\mathcal{X}} \int_{\mathbb{R}^3} (f(u(x)) - E(f))(f(u(x+\lambda)) - E(f))\phi_\lambda(x) dx d\mu(u).$$

Hence after applying Fubini's theorem, and multiplying and dividing by $|B_R \cap (B_R - \lambda)|$, (4.23) can be rewritten as:

$$\frac{1}{|B_R|^2} \int_{B_{2R}\backslash B_\Lambda} |B_R \cap (B_R - \lambda)| \int_{\mathcal{X}} \int_{\mathbb{R}^3} (f(u(x)) - E(f))(f(u(x+\lambda)) - E(f))\phi_\lambda(x) dx d\mu(u) d\lambda$$

(4.24)

$$= \frac{1}{|B_R|^2} \int_{B_{2R}\backslash B_\Lambda} |B_R \cap (B_R - \lambda)| \mathcal{R}_f(\lambda) d\lambda.$$

By the choice of Λ, the last integral of (4.24) can be estimated by

$$\begin{aligned}\left|\frac{1}{|B_R|^2}\int_{B_{2R}\setminus B_\Lambda}|B_R\cap(B_R-\lambda)|\mathcal{R}_f(\lambda)d\lambda\right| &\le \frac{1}{|B_R|^2}\int_{B_{2R}\setminus B_\Lambda}|B_R\cap(B_R-\lambda)||\mathcal{R}_f(\lambda)|d\lambda\\ &\le \frac{1}{|B_R|^2}\int_{B_{2R}\setminus B_\Lambda}|B_R|\varepsilon d\lambda\\ &\le \frac{|B_{2R}|}{|B_R|}\varepsilon\\ &= 8\varepsilon. \end{aligned}\tag{4.25}$$

Putting equations (4.22) and (4.23)-(4.25) together,

$$\begin{aligned}\|S_R-E(f)\|_2^2 \le 8\varepsilon + \frac{1}{|B_R|^2}\int_{\mathcal{X}}\int_{B_\Lambda}\int_{\mathbb{R}^3}&(f(u(x))-E(f))\chi_{B_R}(x)\\ &(f(u(x+\lambda))-E(f))\chi_{B_R}(x+\lambda)dxd\lambda d\mu(u).\end{aligned}\tag{4.26}$$

To estimate the last term of (4.26) use first Hölder and then Fubini's once again to obtain

$$\begin{aligned}&\frac{1}{|B_R|^2}\left|\int_{\mathcal{X}}\int_{B_\Lambda}\int_{\mathbb{R}^3}(f(u(x))-E(f))\chi_{B_R}(x)(f(u(x+\lambda))-E(f))\chi_{B_R}(x+\lambda)dxd\lambda d\mu(u)\right|\\ &\le \frac{1}{|B_R|^2}\int_{\mathcal{X}}\int_{B_\Lambda}\int_{B_R}|(f(u(x))-E(f))(f(u(x+\lambda))-E(f))|dxd\lambda d\mu(u)\\ &\le \frac{1}{|B_R|^2}\left[\int_{\mathcal{X}}\int_{B_\Lambda}\int_{B_R}|(f(u(x))-E(f))|^2dxd\lambda d\mu(u)\right]^{1/2}\\ &\qquad\left[\int_{\mathcal{X}}\int_{B_\Lambda}\int_{B_R}|(f(u(x+\lambda))-E(f))|^2dxd\lambda d\mu(u)\right]^{1/2}\\ &= \frac{|B_\Lambda|^{1/2}}{|B_R|^2}\left[\int_{\mathcal{X}}\int_{B_R}|(f(u(x))-E(f))|^2dxd\mu(u)\right]^{1/2}\\ &\qquad\left[\int_{B_R}\int_{\mathcal{X}}\int_{B_\Lambda}|(f(u(x+\lambda))-E(f))|^2d\lambda d\mu(u)dx\right]^{1/2}.\end{aligned}\tag{4.27}$$

Now there exists a constant C which does not depend on $R\ge 1$ such that

$$\int_{\mathcal{X}}\int_{B_R}|(f(u(x))-E(f))|^2dxd\mu(u)\le C|B_R|\int_{\mathcal{X}}\int_{[0,1]^3}|(f(u(x))-E(f))|^2dxd\mu(u).\tag{4.28}$$

(4.28) follows when Lemma 4.5 is applied to $g(\cdot)=|f(\cdot)-E(f)|$. Now, (4.28) and (2.6) for $p=2$ imply that there exists a constant D which does not depend on R such that

$$\begin{aligned}\int_{\mathcal{X}}\int_{B_R}|(f(u(x))-E(f))|^2dxd\mu(u) &\le C|B_R|\int_{\mathcal{X}}\int_{[0,1]^3}|(f(u(x))-E(f))|^2dxd\mu(u)\\ &\le D|B_R|.\end{aligned}\tag{4.29}$$

Also the homogeneity of μ implies that (perhaps by increasing D which still does not depend on R) for every x,

$$\begin{aligned}\int_{\mathcal{X}}\int_{B_\Lambda} &|(f(u(x+\lambda)) - E(f))|^2 d\lambda d\mu(u) \\ &= \int_{\mathcal{X}}\int_{B_\Lambda} |(f(u(\lambda)) - E(f))|^2 d\lambda d\mu(u) \leq D.\end{aligned} \tag{4.30}$$

By equations (4.29) and (4.30) the right hand side of (4.27) is less than or equal to

$$\frac{|B_\Lambda|^{1/2}}{|B_R|^2} D^{1/2} |B_R|^{1/2} \left[\int_{B_R} D dx\right]^{1/2} = \frac{D|B_\Lambda|^{1/2}}{|B_R|} \to 0 \text{ as } R \to \infty. \tag{4.31}$$

Thus the right hand side of (4.26) tends to 8ε as $R \to \infty$. Since $\varepsilon > 0$ is arbitrary, this finishes the proof of Theorem 4.6. ☐

5. Application to Homogeneous Flows

For homogeneous fluid flows apply the results of the previous sections when $\mathcal{X} = \mathcal{H}^0(r)$, $r < -\frac{3}{2}$. Families of homogeneous measures on such spaces (measures supported on solenoidal trigonometric polynomials and Gaussian measures), are given in [VF] (p. 209 and p. 210 respectively). These include examples satisfying condition (2.6) for $p = 2$.

The choice of space is justified by the following from [VF] which shows that, for almost all initial conditions, weak solutions of the Navier-Stokes equations stay in $\mathcal{H}^0(r)$:

Theorem 5.1. *(cf.* [VF]*, p. 260) For any homogeneous probability measure μ_0 on the divergence-free elements of $\mathcal{H}^0(r)$, $r < -\frac{3}{2}$, such that*

$$\int_{\mathcal{H}^0(r)} |u(x)|^2 \, d\mu_0(u) < \infty,$$

there is a set V of $\mu_0(V) = 1$, such that for any u_0 in V there is $u(t,x)$ in $L^2(0,T;\mathcal{H}^0(r))$, satisfying for any ϕ in $C_0^\infty\left((0,T)\times\mathbb{R}^3\right) \cap C((0,T);\mathcal{H}^0(r))$

$$< u(t,.), \phi >_2 - < u_0, \phi >_2 = \int_0^t \left(< u(\tau,.), \Delta\phi >_2 + \sum_{j=1}^{3} < u_j u, \frac{\partial \phi}{\partial x_j} >_2 \right) d\tau \tag{5.1}$$

for almost all t in $[0,T]$.

[VF] also produce a family of homogeneous measures μ_t, supported for each t by the restrictions at time t of the solutions of (5.1) satisfying condition (2.6) for $p = 2$ whenever μ_0 does. Therefore, whenever μ_t is ergodic or its correlation decays, Batchelor's claim will hold. In work currently in progress, the authors investigate conditions for the ergodicity of the μ_t's and rigorous decay estimates for their correlation functions.

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